DRAFT

6. Derivable functors.

Fix categories $\mathfrak{C}$ and $\mathfrak{D}$, and in $\mathfrak{C}$ a left denominator system S of morphisms.

(6.1) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor. For an object $X \in \mathfrak{C}$ we write $R_S F(X)$ or $RF(X)$ for the inductive limit,

$$RF(X) := \varinjlim_{s \in X/S} F(X_s),$$

provided that the inductive limit exists in $\mathfrak{D}$. If $RF(X)$ exists for every $X \in \mathfrak{C}$, we say that the (*right*) *derived functor* RF *exists* (with respect to S); clearly, RF is a functor,

$$RF\colon \mathfrak{C} \to \mathfrak{D}.$$

Moreover, since a morphism $s\colon X \to Y$ in S induces an isomorphism of ind-objects $X_S \to Y_S$, and hence an isomorphism

$$RF(X) = \varinjlim F(X_S) \xrightarrow{\sim} \varinjlim F(Y_S) = RF(Y),$$

it follows that the functor RF is S-local. Consequently, RF has a unique extension to a functor (also denoted) RF from the localized category,

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{RF} & \mathfrak{D} \\ {\scriptstyle (\,)_S}\downarrow & \nearrow {\scriptstyle RF} & \\ S^{-1}\mathfrak{C}. & & \end{array}$$

(6.2). The transformation $X_1 \to X_S$ induced a transformation $F(X)_1 = F(X_1) \to F(X_S)$ in ind-$\mathfrak{D}$. Consequently, if RF exists, there is a natural transformation $F \to RF$ of functors $\mathfrak{C} \to \mathfrak{D}$.

Proposition. *Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor such that the derived functor RF with respect to S exists. Then for every transformation $F \to G$ from F to an S-local functor $G\colon \mathfrak{C} \to \mathfrak{D}$ there is a unique transformation of functors $RF \to G$ making the following diagram commutative:*

$$\begin{array}{ccc} F & \longrightarrow & RF \\ \downarrow & \nearrow & \\ G. & & \end{array}$$

Proof. In the ind-category ind-$\mathfrak{D}$ there is a commutative diagram

$$\begin{array}{ccc} F(X_1) & \longrightarrow & F(X_S) \\ \downarrow & & \downarrow \\ G(X_1) & \xrightarrow{\sim} & G(X_S), \end{array}$$

where $G(X_1) \to G(X_S)$ is an isomorphism by Remark (5.?). Apply the colimit functor $\varinjlim$ to get the required morphism $RF(X) \to G(X)$. It is easily seen to be unique. ∎

(6.3) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor and X an object of $\mathfrak{C}$. Then F is called (*right*) *derivable at* X (*with respect to* S if the ind-object $F(X_S)$ in ind-$\mathfrak{D}$ is essentially constant, that is, if the colimit $RF(X) = \varinjlim F(X_S)$ exists in $\mathfrak{D}$ and the induced morphism $F(X_S) \to RF(X)_1$ is an isomorphism in ind-$\mathfrak{D}$. If the functor F is *derivable everywhere*, the functor

$$RF\colon \mathfrak{C} \to \mathfrak{D},$$

is called the *right derived* functor of F.

(6.4) Proposition. *The following three conditions on the left denominator system $S \subseteq \mathfrak{C}$ are equivalent:*

(i) *The functor $(\,)_S\colon \mathfrak{C} \to S^{-1}\mathfrak{C}$ has a right adjoint.*
(ii) *For every object X in $\mathfrak{C}$ there exists an S-injective object Q and a morphism $f\colon X \to Q$ such that the induced morphism $f_S\colon X_S \to Q_S$ is an isomorphism in $S^{-1}\mathfrak{C}$.*
(iii) *The identity functor $\mathfrak{C} \to \mathfrak{C}$ is right derivable everywhere with respect to S.*

The three conditions are implied by any of the following two:

(iv) *The class of S-injective objects is S-dense.*
(v) *There exists an S-dense class $\mathfrak{Q}$ of objects in $\mathfrak{C}$ such that, for any commutative diagram,*

$$\begin{array}{ccc} X & \xrightarrow{s'} & Q' \\ {\scriptstyle s}\downarrow & \nearrow{\scriptstyle f} & \\ Q, & & \end{array}$$

if $Q, Q' \in \mathfrak{Q}$ and $s, s' \in S$, then $f \in S$.

If S is saturated then all five conditions are equivalent.

Proof. The equivalence of (i), (ii), and (iii) is the the result in Proposition (5.?). If S is saturated, (iv) is just a restatement of (ii). To prove (iv) $\Rightarrow$ (v), simply observe the the class of S-injective objects has the property in (v).

Finally, we prove the implication (v) $\Rightarrow$ (iii). Let X be an object in $\mathfrak{C}$, and denote by $X/S/\mathfrak{Q}$ the full subcategory of X/S consisting of morphisms $s\colon X \to Q$ with target $Q \in \mathfrak{Q}$. Since $\mathfrak{Q}$ is S-dense, the subcategory $X/S/\mathfrak{Q}$ is final in X/S, and the condition (v) means that the inductive system $X_S\colon X/S \to \mathfrak{C}$ restricts to a constant inductive system on $X/S/\mathfrak{Q}$; therefore X_S is essentially constant. □

(6.5) Definition. Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor. An object Q in $\mathfrak{C}$ is said to be (*right*) *unfolded* for F or F-*acyclic* (with respect to S) if the following canonical morphism is an isomorphism in ind-$\mathfrak{D}$:

$$(FQ)_1 = F(Q_1) \to F(Q_S).$$

Note that an object Q is F-unfolded if and only if F is derivable at Q and $FQ \xrightarrow{\sim} RF(Q)$. Note further that an F-unfolded object is unfolded for any composition GF of F with a functor $G\colon \mathfrak{D} \to \mathfrak{E}$. In particular, an object unfolded for the identity of $\mathfrak{C}$ is unfolded for any functor $F\colon \mathfrak{C} \to \mathfrak{D}$.

Observation. *With respect to the given denominator system an object Q of $\mathfrak{C}$ is unfolded for the identity functor of $\mathfrak{C}$ if and only if Q is S-injective.*

Proof. If Q is S-injective, then the morphism $1\colon Q \to Q$ is the final object in Q/S; hence $Q_1 \to Q_S$ is an isomorphism in ind-$\mathfrak{C}$. Conversely, assume that $Q_1 \to Q_S$ is an isomorphism in ind-$\mathfrak{C}$. Then, for any object X of $\mathfrak{C}$, we have isomorphisms,

$$\mathrm{Hom}_{\mathfrak{C}}(X, Q) = \mathrm{Hom}_{\text{ind-}\mathfrak{C}}(X_1, Q_1) = \mathrm{Hom}_{\text{ind-}\mathfrak{C}}(X_1, Q_S) = \mathrm{Hom}_{S^{-1}\mathfrak{C}}(X_S, Q_S).$$

It follows that the functor $\mathrm{Hom}_{\mathfrak{C}}(\ , Q)$ is S-local. Hence Q is S-injective. ∎

(6.6) Definition. A functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is called *uniformly* (*right*) *derivable* (with respect to S) is there exists an S-dense class $\mathfrak{Q}$ of objects of $\mathfrak{C}$ with the following property for every object X:

($*_X$): For any commutative diagram in $\mathfrak{C}$,

$$\begin{array}{ccc} X & \xrightarrow{s'} & Q' \\ {\scriptstyle s}\downarrow & \nearrow{\scriptstyle f} & \\ Q, & & \end{array}$$

if $Q, Q' \in \mathfrak{Q}$ and $s, s' \in S$, then $Ff\colon FQ \to FQ'$ is an isomorphism in $\mathfrak{D}$.

An S-dense class with the property is said to be (*right*) *F-unfolding* (*with respect to S*.

Note that a uniformly derivable functor is derivable everywhere as it follows from (a generalization) the proof of (v) ⇒ (iii) in (6.4) above.

(6.7) Unfolding Theorem. *Let $F\colon \mathfrak{C} \to \mathfrak{D}$ be a functor and let $\mathfrak{Q}$ be an S-dense class of objects of $\mathfrak{C}$. Then the following three conditions are equivalent:*

(i) *The class $\mathfrak{Q}$ is F-unfolding.*
(ii) *Every object $Q \in \mathfrak{Q}$ is F-unfolding.*
(iii) *For every morphism $s\colon Q \to Q'$ in S, with $Q, Q' \in \mathfrak{Q}$, the morphism $Fs\colon FQ \to FQ'$ is an isomorphism in $\mathfrak{D}$.*

If the three conditions are satisfied, then F is uniformly derivable, and the class of all F-unfolded objects is an F-unfolding class; moreover, for every object $X \in \mathfrak{C}$ and any morphism $s\colon X \to Q$ in S from X to an F-unfolded object Q there is a commutative diagram in $\mathfrak{D}$ with isomorphisms as indicated:

$$\begin{array}{ccc} FX & \longrightarrow & FQ \\ \downarrow & & \downarrow\wr \\ RF(X) & \xrightarrow{\sim} & RF(Q). \end{array}$$

Proof. Since $\mathfrak{Q}$ is S-dense, the condition (i) is that the property ($*_X$) of (6.6) holds for every object X of $\mathfrak{C}$. It follows that (i) ⇒ (iii), and further, that (iii) ⇒ ($*_Q$) for every object Q in

$\mathfrak{Q}$; hence (iii) $\Rightarrow$ (ii). Finally, to prove that (ii) $\Rightarrow$ $(*_X)$ for every object X in $\mathfrak{C}$, consider a commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{s} & Q \\ & \searrow^{s} & \downarrow f \\ & & Q', \end{array} \qquad \text{with } Q,\, Q' \in \mathfrak{Q} \text{ and } s,\, s' \in S.$$

The diagram induces a commutative diagram in ind- $\mathfrak{C}$:

$$\begin{array}{ccccc} X_S & \xrightarrow{\sim} & Q_S & \longleftarrow & Q_1 \\ & \searrow^{\wr} & \downarrow f_S & & \downarrow f_1 \\ & & Q'_S & \longleftarrow & Q'_1, \end{array}$$

and it follows that f_S is an isomorphism. Apply F to obtain a commutative diagram in ind- $\mathfrak{D}$:

$$\begin{array}{ccc} F(Q_S) & \xleftarrow{\sim} & FQ_1 \\ \wr\downarrow F(f_S) & & F(f)_1 \downarrow \\ F(Q'_S) & \xleftarrow{\sim} & FQ'_1. \end{array}$$

It follows that $F(f)\colon FQ \to FQ'$ is an isomorphism in $\mathfrak{D}$.

The remaining assertions of the Theorem are easily proved. ∎

(6.8). Consider composable functors, $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{C}$. Clearly, if F is derivable at X, then so is GF and $R(GF)(X) = G(RF(X))$. Similarly, it F is uniformly derivable, then GF is uniformly derivable, and any F-unfolded object is unfolded for GF.

Note that by Proposition (6.5), the identity functor of $\mathfrak{C}$ is uniformly derivable if and only if the class of S-injective objects is S-dense. In particular, if the class of S-injective objects is S-dense, then every functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is uniformly derivable.

(6.9) Example. Consider for a fixed object $A \in \mathfrak{C}$ the functor,

$$H_A = \operatorname{Hom}_{\mathfrak{C}}(A, \ ) : \mathfrak{C} \to \textbf{Sets}.$$

Then H_A is derivable everywhere with respect to S, and

$$RH_A(X) = \varinjlim_{s \in X/S} H_A(X_s) = \varinjlim_{s \in X/S} \operatorname{Hom}_{\mathfrak{C}}(A, X_s) = \operatorname{Ext}_S(A, X).$$

So $\operatorname{Ext}_S(A, \ )$ is the right derived of $\operatorname{Hom}_{\mathfrak{C}}(A, \ )$:

$$R\operatorname{Hom}_{\mathfrak{C}}(A, \ ) = \operatorname{Ext}_S(A, \ ).$$

Similarly, with respect to a given right denominator system T, the right derived of the functor $\operatorname{Hom}_{\mathfrak{C}}(\ , B)$, as a functor $\mathfrak{C}^{\mathrm{op}} \to \textbf{Sets}$ is equal to $\operatorname{Ext}_T(A, B)$.

By the Unfolding Theorem (6.7)(iii), a class $\mathfrak{Q}$ in $\mathfrak{C}$ is unfolding for all the functors $\operatorname{Hom}_{\mathfrak{C}}(A, \ )$ for $A \in \mathfrak{C}$, if and only if it is unfolding for the identity. In turn, the condition holds if and only if $\mathfrak{Q}$ is S-dense and consists of (up to isomorphism all) S-injective objects.

(6.10). In the applications we will often consider the case when there is given a functor,

$$F\colon \mathfrak{C} \to \mathfrak{D},$$

and, in addition to the given left left denominator system $S \subseteq \mathfrak{C}$, a given left denominator system T in $\mathfrak{D}$. In this situation the preceding definitions will be applied to the composite functor,

$$(\,)_T F\colon \mathfrak{C} \to T^{-1}\mathfrak{D}. \tag{6.10.1}$$

The functor F is called local with respect to S and T if the functor $(\,)_T F$ is S-local. If T is saturated, then F is local, if

$$s \in S \implies F(s) \in T.$$

We say that RF *exists* if $R\big((\,)_T F\big)$ exists with respect to S, and we use

$$RF\colon S^{-1}\mathfrak{C} \to T^{-1}\mathfrak{D}$$

to denote the extension to $S^{-1}\mathfrak{C}$ of the S-local functor $R\big((\,)_T F\big)\colon \mathfrak{C} \to T^{-1}\mathfrak{D}$. The transformation $(\,)_T F \to R\big((\,)_T F\big)$ induces a transformation of functors $\mathfrak{C} \to T^{-1}\mathfrak{D}$:

$$(\,)_T F \to RF(\,)_S.$$

We say that F is *derivable at* X, resp. *uniformly derivable*, if $(\,)_T F$ is derivable at X, resp. $(\,)_T F$ is uniformly derivable, with respect to S. Similarly, the notions of an F*-unfolded object* Q and an F*-unfolding class* $\mathfrak{Q}$ refer to the corresponding notions for the functor $(\,)_T F$. The functor $RF\colon S^{-1}\mathfrak{C} \to T^{-1}\mathfrak{D}$ is the *derived functor* of F.

Note that the considerations of (6.8) yield limited information on a composition GF in this generalized setup; they apply only to a functor with source $T^{-1}\mathfrak{D}$.

(6.11) The Chain Rule. *Consider categories $\mathfrak{C}$ and $\mathfrak{D}$ with left denominator systems S and T and functors $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{E}$. Assume that F is derivable everywhere and that RG exists. In addition, assume that there is a class $\mathfrak{Q}$ of objects in $\mathfrak{C}$ having the following two properties;*

(1) *The class $\mathfrak{Q}$ is S-dense in $\mathfrak{C}$.*
(2) *For every object $Q \in \mathfrak{Q}$, the morphism $G(FQ) \to RG(FQ)$ is an isomorphism in $\mathfrak{E}$.*

Then the composition GF is derivable everywhere and the canonical transformation is an isomorphism,

$$R(GF) \xrightarrow{\sim} RG\,RF. \tag{6.11.1}$$

Moreover, if the class $\mathfrak{Q}$ consists of F-unfolded objects, then it is GF-unfolding; in particular, then GF is uniformly derivable.

Proof. Note that the functor $RG\colon T^{-1}\mathfrak{D} \to \mathfrak{E}$ appearing in (6.11.1) is the extension to $T^{-1}\mathfrak{D}$ of the T-local functor RG. To be precise in the proof, we will use RG to denote the extension. So the restriction to $\mathfrak{D}$ is the functor $RG\,(\,)_T$, and the natural transformation is the transformation $G \to RG\,(\,)_T$ of functors $\mathfrak{D} \to \mathfrak{E}$.

Let X be an object of $\mathfrak{C}$. By (1), the inclusion of categories,

$$\Phi\colon X/S/\mathfrak{Q} \hookrightarrow X/S,$$

is final. Hence we have in ind-$\mathfrak{C}$ an isomorphism $X_S\Phi \xrightarrow{\sim} X_S$, and it induces in ind-$\mathfrak{D}$ the isomorphism $F(X_S)\Phi \xrightarrow{\sim} F(X_S)$. Apply the functors G and $RG\,(\,)_T$ and the transformation $G \to RG\,(\,)_T$ to obtain a commutative diagram in ind-$\mathfrak{D}$:

$$\begin{array}{ccc} GF(X_S)\,\Phi & \xrightarrow{\sim} & GF(X_S) \\ \downarrow & & \downarrow \\ RG\,(\,)_T F(X_S)\,\Phi & \xrightarrow{\sim} & RG\,()_T F(X_S). \end{array}$$

By (2), the left vertical morphism is an isomorphism. Hence, so is the right vertical morphism. Since F is derivable at X, we have in ind $-T^{-1}\mathfrak{D}$ the isomorphism $(\,)_T F(X_S) \xrightarrow{\sim} RF(X)$. So the induced morphisms are isomorphisms in ind-$\mathfrak{E}$:

$$GF(X_S) \xrightarrow{\sim} RG\,(\,)_T F(X_S) \xrightarrow{\sim} RG(RF(X))_1.$$

Thus the first part of the Theorem has been proved.

To prove the last part, consider the following commutative diagram in ind-$\mathfrak{E}$, for $X \in \mathfrak{Q}$:

$$\begin{array}{ccc} GF(X_S) & \longleftarrow & GF(X)_1 \\ \wr\downarrow & & \downarrow \\ RG\,(\,)_T F(X_S) & \longleftarrow & RG\,(\,)_T F(X)_1. \end{array}$$

It follows that X is GF-unfolded(??). ▯

(6.12). The preceding definitions generalize in an obvious way to functors of several variables. For simplicity, consider the case of two variables, that is, a functor,

$$F\colon \mathfrak{C}_1 \times \mathfrak{C}_2 \to \mathfrak{D}$$

from the product $\mathfrak{C} = \mathfrak{C}_1 \times \mathfrak{C}_2$ of two categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ with left denominator systems S_1 and S_2. Then $S := S_1 \times S_2$ is a left denominator system in $\mathfrak{C}$, and there is an isomorphism,

$$S^{-1}\mathfrak{C} = S_1^{-1}\mathfrak{C}_1 \times S_2^{-1}\mathfrak{C}_2.$$

We will say that R_2F *exists*, it $R\big(F(A_1,\ )\big)$ exists with respect to S_2 for every object A_1 in $\mathfrak{C}_1$. Assuming the existence, we may consider R_2F as a functor,

$$R_2F\colon \mathfrak{C}_1 \times S_2^{-1}\mathfrak{C}_2 \to \mathfrak{D}.$$

Similarly, we say that F is derivable everywhere with respect to the second variable if, for every object $A_1 \in \mathfrak{C}_1$, the $F(A_1,\ )\colon \mathfrak{C}_2 \to \mathfrak{D}$ is derivable everywhere with respect to S_2.

Proposition. *If R_2F exists with respect to S_1, then RF exists with respect to $S = S_1 \times S_2$ if and only R_1R_2F exists with respect to S_1. Assuming the existence, there is a canonical isomorphism of functors,*

$$RF \xrightarrow{\sim} R_1R_2F.$$

Similarly, with respect to the appropriate denominator systems, if F is derivable in the second variable, then F is derivable if and only if R_2F is derivable in the first variable.

Proof. The assertions follow from general results about colimits over a product category. ∎

Remark. If there are subclasses $\mathfrak{Q}_1 \subseteq \mathfrak{C}_1$ and $\mathfrak{Q}_2 \subseteq \mathfrak{C}_2$ such that $\mathfrak{Q}_1$ is unfolding for all the functors $F(\;, A_2)$ for $A_2 \in \mathfrak{C}_2$ and $\mathfrak{Q}_2$ is unfolding for all the functors $F(A_1, \;)$ for $A_1 \in \mathfrak{C}_1$ then the product class $\mathfrak{Q} := \mathfrak{Q}_1 \times \mathfrak{Q}_2$ is unfolding for F. Moreover, then F is uniformly derivable, and we have canonical isomorphisms of functors,

$$R_1R_2F \xrightarrow{\sim} RF \xleftarrow{\sim} R_2R_1F.$$

(6.13) Remark. Assume that $\mathfrak{C}$ and $\mathfrak{D}$ are additive categories and that $F\colon \mathfrak{C} \to \mathfrak{D}$ is an additive functor. Assume that RF exists with respect to the left denominator system S. It follows easily that RF is an additive functor. If $\mathfrak{C}$ and $\mathfrak{D}$ have shift automorphisms $U \mapsto U(1)$ and F commutes with the shifts, it follows easily that RF commutes with the shifts.

(6.14). Consider the case of a triangulated category $\mathfrak{K}$ with a left denominator system S and a triangular functor,

$$F\colon \mathfrak{K} \to \mathfrak{K}',$$

from $\mathfrak{K}$ to a triangulated category $\mathfrak{K}'$.

The following easily proved proposition is a complement to the Unfolding Theorem (6.7). The conditions are with respect to the system S.

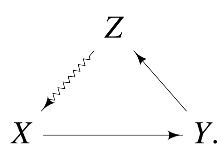
Proposition. *Let $\mathfrak{Q}$ be a triangular S-dense class in $\mathfrak{K}$. Then, with respect to the given denominator system S, the following conditions on $\mathfrak{Q}$ are equivalent:*

(i) *The class $\mathfrak{Q}$ is F-unfolding.*
(ii) *Every object in $\mathfrak{Q}$ is F-unfolded.*
(iii) *If Q in $\mathfrak{Q}$ is acyclic, then $F(Q) = 0$ in $\mathfrak{K}'$.*

Often the conditions are applied when the target category is of the form $(S')^{-1}\mathfrak{K}'$, obtained by localizing $\mathfrak{K}'$ with respect to a triangular denominator system S' (and the functor is the functor $(\;)_{S'}F$). In this case that last condition (iii) takes the following form: If Q in $\mathfrak{Q}$ is acyclic, then $(\;)_{S'}F(Q) = 0$. If the system S' is saturated, the condition is equivalent to the following:

(iii') If Q in $\mathfrak{Q}$ is acyclic, then $F(Q)$ is acyclic.

(6.15) Theorem. *In the setup of* (6.14)*, consider an exact triangle in $\mathfrak{K}$,*



Assume that F is derivable at X and at Y. Then F is derivable at Z and at $X(1)$, and the following triangle in $\mathfrak{K}'$ is exact:

$$\begin{array}{ccc} & RFZ & \\ \swarrow\!\!\sim & & \nwarrow \\ RFX & \longrightarrow & RFY. \end{array}$$

It follows from the Theorem that if F is derivable everywhere then RF is a triangular functor. In addition, if F is uniformly derivable, then the class of F-unfolded objects is a triangular subclass of $\mathfrak{K}$.

We are not going to make any use of the result in the stated generality, but only the special case considered in the following corollary. Therefore we will give a direct proof of the special case, and postpone the proof of the general case.

Corollary. *Assume in the setup of the Theorem that there is a triangular F-unfolding class $\mathfrak{Q}$. Then the functor RF is a triangular functor.*

Direct proof of the Corollary. Let $X \to X' \to X'' \to X(1)$ be an exact triangle in $\mathfrak{K}$. Since $\mathfrak{Q}$ is S-dense, it follows from the denominator conditions that there is a commutative diagram,

$$\begin{array}{ccc} X' & \xrightarrow{s'} & Q' \\ \uparrow & & \uparrow \\ X & \xrightarrow{s} & Q. \end{array} \qquad \text{with } Q,\, Q' \in \mathfrak{Q} \text{ and } s,\, s' \in S.$$

Embed $Q \to Q'$ into an exact triangle with vertex Q'' in $\mathfrak{Q}$, By (LOC 4), the pair (s, s') may be extended to a morphism (s, s', s'') of triangles, with $s'' \in S$,

$$\begin{array}{ccccc} & X' & \xrightarrow{s'} & Q' & \\ \swarrow & \uparrow & \swarrow & \uparrow & \\ X'' & \xrightarrow{s''} & Q'' & & \\ & \searrow\!\!\sim & & \searrow\!\!\sim & \\ & X & \xrightarrow[s]{} & Q. & \end{array}$$

Apply F and RF and the transformation $F \to RF$. The result is a commutative diagram of triangles in $\mathfrak{K}'$:

$$\begin{array}{ccccc} RFX' & \xrightarrow{\sim} & RFQ' & \xleftarrow{\sim} & FQ' \\ \swarrow\ \ \uparrow & & \swarrow\ \ \uparrow & & \swarrow\ \ \uparrow \\ RFX'' \xrightarrow{\sim} & & RFQ'' \xleftarrow{\sim} & & FQ'' \\ \searrow\!\!\sim & & \searrow\!\!\sim & & \searrow\!\!\sim \\ RFX & \xrightarrow[\sim]{} & RFQ & \xleftarrow[\sim]{} & FQ. \end{array}$$

The horizontal morphisms are isomorphisms, and the right triangle is exact. Hence the left triangle is exact. Consequently, RF is a triangular functor. ∎

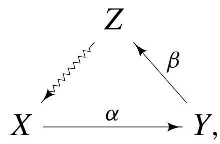
(6.16) Remark. There is a result similar to Corollary (6.15) for the case of a triangular functor $F: \mathfrak{K} \to \mathfrak{A}$ from the triangulated category $\mathfrak{K}$ to an abelian category $\mathfrak{A}$. For the case $\mathfrak{A} = \mathbf{Ab}$ the following result is more precise.

Proposition. *Let $\mathfrak{K}$ be a triangulated category with a triangular denominator system S. Let $G: \mathfrak{K} \to \mathbf{Ab}$ be a triangular functor. Then RG exists and RG is a triangular functor.*

Proof. For every object X of $\mathfrak{K}$ we have by definition

$$RG(X) = \varinjlim_{s \in X/S} G(X_s).$$

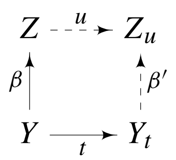
So we have to prove for any exact triangle in $\mathfrak{K}$,



that the induced sequence of abelian groups is exact:

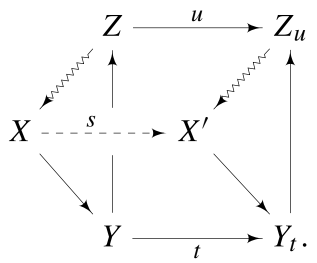
$$\varinjlim_{s \in X/S} G(X_s) \xrightarrow{\alpha_*} \varinjlim_{t \in Y/S} G(Y_t) \xrightarrow{\beta_*} \varinjlim_{u \in Z/S} G(Z_u).$$

Let η be an element in $\operatorname{Ker} \beta_*$, and represent η by an element $y \in G(Y_t)$ for some index $t \in Y/S$. Then there is a commutative diagram,



with $u \in Z/S$,

and then $G(\beta')(y) \in G(Z_u)$ represents $\beta_*(\eta)$. Since $\beta_*(y) = 0$, we may modify u in the diagram and assume that $G(\beta')(y) = 0$. Now embed $\beta': Y_t \to Z_u$ in an exact triangle $X' \to Y_t \to Z_u \to X'(1)$. By (LOC4), the pair (t, u) may be extended to a morphism of triangles (s, t, u) with $s \in S$,



Then $s: X \to X'$ is an index in X/S with $X_s = X'$. Since G is triangular, the following sequence in $\mathbf{Ab}$ is exact:

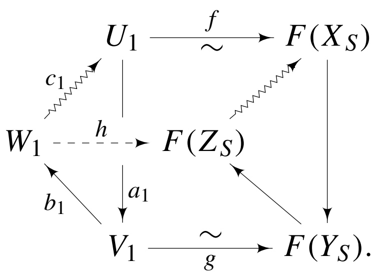
$$G(X_s) \to G(Y_t) \to G(Z_u).$$

Moreover, $G(\beta')(y) = 0$. Hence there is an element $x \in G(X_s)$ such that $G(\alpha')(x) = y$. Clearly, then x represents an element $\xi \in \varinjlim_{s \in X/S} G(X_s)$ such that $\alpha_*(\xi) = \eta$. ∎

(6.17). *Proof of Theorem 6.15* Consider an exact triangle $(X, Y, Z, \alpha, \beta, \gamma)$ in $\mathfrak{K}$. Set $U := RF(X)$ and $V = RF(Y)$ and let $a = RF(\alpha)\colon U \to V$. Embed $a\colon U \to V$ in an exact triangle (U, V, W, a, b, c) of $\mathfrak{K}'$. The inductive systems $F(X_S)$ and $F(Y_S)$ are essentially constant. Hence, in the ind-category ind-$\mathfrak{K}'$ there is a commutative diagram with horizontal isomorphisms:

$$\begin{array}{ccc} U_1 & \xrightarrow[\sim]{f} & F(X_S) \\ {\scriptstyle a_1}\downarrow & & \downarrow \\ V_1 & \xrightarrow[g]{\sim} & F(Y_S). \end{array} \tag{6.17.1}$$

The shifts in $\mathfrak{K}'$ define natural shifts in the ind-category ind-$\mathfrak{K}'$. Let us first show (with respect to these shifts) that the pair f, g extends to a morphism of triangles in ind-$\mathfrak{K}'$:

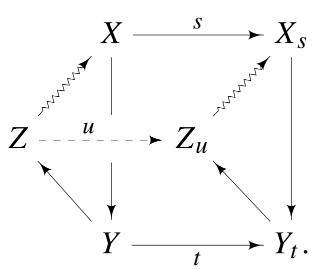


(6.17.2)

The morphism $f_1\colon U_1 \to F(X_S)$ in the ind-category is represented by a morphism $\varphi\colon U \to F(X_s)$ for some $s \in X/S$. Similarly, g is represented by a morphism $\psi\colon V \to F(Y_t)$ for $t \in Y/S$. The square (6.17.1) is commutative. Hence, replacing (t, ψ) by an other representative if necessary, we may assume that there are commutative squares, in $\mathfrak{K}$ and in $\mathfrak{K}'$: commutative.

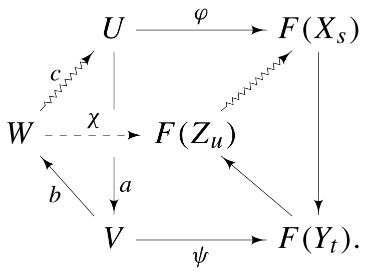
$$\begin{array}{ccc} X & \xrightarrow{s} & X_s \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ Y & \xrightarrow[t]{} & Y_t, \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{\varphi} & F(X_s) \\ {\scriptstyle a}\downarrow & & \downarrow \\ V & \xrightarrow[\psi]{} & F(Y_t). \end{array}$$

Now, let Z_u be the cone of $\alpha'\colon X_s \to Y_t$, and consider the morphisms s, t in S. By (LOC4), the pair (s, t) may be extended to a morphism of triangles (s, t, u) with $u \in S$:



Since F is triangular, we may extend the pair φ, ψ of morphisms in $\mathfrak{K}'$ to a morphism (φ, ψ, χ)

of exact triangles in $\mathfrak{K}'$:



(6.17.3)

Then χ represents a morphism $W_1 \to F(Z_S)$ in ind-$\mathfrak{K}'$ which is the morphism required in (6.17.2).

To finish the proof we show that h is an isomorphism in ind-$\mathfrak{K}'$: It is enough to show for any object A of $\mathfrak{K}'$ that the morphism induced by h is an isomorphism of abelian groups:

$$\operatorname{Hom} \mathfrak{K}'(A, W) \xrightarrow{\sim} \varinjlim_{u \in Z/S} \operatorname{Hom}_{\mathfrak{K}'}(A, F(Z_u)). \tag{6.17.4}$$

Consider the triangular functor $G\colon \mathfrak{K} \to \mathbf{Ab}$ defined by $G(X) = \operatorname{Hom}_{\mathfrak{K}'}(A, F(X))$. By the previous result, RG is a triangular functor. Now, the inductive limit on the right side of (6.17.4) is the value $G(Z)$. Let $H_A : \mathfrak{K}' \to \mathbf{Ab}$ denote the functor $H_A(\,) = \operatorname{Hom}_{\mathfrak{K}}(A, \,)$. Then $G = H_A F$, and the commutative diagram (6.17.3) with exact triangles induces a commutative diagram in **Ab** with exact rows,

$$\begin{array}{ccccccccc}
H_A(U) & \longrightarrow & H_A(V) & \longrightarrow & H_A(W) & \longrightarrow & H_A(U(1)) & \longrightarrow & H_A(V(1)) \\
{\scriptstyle f}\downarrow\wr & & {\scriptstyle g}\downarrow\wr & & {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow\wr & & {\scriptstyle g}\downarrow\wr \\
RG(U) & \longrightarrow & RG(V) & \longrightarrow & RG(W) & \longrightarrow & RG(U(1)) & \longrightarrow & RG(V(1)).
\end{array}$$

The four maps induced by f and g are isomorphisms. So, by the 5-Lemma, the map induced by h is bijective. Therefore, h is an isomorphism, and the proof is complete. ∎