Chapter 5

Topological properties of manifolds

In this chapter we investigate the consequences for a manifold, when certain topological properties are assumed. In particular, we develop an important analytical tool, called partition of unity.

5.1 Compactness

Recall that in a metric space $X$, a subset $K$ is said to be compact, if every sequence from $K$ has a subsequence which converges to a point in $K$. Recall also that every compact set is closed and bounded, and that the converse statement is valid for $X = \mathbb{R}^n$ with the standard metric, that is, the compact subsets of $\mathbb{R}^n$ are precisely the closed and bounded subsets.

The generalization of compactness to an arbitrary topological space $X$ does not invoke sequences. It originates from another important property of compact sets in a metric space, called the Heine-Borel property, which concerns coverings of $K$.

Let $X$ be a Hausdorff topological space, and let $K \subset X$.

**Definition 5.1.1.** An open covering of $K$ is a collection of open sets $U_i \subset X$, where $i \in I$, whose union $\bigcup_i U_i$ contains $K$. A subcovering is a subcollection $(U_j)_{j \in J}$, where $J \subset I$, which also is an open covering of $K$.

**Definition 5.1.2.** The set $K$ is said to be compact if every open covering has a finite subcovering.

It is a theorem that for a metric space the property in Definition 5.1.2 is equivalent with the property that $K$ is compact (according to the definition with sequences), hence there is no conflict of notations. The space $X$ itself is called compact, if it satisfies the above definition with $K = X$.

The following properties of compact sets are well known and easy to prove. Let $X$ and $Y$ be Hausdorff topological spaces.

**Lemma 5.1.1.** Let $f: X \to Y$ be a continuous map. If $K \subset X$ is compact, then so is the image $f(K) \subset Y$.

**Lemma 5.1.2.** Let $K \subset X$ be compact. Then $K$ is closed, and every closed subset of $K$ is compact.
Lemma 5.1.3. Assume that $X$ is compact, and let $f:X \to Y$ be a continuous bijection. Then $f$ is a homeomorphism.

Proof. We have to show that $f^{-1}$ is continuous, or equivalently, that $f$ carries open sets to open sets. By taking complements, we see that it is also equivalent to show that $f$ carries closed sets to closed sets. This follows from Lemma 5.1.1, in view of Lemma 5.1.2. □

5.2 Countable exhaustion by compact sets

The following property of a manifold with a countable atlas will be used in the next section.

Theorem 5.2. Let $M$ be an abstract manifold. The following conditions are equivalent.

(i) There exists a countable atlas for $M$.

(ii) There exists a sequence $K_1, K_2, \ldots$ of compact sets with union $M$.

(iii) There exists a sequence $D_1, D_2, \ldots$ of open sets with union $M$, such that each $D_n$ has compact closure $\bar{D}_n$.

(iv) There exists a sequence $K_1, K_2, \ldots$ of compact sets with union $M$ such that

$$K_1 \subset K_2^o \subset \cdots \subset K_n \subset K_{n+1}^o \subset \ldots.$$ 

Proof. The implications (iv)$\Rightarrow$(iii)$\Rightarrow$(ii) are easily seen. The implication (ii)$\Rightarrow$(i) follows from the fact, seen from Definition 5.1.2, that every compact set $K \subset M$ can be covered by finitely many charts.

We establish (i)$\Rightarrow$(iii). For each chart $\sigma:U \to M$ the collection of closed balls $\bar{B}(x,r)$ in $U$ with rational center and rational radius is countable, and the corresponding open balls cover $U$ (see Example 2.9). Since $\sigma$ is a homeomorphism, the collection of all the images $\sigma(B(x,r))$ of these balls, for all charts in a countable atlas, is a countable collection of open sets with the desired property.

Finally we prove that (iii)$\Rightarrow$(iv). Let $D_1, D_2, \ldots$ be as in (iii). Put $K_1 = \bar{D}_1$. By the compactness of $K_1$ we have

$$K_1 \subset D_1 \cup D_2 \cup \cdots \cup D_{i_1}$$

for some number $i_1$. Put

$$K_2 = \bar{D}_1 \cup \bar{D}_2 \cup \cdots \cup \bar{D}_{i_1},$$

then $K_2$ is compact and $K_1 \subset K_2^o$. Again by compactness we have

$$K_2 \subset D_1 \cup D_2 \cup \cdots \cup D_{i_2}$$

for some number $i_2 > i_1$. Put

$$K_3 = \bar{D}_1 \cup \bar{D}_2 \cup \cdots \cup \bar{D}_{i_2},$$

then $K_3$ is compact and $K_2 \subset K_3^o$. Proceeding inductively in this fashion we obtain the desired sequence. □
5.3 Locally finite atlas

**Definition 5.3.** A collection of subsets of a topological space \( X \) is said to be **locally finite**, if for each element \( x \in X \) there exists a neighborhood which intersects non-trivially with only finitely many of the subsets.

An atlas of an abstract manifold \( M \) is said to be locally finite if the collection of images \( \sigma(U) \) is locally finite in \( M \).

In the following lemma we give a useful criterion for the existence of a locally finite atlas.

**Lemma 5.3.1.** Let \( M \) be an abstract manifold. There exists a locally finite atlas for \( M \) if and only if the following criterion holds.

There exists a covering \( M = \bigcup_{\alpha \in A} K\alpha \) of \( M \) by compact sets \( K\alpha \subset M \), and a locally finite covering \( M = \bigcup_{\alpha \in A} W\alpha \) by open sets \( W\alpha \subset M \) (with the same set \( A \) of indices), such that \( K\alpha \subset W\alpha \) for each \( \alpha \in A \).

Before proving the lemma, we shall verify that the criterion holds in case of a manifold with a countable atlas.

**Lemma 5.3.2.** Let \( M \) be an abstract manifold for which there exists a countable atlas. Then the criterion in Lemma 5.3.1 holds for \( M \).

*Proof.* Let \( L_1, L_2, \ldots \) be an increasing sequence of compact sets in \( M \) as in Theorem 5.2(iv). Put \( K_1 = L_1 \) and \( K_n = L_n \setminus L_{n-1}^c \) for \( n > 1 \), and put \( W_1 = \sigma L_2 \), \( W_2 = \sigma L_3 \) and \( W_n = \sigma L_{n+1} \setminus L_{n-2} \) for \( n > 2 \). It is easily seen that the criterion is satisfied by these collections of sets. \( \square \)

*Proof of Lemma 5.3.1.* Assume that \( M \) has a locally finite atlas. For each chart \( \sigma \) in this atlas, we can apply the preceding lemma to the manifold \( \sigma(U) \) (which has an atlas of a single chart). It follows that there exist collections of sets as described, which cover \( \sigma(U) \). The combined collection of these sets, over all charts in the atlas, satisfies the desired criterion for \( M \).

Assume conversely that the above condition holds. For each \( \alpha \in A \) we can cover the compact set \( K\alpha \) by finitely many charts \( \sigma \), each of which maps into the open set \( W\alpha \). The combined collection of all these charts for all \( \alpha \) is an atlas because \( \bigcup_{\alpha} K\alpha = M \). Each \( p \in M \) has a neighborhood which is disjoint from all but finitely many \( W\alpha \), hence also from all charts except the finite collection of those which maps into these \( W\alpha \)’s. Hence this atlas is locally finite. \( \square \)

**Corollary 5.3.** Every abstract manifold with a countable atlas has a locally finite atlas.

*Proof.* Follows immediately from the two lemmas (in fact, by going through the proofs above, one can verify that there exists an atlas which is both countable and locally finite). \( \square \)
5.4 Partition of unity

Recall that the support, supp$f$, of a function $f: M \to \mathbb{R}$ is the closure of the set where $f$ is non-zero.

**Definition 5.4.** Let $M$ be an abstract manifold. A partition of unity for $M$ is a collection $(f_\alpha)_{\alpha \in A}$ of functions $f_\alpha \in C^\infty(M)$ such that:

1. $0 \leq f_\alpha \leq 1$,
2. the collection of the supports supp$f_\alpha$, $\alpha \in A$, is locally finite in $M$,
3. $\sum_{\alpha \in A} f_\alpha(x) = 1$ for all $x \in M$.

Notice that because of condition (2) the (possibly infinite) sum in (3) has only finitely many non-zero terms for each $x$ (but the non-zero terms are not necessarily the same for all $x$).

**Theorem 5.4.** Let $M$ be an abstract manifold for which there exists a locally finite atlas, and let $M = \bigcup_{\alpha \in A} \Omega_\alpha$ be an arbitrary open cover of $M$. Then there exists a partition of unity $(f_\alpha)_{\alpha \in A}$ for $M$ (with the same set of indices), such that $f_\alpha$ has support inside $\Omega_\alpha$ for each $\alpha$.

**Proof.** Let $M = \bigcup_{\beta \in B} K_\beta$ and $M = \bigcup_{\beta \in B} W_\beta$ be coverings as in Lemma 5.3.1, with $K_\beta \subset W_\beta$. Let $\beta \in B$ be arbitrary. For each $p \in K_\beta$ we choose $\alpha \in A$ such that $p \in \Omega_\alpha$, and we choose a chart $\sigma: U \to M$ (not necessarily from the given atlas) with $\sigma(x) = p$ and with image $\sigma(U) \subset \Omega_\alpha \cap W_\beta$. Furthermore, we choose a pair of concentric open balls $B(x, r) \subset B(x, s)$ around $x$, such that the the closure of the larger ball $B(x, s)$ is contained in $U$. Since $K_\beta$ is compact a finite collection of the images of the smaller balls $\sigma(B(x, r))$ cover it. We choose such a finite collection of concentric balls for each $\beta$.

Since the sets $K_\beta$ have union $M$, the combined collection of all the images of inner balls, over all $\beta$, covers $M$. Furthermore, for each $p \in M$ there exists a neighborhood which meets only finitely many of the images of the outer balls, since the $W_\beta$ are locally finite. That is, the collection of the images of the outer balls is locally finite.

The rest of the proof is based on Lemma 3.5.3. For each of the above mentioned pairs of concentric balls we choose a smooth function $g \in C^\infty(M)$ with the properties mentioned in this lemma. Let $(g_i)_{i \in I}$ denote the collection of all these functions. By the remarks in the preceding paragraph, we see that for each $p \in M$ there exists some $g_i$ with $g_i(p) = 1$, and only finitely many of the functions $g_i$ are non-zero at $p$.

Hence the sum $g = \sum_i g_i$ has only finitely many terms in a neighborhood of each point $p$. It follows that the sum makes sense and defines a positive smooth function. Let $f_i = g_i/g$, then this is a partition of unity. Moreover, for each $i$ there exists an $\alpha \in A$ such that $g_i$, hence also $f_i$, is supported inside $\Omega_\alpha$.

We need to fix the index set for the $g_i$ so that it is $A$. For each $i$ choose $\alpha_i$ such that $f_i$ has support inside $\Omega_{\alpha_i}$. For each $\alpha \in A$ let $f_\alpha$ denote the sum
of those $f_i$ for which $\alpha_i = \alpha$, if there are any. Let $f_\alpha = 0$ otherwise. The result follows easily. □

**Corollary 5.4.** Let $M$ be an abstract manifold with a locally finite atlas, and let $C_0, C_1$ be closed, disjoint subsets. There exists a smooth function $f \in C^\infty(M)$ with values in $[0, 1]$, which is 1 on $C_0$ and 0 on $C_1$.

*Proof.* Apply the theorem to the covering of $M$ by the complements of $C_1$ and $C_0$. □

**5.5 Embedding in Euclidean space**

As an illustration of the use of partitions of unity, we shall prove the following theorem, which is a weak version for compact manifolds of Whitney’s Theorem 2.10. Notice that the theorem applies to all compact manifolds.

**Theorem 5.5.** Let $M$ be an abstract manifold, for which there exists a finite atlas. Then there exists a number $N \in \mathbb{N}$ and a diffeomorphism of $M$ onto a manifold in $\mathbb{R}^N$.

*Proof.* Let $\sigma_i: U_i \to M$, where $i = 1, \ldots, n$ be a collection of charts on $M$, which comprise an atlas. We will prove the theorem with the value $N = n(m + 1)$ where $m = \dim M$.

Let $f_1, \ldots, f_n$ be a partition of unity for $M$ such that $\text{supp} f_i \subset \sigma_i(U_i)$ for each $i$. Its existence follows from Theorem 5.4. Then $f_i \in C^\infty(M)$ and $f_1 + \cdots + f_n = 1$.

For each $i$, let $\varphi_i = \sigma_i^{-1}: \sigma_i(U_i) \to U_i \subset \mathbb{R}^m$ denote the inverse of $\sigma_i$. It is a smooth map. We define a function $h^i: M \to \mathbb{R}^m$ by

$$h^i(p) = f_i(p)\varphi_i(p)$$

for $p \in \sigma_i(U_i)$ and by $h^i(p) = 0$ outside this set. Then $h^i$ is smooth since the support of $f_i$ is entirely within $\sigma_i(U_i)$, so that every point in $M$ has a neighborhood either entirely inside $\sigma_i(U_i)$ or entirely inside the set where $f_i$, and hence also $h^i$, is 0.

We now define a smooth map $F: M \to \mathbb{R}^N = \mathbb{R}^m \times \cdots \times \mathbb{R}^m \times \mathbb{R}^n$ by

$$F(p) = (h^1(p), \ldots, h^n(p), f_1(p), \ldots, f_n(p)).$$

We will show that $F$ is injective and a homeomorphism onto its image $F(M)$. Let $\Omega_i = \{p \in M \mid f_i(p) \neq 0\}$, for $i = 1, \ldots, n$, then these sets are open and cover $M$. Likewise, let

$$W_i = \{(x^1, \ldots, x^n, y_1, \ldots, y_n) \in \mathbb{R}^N \mid y_i \neq 0, \frac{x_i}{y_i} \in U_i\},$$
where \( x^i \in \mathbb{R}^m \) and \( y_i \in \mathbb{R} \) for \( i = 1, \ldots, n \). Then \( W_i \) is open in \( \mathbb{R}^N \), and \( F(\Omega_i) \subset W_i \).

Define \( g_i : W_i \to M \) by

\[
g_i(x^1, \ldots, x^n, y_1, \ldots, y_n) = \sigma(x^i/y_i)
\]

then \( g_i \) is clearly smooth, and we see that \( g_i(F(p)) = p \) for \( p \in \Omega_i \).

It follows now that \( F \) is injective, for if \( F(p) = F(q) \), then since the values \( f_1(p), \ldots, f_n(p) \) are among the coordinates of \( F(p) \), we conclude that \( f_i(p) = f_i(q) \) for each \( i \). In particular, \( p \in \Omega_i \) if and only if \( q \in \Omega_i \). We choose \( i \) such that \( p, q \in \Omega_i \) and conclude \( p = g_i(F(p)) = g_i(F(q)) = q \).

Furthermore, on the open set \( F(M) \cap W_i \) in \( F(M) \), the inverse of \( F \) is given by the restriction of \( g_i \), which is continuous. Hence \( F \) is a homeomorphism onto its image.

For each \( i = 1, \ldots, n \) we now see that the restriction of \( F \circ \sigma_i \) to \( \sigma_i^{-1}(\Omega_i) \) is an embedded \( m \)-dimensional parametrized manifold in \( \mathbb{R}^N \), whose image is \( F(M) \cap W_i \). It follows that \( F(M) \) is a manifold in \( \mathbb{R}^N \).

The map \( F \) is smooth and bijective \( M \to F(M) \), and we have seen above that the inverse has local smooth extensions \( g_i \) to each set \( W_i \), hence the inverse is smooth, and \( F \) is a diffeomorphism onto its image. \( \square \)

5.6 Connectedness

In this section two different notions of connectedness for subsets of a topological space are introduced and discussed. Let \( X \) be a non-empty topological space.

**Definition 5.6.** (1) \( X \) is said to be connected if it cannot be separated in two disjoint non-empty open subsets, that is, if \( X = A_1 \cup A_2 \) with \( A_1, A_2 \) open and disjoint, then \( A_1 \) or \( A_2 \) is empty (and \( A_2 \) or \( A_1 \) equals \( X \)).

(2) \( X \) is called pathwise connected if for each pair of points \( a, b \in S \) there exists real numbers \( \alpha \leq \beta \) and a continuous map \( \gamma : [\alpha, \beta] \to X \) such that \( \gamma(\alpha) = a \) and \( \gamma(\beta) = b \) (in which case we say that \( a \) and \( b \) can be joined by a continuous path in \( X \)).

(3) A non-empty subset \( E \subset X \) is called connected or pathwise connected if it has this property as a topological space with the inherited topology.

The above definition of ”connected” is standard in the theory of topological spaces. However, the notion of ”pathwise connected” is unfortunately sometimes also referred to as ”connected”. The precise relation between the two notions will be explained in this section. The empty set was excluded in the definition, let us agree to call it both connected and pathwise connected.

**Example 5.6.1** A singleton \( E = \{x\} \subset X \) is clearly both connected and pathwise connected.
Example 5.6.2 A convex subset \( E \subset \mathbb{R}^n \) is pathwise connected, since by definition any two points from \( E \) can be joined by a straight line, hence a continuous curve, inside \( E \). It follows from Theorem 5.6.3 below that such a subset is also connected.

Example 5.6.3 It is a well-known fact, called the intermediate value property, that a continuous real function carries intervals to intervals. It follows from this fact that a subset \( E \subset \mathbb{R} \) is pathwise connected if and only if it is an interval. We shall see below in Theorem 5.6.1 that likewise \( E \) is connected if and only if it is an interval. Thus for subsets of \( \mathbb{R} \) the two definitions agree.

Lemma 5.6. Let \( (E_i)_{i \in I} \) be a collection of subsets of \( X \), and let \( E_0 \subset X \) be a subset with the property that \( E_i \cap E_0 \neq \emptyset \) for all \( i \).

If both \( E_0 \) and all the sets \( E_i \) are connected, respectively pathwise connected, then so is their union \( E = E_0 \cup (\cup_i E_i) \).

Proof. Assume that \( E_0 \) and the \( E_i \) are connected, and assume that \( E \) is separated in a disjoint union \( E = A_1 \cup A_2 \) where \( A_1, A_2 \) are relatively open in \( E \). Then \( A_1 = W_1 \cap E \) and \( A_2 = W_2 \cap E \), where \( W_1, W_2 \) are open in \( X \). Hence the intersections \( A_1 \cap E_i = W_1 \cap E_i \) and \( A_2 \cap E_i = W_2 \cap E_i \) are relatively open in \( E_i \) for all \( i \) (including \( i = 0 \)).

It follows that \( E_i = (A_1 \cap E_i) \cup (A_2 \cap E_i) \) is a disjoint separation in open sets, and since \( E_i \) is connected, one of the sets \( A_1 \cap E_i \) and \( A_2 \cap E_i \) must be empty, for each \( i \) (including \( i = 0 \)). If for example \( A_1 \cap E_0 = \emptyset \), then \( E_0 \) is contained in \( A_2 \), and since all the other \( E_i \) have nontrivial intersection with \( E_0 \), we conclude that then \( E_i \cap A_2 \neq \emptyset \) for all \( i \). Hence the intersection \( E_i \cap A_1 \) is empty for all \( i \), and we conclude that \( A_1 \) is empty. We have shown that \( E \) is connected.

Assume next that \( E_0 \) and all the \( E_i \) are pathwise connected. Since all the \( E_i \) have non-trivial intersection with \( E_0 \), every point in \( E \) can be joined to a point in \( E_0 \) by a continuous path. Hence any two points of \( E \) can be joined by a continuous path, composed by the paths that join them to two points in \( E_0 \), and a path in \( E_0 \) that joins these two points. \( \square \)

Example 5.6.4 Let \( H \subset \mathbb{R}^3 \) be the surface \( \{(x, y, z) \mid x^2 + y^2 - z^2 = 1\} \), called the one-sheeted hyperboloid. We will prove that it is pathwise connected. Let

\[ H_0 = \{(x, 0, z) \mid x^2 - z^2 = 1, x > 0\} \subset H \]

and for each \( t \in \mathbb{R} \),

\[ C_t = \{(x, y, t) \mid x^2 + y^2 = 1 + t^2\} \subset H. \]

Then \( H \) is the union of these sets. The set \( H_0 \) is the trace of the continuous curve \( t \mapsto (\sqrt{1+t^2}, 0, t) \), hence it is pathwise connected, and each set \( C_t \) is a circle, hence also pathwise connected. Finally, \( H_0 \cap C_t \) is non-empty for all \( t \), as it contains the point \( (\sqrt{1+t^2}, 0, t) \).
Theorem 5.6.1. Let $E \subset \mathbb{R}$ be non-empty. Then $E$ is connected if and only if it is an interval.

Proof. Assume that $E$ is connected. Let $a = \inf E$ and $b = \sup E$. For each element $c \in \mathbb{R}$ with $a < c < b$ the sets $E \cap ]-\infty, c[$ and $E \cap ]c, \infty[$ are disjoint and open in $E$, and it follows from the definitions of $a$ and $b$ that they are both non-empty. Since $E$ is connected, their union cannot be $E$, hence we conclude that $c \in E$. Hence $E$ is an interval with endpoints $a$ and $b$.

Assume conversely that $E$ is an interval, and that $E = A \cup B$ where $A$ and $B$ are open, disjoint and non-empty. Since they have open complements, $A$ and $B$ are also closed in $E$. Choose $a \in A$ and $b \in B$, and assume for example that $a < b$. Then $[a, b] \subset E$, since $E$ is an interval. The set $[a, b] \cap A$ is non-empty, since it contains $a$, let $c$ be its supremum. Since $A$ is closed it contains $c$, hence $c < b$ and $]c, b[ \subset B$. Since $B$ is closed it also contains $c$, contradicting that $A$ and $B$ are disjoint. □

One of the most fundamental property of connected sets is expressed in the following theorem, which generalizes the intermediate value property for real functions on $\mathbb{R}$ (see Example 5.6.3).

Theorem 5.6.2. Let $f : X \to Y$ be a continuous map between topological spaces. If $E \subset X$ is connected, then so is the image $f(E) \subset Y$. Likewise, if $E$ is pathwise connected then so is $f(E)$.

Proof. We may assume $E = X$ (otherwise we replace $X$ by $E$).

1) Assume $f(X) = B_1 \cup B_2$ with $B_1, B_2$ open and disjoint, and let $A_i = f^{-1}(B_i)$. Then $A_1, A_2$ are open, disjoint and with union $X$. Hence if $X$ is connected then $A_1$ or $A_2$ is empty, and hence $B_1$ or $B_2$ is empty.

2) If $a, b \in X$ can be joined by a continuous path $\gamma$, then $f(a)$ and $f(b)$ are joined by the continuous path $f \circ \gamma$. □

Theorem 5.6.3. A pathwise connected topological space is also connected.

Proof. Suppose $X$ were pathwise connected but not connected. Then $X = A \cup B$ with $A, B$ open, disjoint and nonempty. Let $a \in A$, $b \in B$, then there exists a continuous path $\gamma : [\alpha, \beta] \to X$ joining $a$ to $b$. The image $C = \gamma([\alpha, \beta])$ is the disjoint union of $C \cap A$ and $C \cap B$. These sets are open subsets of the topological space $C$, and they are nonempty since they contain $a$ and $b$, respectively. Hence $C$ is not connected. On the other hand, since the interval $[\alpha, \beta]$ is connected, it follows from Theorem 5.6.2 that $C = \gamma([\alpha, \beta])$ is connected, so that we have reached a contradiction. □

The converse statement is false. There exists subsets of, for example $\mathbb{R}^n$ ($n \geq 2$), which are connected but not pathwise connected (an example in $\mathbb{R}^2$ is given below). However, for open subsets of $\mathbb{R}^n$ the two notions of connectedness agree. This will be proved in the following section.
Example 5.6.5. The graph of the function

\[ f(x) = \begin{cases} 
\sin(1/x) & x \neq 0 \\
0 & x = 0
\end{cases} \]

is connected but not pathwise connected.

5.7 Connected manifolds

In this section we explore the relation between the two types of connectedness, with the case in mind that the topological space \( X \) is an abstract manifold.

Definition 5.7. A topological space \( X \) is said to be locally pathwise connected if it has the following property. For each point \( x \in X \) and each neighborhood \( V \) there exists an open pathwise connected set \( U \) such that \( x \in U \subset V \).

Example 5.7. The space \( X = \mathbb{R}^n \) is locally pathwise connected, since all open balls are pathwise connected. The same is valid for an abstract manifold, since each point has a neighborhood, which is the image by a chart of a ball in \( \mathbb{R}^m \), hence pathwise connected.

Lemma 5.7. In a locally pathwise connected topological space, all open connected sets \( E \) are pathwise connected.

Proof. For \( a, b \in E \) we write \( a \sim b \) if \( a \) and \( b \) can be joined by a continuous path in \( E \). It is easily seen that this is an equivalence relation. Since \( E \) is open there exists for each \( a \in E \) an open neighborhood \( V \subset E \), hence an open pathwise connected set \( U \) with \( a \in U \subset E \). For all points \( x \) in \( U \) we thus have \( a \sim x \). It follows that the equivalence classes for \( \sim \) are open. Let \( A \) be an arbitrary of these equivalence classes, and let \( B \) denote the union of all other equivalence classes. Then \( A \) and \( B \) are open, disjoint and have union \( E \). Since \( E \) is connected, \( A \) or \( B \) is empty. Since \( a \in A \), we conclude that \( B = \emptyset \) and \( A = E \). Hence all points of \( E \) are equivalent with each other, which means that \( E \) is pathwise connected. \( \Box \)

Theorem 5.7. Let \( M \) be an abstract smooth manifold. Each open connected subset \( E \) of \( M \) is also pathwise connected.

Proof. Follows immediately from Lemma 5.7, in view of Example 5.7. \( \Box \)

In particular, every open connected subset of \( \mathbb{R}^n \) is pathwise connected.

5.8 Components

Let \( X \) be topological space. We shall determine a decomposition of \( X \) as a disjoint union of connected subsets. For example, the set \( \mathbb{R}^\times = \mathbb{R} \setminus \{0\} \) is the disjoint union of the connected subsets \( ]-\infty, 0[ \) and \( ]0, \infty[ \).
**Definition 5.8.** A component (or connected component) of \( X \) is a subset \( E \subset X \) which is maximal connected, that is, it is connected and not properly contained in any other connected subset.

Components are always closed. This is a consequence of the following lemma (notice that in the example above, the components \([-\infty, 0[ \) and \([0, \infty[\) are really closed in \( \mathbb{R}^\times \).

**Lemma 5.8.** The closure of a connected subset \( E \subset X \) is connected.

**Proof.** Assume that \( \overline{E} \) is separated in a disjoint union \( \overline{E} = A_1 \cup A_2 \) where \( A_1, A_2 \) are relatively open in \( \overline{E} \). Then \( A_i = W_i \cap \overline{E} \) for \( i = 1, 2 \), where \( W_1 \) and \( W_2 \) are open in \( X \). Hence each set \( W_i \cap E = A_i \cap E \) is relatively open in \( E \), and these two sets separate \( E \) in a disjoint union. Since \( E \) is connected, one of the two sets is empty. If for example \( W_1 \cap E = \emptyset \), then \( E \) is contained in the complement of \( W_1 \), which is closed in \( X \). Hence also \( \overline{E} \) is contained in this complement, and we conclude that \( A_1 = \overline{E} \cap W_1 \) is empty. \( \square \)

**Theorem 5.8.** \( X \) is the disjoint union of its components. If \( X \) is locally pathwise connected, for example if it is an abstract manifold, then the components are open and pathwise connected.

**Proof.** Let \( x \in X \) be arbitrary. It follows from Lemma 5.6 with \( E_0 = \{x\} \), that the union of all the connected sets in \( X \) that contain \( x \), is connected. Clearly this union is maximal connected, hence a component. Hence \( X \) is the union of its components. The union is disjoint, because if two different components overlapped, their union would be connected, again by Lemma 5.6, and hence none of them would be maximal.

Assume that \( X \) is locally pathwise connected, and let \( E \subset X \) be a component of \( X \). For each \( x \in E \) there exists a pathwise connected open neighborhood of \( x \) in \( X \). This neighborhood must be contained in \( E \) (by maximality of \( E \)). Hence \( E \) is open. Now Theorem 5.7 implies that \( E \) is pathwise connected. \( \square \)

**Example 5.8** Let \( H \subset \mathbb{R}^3 \) be the surface \( \{(x, y, z) \mid x^2 + y^2 - z^2 = -1\} \), called the two-sheeted hyperboloid. We claim it has the two components

\[
H^+ = \{(x, y, z) \in H \mid z > 0\}, \quad H^- = \{(x, y, z) \in H \mid z < 0\}.
\]

It is easily seen that \( z \geq 1 \) for all \( (x, y, z) \in H \), hence \( H = H^+ \cup H^- \), a disjoint union. The verification that \( H^+ \) and \( H^- \) are pathwise connected is similar to that of Example 5.6.4.

**Corollary 5.9.** Let \( M \) be an abstract manifold. The components of \( M \) are abstract manifolds (of the same dimension). If an atlas is given for each component, then the combined collection of the charts comprises an atlas for \( M \).
Proof. Follows from the fact that the components are open and cover $M$. □

In particular, since there is no overlap between the components, $M$ is orientable if and only if each of its components is orientable.

5.9 The Jordan-Brouwer theorem

The proof of the following theorem is too difficult to be given here.

**Theorem 5.9.** Let $M \subset \mathbb{R}^n$ be an $n-1$-dimensional compact connected manifold in $\mathbb{R}^n$. The complement of $M$ in $\mathbb{R}^n$ consists of precisely two components, of which one, called the outside is unbounded, and the other, called the inside is bounded. Each of the two components is a domain with smooth boundary $M$.

The Jordan curve theorem for smooth plane curves is obtained in the special case $n = 2$.

**Example** Let $M = S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. Its inside is the open $n$-ball $\{x \mid \|x\| < 1\}$, and the outside is the set $\{x \mid \|x\| > 1\}$.

**Corollary 5.9.** Let $M$ be an $n-1$-dimensional compact manifold in $\mathbb{R}^n$. Then $M$ is orientable.

*Proof.* It suffices to prove that each component of $M$ is orientable, so we may as well assume that $M$ is connected. The inside of $M$ is orientable, being an open subset of $\mathbb{R}^n$. Hence it follows from Theorem 4.8 that $M$ is orientable. □

In particular, all compact surfaces in $\mathbb{R}^3$ are orientable.