16. Cohomology of some special groups and monoids.

(16.0) Exercise. Read the examples on the cohomology of a group G via standard resolution assuming only that G is a monoid. Did you find any reservations?

(16.1) Setup. Let G be a monoid and let $\mathfrak{A}$ be an abelian category with exact $\prod_G$'s. A standard complex defines a resolvent complex for the limit functor $\Gamma^G = \varprojlim_G \colon \mathfrak{A}^G \to \mathfrak{A}$, defined on the category $\mathfrak{A}^G$ of co-G-objects.

For special groups or monoids there may be special constructions of a coaugmented resolvent complex for Γ^G:

$$\widetilde{\Pi}A : \qquad 0 \to \Gamma^G A \to \Pi^0 A \to \Pi^1 A \to \cdots ,$$

where each Π^i is a functor $A \mapsto \Pi^i A$, defined on co-G-objects of $\mathfrak{A}$. In each of the examples below we construct such a complex where each Π^i is exact and such that the coaugmented complex is contractible when evaluated on a trivial G-object. Then it results from the general theory that the cohomology $H^p(\Pi A)$ of the complex ΠA is the cohomology $\varprojlim^{(p)}_G A = H^p(G, A)$.

(16.2) Example. Let G be the free (multiplicative) monoid with a single generator f (in additive notation, G is the monoid $\mathbb{N}_0$ of nonnegative integers). Then a G-object A is an object A of $\mathfrak{A}$ with a given endomorphism $f = f_A \colon A \to A$. The following coaugmented complex:

$$\widetilde{\Pi}A : \qquad 0 \to \Gamma^G A \to A \xrightarrow{f-1} A \to 0 \to \cdots ,$$

is a resolvent complex. Indeed, the functors $\Pi^0 = \Pi^1$ are exact, given by $A \mapsto A$ (and forget the endomorphism f). A coinduced object has the form $\rho B = \prod_{n\geqslant 0} B$, where the endomorphism $f = f_{\rho B}$ is determined by $\mathrm{pr}_n\, f = \mathrm{pr}_{n+1}$. The complex $\widetilde{\Pi}(\rho B)$ is the following:

$$\begin{tikzcd} \widetilde{\Pi}(\rho B) : \quad 0 \arrow[r] & B \arrow[r, shift left, "\epsilon"] & \prod\limits_{n\geqslant 0} B \arrow[l, shift left, dashed, "\tau"] \arrow[r, shift left, "f-1"] & \prod\limits_{n\geqslant 0} B \arrow[l, shift left, dashed, "\sigma"] \arrow[r] & 0 \arrow[r] & \cdots , \end{tikzcd}$$

split by the indicated morphisms defined by $\tau = \mathrm{pr}_0$ and $\mathrm{pr}_n\, \sigma = \sum_{j<n} \mathrm{pr}_n$. Do check it! As a consequence, for this monoid G,

$$H^0(G, A) = \Gamma^G A = \mathrm{Ker}(A \xrightarrow{f-1} A), \qquad H^1(G, A) = \mathrm{Cok}(A \xrightarrow{f-1} A) = A/G,$$

and $H^p = 0$ for $p > 1$.

(16.3) Example. Let G be the free group with a single generator e (in additive notation, G is the group $\mathbb{Z}$ of integers). Then a G-object A is an object A of $\mathfrak{A}$ with a given automorphism $e = e_A \colon A \to A$. The the following coaugmented complex,

$$\Pi A : 0 \to \Gamma^G A \to A \xrightarrow{e-1} A \to 0 \to \cdots ,$$

is a resolvent complex. Indeed, the functors $\Pi^0 = \Pi^1$ are exact, given by $A \mapsto A$ (and forget the automorphism e). A coinduced object has the form $\rho B = \prod_n B$ (where the product is over $n \in \mathbb{Z}$), where the endomorphism $e = e_{\rho B}$ is determined by $\mathrm{pr}_n\, e = \mathrm{pr}_{n+1}$. The complex $\widetilde{\Pi}(\rho B)$ is the following:

$$\widetilde{\Pi}(\rho B): \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left, "\epsilon"] & \prod\limits_n B \arrow[l, dashed, shift left, "\tau"] \arrow[r, shift left, "e-1"] & \prod\limits_n B \arrow[l, dashed, shift left, "\sigma"] \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd} ,$$

split by the indicated morphisms defined by $\tau = \mathrm{pr}_0$ and $\mathrm{pr}_n\, \sigma = \sum_{0 \leqslant j < n} \mathrm{pr}_n$ for $n \geqslant 0$ and $\mathrm{pr}_n\, \sigma = -\sum_{n \leqslant j < 0}$ for $n \leqslant 0$. Do check it! As a consequence, for this group G,

$$H^0(G, A) = \Gamma^G A = \mathrm{Ker}(A \xrightarrow{e-1} A), \qquad H^1(G, A) = \mathrm{Cok}(A \xrightarrow{e-1} A) = A/G,$$

and $H^p = 0$ for $p > 1$.

(16.4) Example. Let G be a free (noncommutative) group with generators e_i for $i \in I$; denote by $\emptyset$ the neutral element of G (the empty word). Then a G-object A is an object $A \in \mathfrak{A}$ with a given family of automorphisms $e_i = e_{i,A}$ for $i \in I$. Consider the complex,

$$0 \to \Gamma^G A \xrightarrow{\epsilon} A \xrightarrow{d} \prod_{i \in I} A \to 0 \to \cdots,$$

where d is determined by its projections: $\mathrm{pr}_i\, d = e_{i,A} - 1_A$ for $i \in I$. Assume that $A = \rho B$ is the trivial G-object determined by an object B of $\mathfrak{A}$. Then $A = \prod_{w \in G} B$, and G acts by permutation of the coordinates. We want to prove that the complex is contractible when evaluated on ρB, so we want to define homotopies σ, τ:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left, "\epsilon"] & \prod\limits_{w \in G} B \arrow[l, shift left, "\sigma"] \arrow[r, shift left, "d"] & \prod\limits_{i \in I} \prod\limits_{w \in G} B \arrow[l, shift left, "\tau"] \arrow[r] & 0 \end{tikzcd},$$

The morphism σ is the projection on the index $\emptyset$ (the unit of the group G), that is, $\sigma = \mathrm{pr}_\emptyset$. The morphism τ is determined by its projections $\mathrm{pr}_w\, \tau$, and they are defined inductively on the length of the word w. For the empty word $\mathrm{pr}_\emptyset\, \tau = 0$, and

$$\mathrm{pr}_{e_i w}\, \tau = \mathrm{pr}_w\, \tau + \mathrm{pr}_{i,w}, \qquad \mathrm{pr}_{e_i^{-1} w}\, \tau = \mathrm{pr}_w\, \tau - \mathrm{pr}_{i, e_i^{-1} w}.$$

(16.5) Example. Let G be the free abelian (multiplicative) monoid with basis $f_1, \dots, f_r$ (the additive version of G is the monoid $\mathbb{N}_0^r$ of r-sets $(n_1, \dots, n_r)$ of nonnegative integers). Then a G-object is an object A of $\mathfrak{A}$ with a given family of commuting endomorphisms f_i. The trivial G-object ρB determined by an object B is the product $\rho B = \prod_{n_1, \dots, n_r} B$, over $\mathbb{N}_0^r$; the endomorphism $f_i \colon \rho B \to \rho B$ is determined by its projections,

$$\mathrm{pr}_{n_1, \dots, n_r}\, f_i = \mathrm{pr}_{n_1, \dots, n_i + 1, \dots, n_r}. \tag{16.5.1}$$

Consider the functor $\Gamma_{\mathbf{f}} = \Gamma_{f_1,\dots,f_r} \colon \mathfrak{A}^G \to \mathfrak{A}$ defined by

$$\Gamma_{f_1,\dots,f_r} A = \bigcap \operatorname{Ker}(f_{i,A}) = \operatorname{Ker}(A \xrightarrow{\mathbf{f}} A^{\oplus r}),$$

where $\mathbf{f} = \mathbf{f}_A \colon A \to A^{\oplus r}$ is the morphism with coordinates $f_{i,A}$. Clearly, in this notation,

$$\Gamma^G A = \Gamma_{f_1 - 1,\dots,f_r - 1}(A).$$

Note that the Koszul cochain complex $K^\bullet(X) = K^\bullet(f_1,\dots,f_r; X)$ is defined in this general setup for any complex X of G-objects. For the G-object A, viewed as a complex in degree 0, the morphism $K^0(A) \to K^1(A)$ is the morphism $\mathbf{f} \colon A \to A^{\oplus r}$; hence $\Gamma_{\mathbf{f}}(A) = H^0(\mathbf{f}, A)$.

Lemma. *The following coaugmented complex* the reduced Koszul complex)*,*

$$\widetilde{K}_{\mathbf{f}}(A): \qquad 0 \to \Gamma_f A \to K^0(A) \to \cdots \to K^r(A) \to 0 \to \cdots, \tag{16.5.2}$$

defines a resolvent complex for the functor $\Gamma_{\mathbf{f}}$*.*

Hint. In degree i the functor K^i is given by $K^i(A) = A^{\oplus\binom{r}{i}}$; it is clearly exact. So it remains to prove that the *reduced Koszul complex* (16.5.1) is exact when evaluated at a coinduced object ρB. Note that $\Gamma_{\mathbf{f}}(\rho B) = B$, as it follows from the description (16.5.1).

Exactness is proved by induction on r. It $r = 1$, the coinduced object has the form $\rho_1 B = \prod_{n\geqslant 0} B$, and the reduced Koszul complex has the following form:

$$\begin{tikzcd}
\widetilde{K}(\rho_1 B): & 0 \arrow[r] & B \arrow[r, shift left, "\iota"] & \prod\limits_{n\geqslant 0} B \arrow[l, dashed, shift left, "\tau"] \arrow[r, shift left, "f"] & \prod\limits_{n\geqslant 0} B \arrow[l, dashed, shift left, "\sigma"] \arrow[r] & 0 \arrow[r] & \cdots,
\end{tikzcd}$$

where $\operatorname{pr}_0 \iota = 1$ and $\operatorname{pr}_n \iota = 0$ for $n > 0$. It is split by the indicated morphisms, defined by $\tau = \operatorname{pr}_0$ and $\operatorname{pr}_0 \sigma = 0$ and $\operatorname{pr}_n \sigma = \operatorname{pr}_{n-1}$ for $n > 0$. Do check it!

Now, for $r > 1$, the monoid G is the product $G = G' \times G_1$ where G' is the submonoid generated by $f_1,\dots,f_{r-1}$ and G_1 is generated by f_r; both submonoids are free. Accordingly, a co-G-object may be viewed as a co-G_1-object in the category of co-G'-objects, $\mathfrak{A}^G = (\mathfrak{A}^{G'})^{G_1}$, and the functor ρ is a composition $\rho = \rho_1\rho'$:

$$\rho \colon \mathfrak{A} \xrightarrow{\rho'} \mathfrak{A}^{G'} \xrightarrow{\rho_1} \big(\mathfrak{A}^{G'}\big)^{G_1} = \mathfrak{A}^G.$$

The Koszul cochain complex may be defined by a similar recursion: $K_{\mathbf{f}} X = K_{\mathbf{f}', f_r} X = K_{\mathbf{f}'} K_{f_r} X$. Now, let B be an object of $\mathfrak{A}$, and set $B_1 := \rho' B$; then $\rho B = \rho_1 B_1$. The following two morphisms of complexes are homotopy equivalences,

$$B \to K_{\mathbf{f}'}(\rho' B) = K_{\mathbf{f}'}(B_1) \to K_{\mathbf{f}'} K_{f_r}(\rho_1 B_1) = K_{\mathbf{f}}(\rho B),$$

the first by the induction hypothesis, the second because it is obtained by applying $K_{\mathbf{f}'}$ to the morphism $B_1 \to K_{f_r}(\rho_1 B_1)$ which is a homotopy equivalence by the case $r = 1$. ▯

Corollary. *For the free abelian monoid G with generators $f_1, \dots, f_r$, the cohomology with coefficients in a co-G-object A is equal to the Koszul cohomology with respect to the sequence $f_1 - 1, \dots, f_r - 1$,*

$$H^p(G, A) = H^p_{f_1-1,\dots,f_r-1}(A).$$

(16.6) Example. Let $G = \mathbb{Z}^r$ be the free rank-r abelian group, with multiplicative generators $e_1, \dots, e_r$. As in (16.5), the group cohomology is given by the Koszul cohomology,

$$H^p(G, A) = H^p_{e_1-1,\dots,e_r-1}(A).$$

(16.7) Example. If A is a co-G-object in $\mathfrak{A}$, then the group homomorphism $G \to \mathrm{Aut}(A)$ extends to a ring homomorphism $\mathbb{Z}G \to \mathrm{End}(A)$, denoted $\lambda \mapsto \lambda_A$, from the group ring of G to the endomorphism ring of A.

Assume that G is a finite group. Then the *norm* N is the element $N = \sum_{u \in G} u \in \mathbb{Z}G$. So the norm defines an endomorphism $N = N_A \colon A \to A$. Again, since G is finite, there is a morphism $D = D_A \colon \prod_{s \in G} A \to A$ defined by $D = \sum_{s \in G}(s_A - 1_A)\,\mathrm{pr}_s$.

Clearly, since $N(s-1) = (s-1)N$ in the group algebra, it follows that $ND \colon \prod_s A \to A$ and $\varepsilon N \colon A \to \prod_s A$ are zero. So the image $NA := \mathrm{Im}\, N_A$ of the norm is a subobject of $\Gamma^G A$, and the image $DA := \mathrm{Im}\, D_A$ is contained in the kernel ${}_N A := \mathrm{Ker}\, N_A$ of the norm.

Lemma. *If A is co-G-induced, $A = \rho B$, then the following two zero sequences are split by the morphisms defined in the proof:*

$$A \xrightarrow{N} A \xrightarrow{\varepsilon} \prod_{s \in G} A, \qquad \prod_{s \in G} \xrightarrow{D} A \xrightarrow{N} A,$$

Proof. With $A = \rho B$ the first sequence is the following, split by the indicated morphisms:

$$\begin{tikzcd} \prod\limits_t B \arrow[r, shift left, "N"] & \prod\limits_t B \arrow[l, shift left, dashed, "\tau"] \arrow[r, shift left, "\epsilon"] & \prod\limits_s \prod\limits_t B; \arrow[l, shift left, dashed, "\sigma"] \end{tikzcd}$$

here N and ε satisfy the equations $\mathrm{pr}_t\, N = \sum_u \mathrm{pr}_{tu}$ and $\mathrm{pr}_t\, \mathrm{pr}_s\, \varepsilon = \mathrm{pr}_{ts} - \mathrm{pr}_t$. The morphisms τ and σ are determined by the projections, $\mathrm{pr}_1\, \tau = \mathrm{pr}_1$ and $\mathrm{pr}_t\, \tau = 0$ when $t \neq 1$ and $\mathrm{pr}_t\, \sigma = \mathrm{pr}_1\, \mathrm{pr}_t$. It is easy to verify the equation $N\tau + \sigma\varepsilon = 1$.

The second sequence is the dual of the first; so the result for the second second is a consequence of the first. □

(16.8) Example. Let $G = \mathrm{C}_d$ be the finite cyclic group of order d with a generator e. (So the additive version of G is the group $\mathbb{Z}/d\mathbb{Z}$.) Then the following coaugmented complex defines a resolvent complex for Γ^G:

$$0 \to \Gamma^G A \to A \xrightarrow{e-1} A \xrightarrow{N} A \xrightarrow{e-1} A \xrightarrow{N} \cdots .$$

As a consequence, for $p > 0$,

$$H^{2p-1}(G,A) = {}_N A/DA, \qquad H^{2p}(G,A) = \Gamma^G A/NA.$$

(16.9) Exercises.

1. Prove for a finite group G and a G-object A that the p'th cohomology $H^p(G,A)$ for $p > 0$ is killed by the order of G. [Hint: Consider for the standard complex ΠA the morphisms $\sigma^{p+1}\colon \Pi^{p+1} \to \Pi^p$ determined by the projections,

$$\mathrm{pr}_{s_1,\dots,s_p}\,\sigma^{p+1} = (-1)^{p+1} \sum_{s \in G} \mathrm{pr}_{s_1,\dots,s_p,s}\,.$$

Prove that $\sigma\partial + \partial\sigma = |G|$. Is it unfair not to specify the range of p?]

2. (1) Let G be the group of order 2. Give an example of a commutative group with a G-action such that $H^p(G,A) \neq 0$ for every $p \geqslant 0$.

Let G be the monoid of order 2 generated by an f with $f^2 = f$. Prove for any co-G-object A that $H^p(G,A) = 0$ for all p. [Hint: prove that Γ^G is exact.]

17. Thge Lyndon spectral sequence.

18. The spectral sequence of a Galois covering.

(18.1) Setup. Consider for a topological space X the set of singular p-simplices in X,

$$\Delta_p(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^p, X) \qquad p \geqslant -1.$$

Let $\mathfrak{A}$ be an abelian category with $\prod$'s, and A an object in $\mathfrak{A}$. We write $C_{\mathrm{sing}}(X, A)$ for the product $A^{\Delta_p(X)}$. Then there is a positive complex $C = C_{\mathrm{sing}}(X, A)$ with differentials defined by formulas analogous to those defining the chain complex $C^{\mathrm{sing}}(X, \mathbb{Z})$. With an obvious coaugmentation from $C^{-1} = A$ there is a similar *reduced singular cochain complex* $\widetilde{C}_{\mathrm{sing}}(X, A)$.

Alternatively the differentials between the objects of $C_{\mathrm{sing}}(X, A)$, and other related morphisms, may be defined by the following process of of transposing linear maps between the modules in the chain complex $C^{\mathrm{sing}}(X, \mathbb{Z})$: For any set I the projections $\mathrm{pr}_i \colon A^I \to A$ for $i \in I$ form a family of morphisms in the set $\mathrm{Hom}_{\mathfrak{A}}(A^I, A)$, that is, $i \mapsto \mathrm{pr}_i$ is a map of sets from I to $\mathrm{Hom}_{\mathfrak{A}}(A^I, A)$. So it extends to a homomorphism of abelian groups,

$$\mathbb{Z}^{\oplus I} \to \mathrm{Hom}_{\mathfrak{A}}(A^I, A);$$

naturally, the image of en element $c \in \mathbb{Z}^{\oplus I}$ will be denoted $\mathrm{pr}_c \colon A^I \to I$. If c is the finite linear combination $c = \sum_i c_i i$, then pr_c is the sum morphism $\mathrm{pr}_c = \sum_{i \in I} c_i \,\mathrm{pr}_i$ in the group $\mathrm{Hom}_{\mathfrak{A}}(A^I, A)$. With this notation there is for every linear map $\varphi \colon \mathbb{Z}^{\oplus I} \to \mathbb{Z}^{\oplus J}$ an associated *transposed morphism,*

$$\varphi^{\mathrm{tr}} \colon A^J \to A^I, \qquad \text{defined by } \mathrm{pr}_i \,\varphi^{\mathrm{tr}} = \mathrm{pr}_{\varphi i}\, .$$

It is easy to see that transposing is functorial:

$$(\varphi\psi)^{\mathrm{tr}} = \psi^{\mathrm{tr}}\varphi^{\mathrm{tr}}.$$

The differentials in the cochain complex $C_{\mathrm{sing}}(X, A)$ may by obtained by transposing the differentials of $C^{\mathrm{sing}}(X, \mathbb{Z})$.

(18.2) Example. Other linear maps may be transposed. For instance: For the n-sphere ($n \geqslant 0$) there is a homotopy equivalence $C^{\mathrm{sing}}(X, \mathbb{Z}) = \mathbb{Z}(0) \oplus \mathbb{Z}(n)$. Consequently, for the general cochain complex there is a homotopy equivalence

$$C_{\mathrm{sing}}(X, A) = A(0) \oplus A(-n).$$

(18.3) Setup. A map $f \colon X \to Y$ (of topological spaces) is a *covering projection* if Y may be covered by open subsets U such that the restricted map $f_U \colon f^{-1}U \to U$ is isomorphic to a projection $U \times J \to U$ with a discrete set J (equivalently, if $f^{-1}U$ is a disjoint union of open sets U_α each being mapped homeomorphically onto U. The covering is *trivial,* if f is isomorphic to a projection $Y \times J \to Y$.

It is a standard fact that every covering of the unit square $[0.1] \times [0, 1]$ is trivial. It is a consequence that every covering of a 1-connected (i.e., path connected and simply connected) space Y_0 trivial. It follows that a covering has the following *lifting property*: for every pair of based maps p, φ:

$$\begin{tikzcd} & (X, x) \arrow[d, "p"] \\ (Y_0, y_0) \arrow[ur, dashed] \arrow[r, "\varphi"'] & (Y, y), \end{tikzcd}$$

where p is a covering and Y_0 is 1-connected there is a unique map $(Y_0, y_0) \to (X, x)$ making the diagram commutative.

Consider a topological space X with a *properly discontinuous* action of a group G, in other words, every point $x \in X$ has on open neighborhood U such that $U \cap sU = \emptyset$ for all elements $s \neq 1$ in G. It follows easily that the quotient map,

$$X \to X/G,$$

is a covering projection.

Clearly, G acts on each set $\Delta_p(X)$ of singular p-simplices. By the lifting property, the map induced by $X \to X/G$ is surjective:

$$\Delta_p(X) \to \Delta_p(X/G),$$

Let $T_p \subseteq \Delta_p(X)$ be a subset mapped bijectively onto $\Delta_p(X/G)$. Assume that $p \geqslant 0$ so that $\Delta^p \neq \emptyset$. Then, by uniqueness of the lifting, $\Delta_p(X)$ is the disjoint union of 'translates' of T_p,

$$\Delta_p(X) = \bigvee_{t \in G} t(T_p).$$

The action of G on the set $\Delta_p(X)$ induces an action of G on the product $A^{\Delta_p(X)}$. Moreover, it is easy to see that the differentials in the complex commute with the action of G; hence the cochain complex C_{sing} may be viewed as a complex of objects from $\mathfrak{A}^G$. Moreover, by the description above,

$$A^{\Delta_p(X)} = \prod_{t \in G} A^{T_p};$$

hence each $A^{\Delta_p(X)}$ is a trivial G-object, induced by the object $A^{T_p} = A^{\Delta_p(X/G)}$. In particular, there is an isomorphism,

$$\Gamma^G C_{\text{sing}}(X, A) = C_{\text{sing}}(X/G, A). \tag{18.3.1}$$

As the objects of $C_{\text{sing}}(X, A)$ are co-induced G-objects, and hence acyclic for Γ^G, the left side of (18.3.1) is the hyper derived of Γ^G evaluated at the complex $C_{\text{sing}}(X, A)$:

$$R\Gamma^G C_{\text{sing}}(X, A) = C_{\text{sing}}(X/G, A). \tag{18.3.2}$$

A 2-spectral sequence falls out:

$$H^p(G, H^q(X, A) \Rightarrow H^n(X/G, A). \tag{18.3.3}$$

19. Some special Galois coverings.

(19.1) Setup. Let A be an object in an abelian category $\mathfrak{A}$. Assume that the group G acts properly discontinuously on a topological space X. Then there is an isomorphism of complexes,

$$R\Gamma^G C_{\text{sing}}(X, A) = \Gamma^G C_{\text{sing}}(X, A) \simeq C_{\text{sing}}(X/G, A); \tag{19.1.1}$$

the first equality because the complex $C_{\text{sing}}(X, A)$ on the left side is consists of co-induced G-objects which are acyclic for Γ^G. The functor $R\Gamma^G$ respects quasi-isomorphisms and exact triangles. Hence, from the mapping cone,

$$\begin{tikzcd}
 & \widetilde{C}_{\text{sing}}(X, A) \arrow[dl, squiggly] & \\
A(0) \arrow[rr] & & C_{\text{sing}}(X, A), \arrow[ul]
\end{tikzcd}$$

and the isomorphism (19.1.1), there is an induced exact triangle,

$$\begin{tikzcd}
 & R\Gamma^G \widetilde{C}_{\text{sing}}(X, A) \arrow[dl, squiggly] & \\
R\Gamma^G A(0) \arrow[rr] & & C_{\text{sing}}(X/G, A). \arrow[ul]
\end{tikzcd} \tag{19.1.2}$$

(19.2) Example. The group $G = \mathbb{Z}$ acts as translations on the space $\mathbb{R}$ of reals. The quotient is the 1-sphere: $\mathbb{R}/\mathbb{Z} = S^1$. The space $\mathbb{R}$ is contractible, and so there is a homotopy equivalence of chain complexes $C^{\text{sing}}(\mathbb{R}, \mathbb{Z}) \simeq \mathbb{Z}(0)$. Hence there is an induce equivalence of cochain complexes $C_{\text{sing}}(\mathbb{R}, A) \simeq A(0)$. It is easily seen to be G-invariant, when A is viewed as a constant G-object. Consequently, by (19.1.1),

$$C_{\text{sing}}(\mathbb{R}/\mathbb{Z}, A) = R\Gamma^G A(0).$$

So the cohomology of S^1 is the cohomology $H^*(G, A)$ which, with the constant action of $G = \mathbb{Z}$ on A, is the following:

$$H^0_{\text{sing}}(S^1, A) = H^1_{\text{sing}}(S^1, A) = A, \quad H^p_{\text{sing}}(S^1, A) = 0 \text{ for } p > 1.$$

(19.3) Example. The group $G = \mathbb{Z}^r$ acts on r-space $\mathbb{R}^r$. The quotient is a product of 1-spheres: $\mathbb{R}^r/\mathbb{Z}^r = (S^1)^r$. Hence $H^p_{\text{sing}}((S^1)^r, A) = H^p(G, A)$ (where A is the constant G-object); the latter cohomology is the p'th Koszul cohomology of A corresponding to the sequence $\mathbf{f} = \mathbf{0}$. Hence,

$$H^p_{\text{sing}}((S^1)^r) = A^{\binom{r}{p}}.$$

(19.4) Example. The cyclic group $G = \pm 1$ operates on S^r via the antipodal map $x \mapsto -x$; the quotient $S^r/\pm$ is the real projective r-space $\mathbb{P}^r = \mathbb{P}^r(\mathbb{R})$. There is a natural homotopy equivalence $\mathbb{Z}(r) \xrightarrow{\sim} \widetilde{C}^{\text{sing}}(S^r, \mathbb{Z})$ and hence a homotopy equivalence $A(-r) \xrightarrow{\sim} \widetilde{C}_{\text{sing}}(X, A)$.

The equivalence is *not* G-invariant. In fact, it is easy to see that the induced action of the element $-1 \in G$ on $H^r_{\text{sing}}(S^r, A) = A$ is multiplication by $(-1)^{r+1}$.

Let us write $A^{\pm}$ for A with this G-action (if r is odd, it is the constant action of G on A, and when r is even, the element $-1 \in G$ acts as multiplication by -1 on A). Then there is a quasi-isomorphism of complexes of G-objects $\widetilde{C}_{\text{sing}}(X, A) \xrightarrow{\sim} A^{\pm}(-r)$. So the exact triangle (19.1.2) takes the following form,

$$\begin{tikzcd}
& R\Gamma^G A^{\pm}(-r) \arrow[dl, squiggly] & \\
R\Gamma^G A(0) \arrow[rr] & & C_{\text{sing}}(\mathbb{P}^r, A). \arrow[ul]
\end{tikzcd} \tag{19.4.1}$$

The pth cohomology of the top vertex is $H^{p-r}(G, A^{\pm})$, and it vanishes when $p < r$. So the long exact cohomology sequence of the triangle yields isomorphisms,

$$H^p_{\text{sing}}(\mathbb{P}^r, A) = H^p(G, A) = \begin{cases} A & \text{when } p = 0; \\ {}_2A & \text{when } 0 < p < r,\ p \text{ odd}; \\ A/2A & \text{when } 0 < p < r,\ p \text{ even}. \end{cases}$$

Without knowledge of the morphisms in the triangle, the exact sequence does not determine the cohomology $H^p_{\text{sing}}(\mathbb{P}^r, A)$ for $p \geqslant r$. A triangulation of $\mathbb{P}^r$ may be obtained from a G-invariant triangulation of S^r; it is a consequence that $H^p_{\text{sing}}(\mathbb{P}^r, A) = 0$ for $p > r$. Given this fact, the long exact sequence reduces to isomorphism $H^{p-r}(G, A^{\pm}) \xrightarrow{\sim} H^{p+1}(G, A)$ for $p > r$ and an exact sequence:

$$0 \to H^r(G, A) \to H^r_{\text{sing}}(\mathbb{P}^r, A) \to H^0(G, A^{\pm}) \to H^{r+1}(G, A) \to 0.$$

In turn, depending on the parity of r, the exact sequence is the following:

$$\begin{gathered} 0 \to {}_2A \to H^r_{\text{sing}}(\mathbb{P}^r, A) \to A \to A/2A \to 0 \quad (r \text{ odd}), \\ 0 \to A/2A \to H^r_{\text{sing}}(\mathbb{P}^r, A) \to {}_2A \to {}_2A \to 0. \quad (r \text{ even}). \end{gathered} \tag{19.4.2}$$

The exact sequence determines the cohomology in important cases, like $A = \mathbb{Z}$, $A = \mathbb{R}$, or $A = \mathbb{F}_2$. It is natural to expect from (19.4.2) in general that $H^r_{\text{sing}}(\mathbb{P}^r, A) = A$ when r is odd, and $H^r_{\text{sing}}(\mathbb{P}^r, A) = A/2A$ when r is even. In fact, there is an isomorphism $H^p_{\text{sing}}(\mathbb{P}^r, A) = H^p(C^{\leqslant r})$ for all p, where $C^{\leqslant r}$ is the r'th cochain truncation of the following positive complex (with the first A in degree 0):

$$C: \quad 0 \to A \xrightarrow{0} A \xrightarrow{2} A \xrightarrow{0} A \xrightarrow{2} A \xrightarrow{0} A \longrightarrow \cdots .$$

20. Local systems; homotopy groups.

(20.1) Setup. Fix a topological space X and a decent category $\mathfrak{C}$. Assume in particular the $\mathfrak{C}$ has small limits, and denote by 0 the initial object of $\mathfrak{C}$. A $\mathfrak{C}$-valued *local system* on X is a functor,

$$\mathcal{G}\colon \mathcal{P}(X) \to \mathfrak{C},$$

where $\mathcal{P}(X)$, the *fundamental groupoid* of X, is the following category: The objects of $\mathcal{P} = \mathcal{P}(X)$ are the points of X, the morphisms in $\mathcal{P}$ from $a \in X$ to $b \in X$ are homotopy classes of paths from a to b, and composition in $\mathcal{P}$ is concatenation of paths. The category $\mathcal{P} = \mathcal{P}(X)$ is indeed a groupoid: every morphism is an isomorphism.

Fix a point b in X. The group $\operatorname{Aut}_{\mathcal{P}}(b)$ (equal to $\operatorname{End}_{\mathcal{P}}(b)$) is the *fundamental group* $\pi = \pi_1(X, b)$ of X at b. View the group π as a category with one object. Then the inclusion is a functor,

$$b\colon \pi \hookrightarrow \mathcal{P}, \tag{20.1.1}$$

from the fundamental group to the fundamental groupoid $\mathcal{P} = \mathcal{P}(X)$. The corresponding restriction functor,

$$b^*\colon \mathfrak{C}^{\mathcal{P}} \to \mathfrak{C}^{\pi}, \tag{20.1.2}$$

associates to a local system $\mathcal{G}$ the co-π-object $\mathcal{G}(b)$. By the Kan-construction, the restriction functor has a right adjoint functor $\rho_b\colon \mathfrak{C}^{\pi} \to \mathfrak{C}^{\mathcal{P}}$. It associates with a co-π-object A the local system given as a limit,

$$(\rho_b A)(a) = \varprojlim_{a/\pi} A, \tag{20.1.3}$$

where the index category a/π is the right fiber at a of the inclusion $\pi \to \mathcal{P}$: Its objects are the paths $\xi\colon a \to b$, and there is only one morphism from $\xi\colon a \to b$ to $\eta\colon a \to b$, which is the loop $\eta\xi^{-1}$. Consequently,

$$(\rho_b A)(a) = \begin{cases} A & \text{if } a, b \text{ belong to the same path component,} \\ 0 & \text{otherwise;} \end{cases}$$

an explicit isomorphism in the first case being given by a choice of a path class $a \to b$.

Hence. *, when X is path connected: For any $b \in X$ the functor $\mathfrak{F} \mapsto \mathfrak{F}(b)$ is an equivalence between local systems on X and co-$\pi(X, b)$-objects.*

If a local system $\mathfrak{G}$ has values in an abelian category with $\prod$'s we may form the complex $C(\mathfrak{P}, \mathfrak{G})$ with cohomology $H^p(\mathfrak{P}, \mathfrak{G})$. If $b \in X$ (and $\pi := \pi(X, b)$), we have the restriction,

$$C(\mathfrak{P}, \mathfrak{G}) \to C(\pi, \mathfrak{G}(b)),$$

and, for a co-π-object A the right adjunction,

$$C(\mathfrak{P}, \rho_b A) \to C(\pi, A).$$

As noted above these two maps of complexes are homotopy equivalences when X is path connected. In particular, in the path connected case, the cohomology $H^p(\mathfrak{P}, \mathfrak{G})$ is isomorphic to the group cohomology $H^p(\pi(X, b), \mathfrak{G}(b))$.

(20.2). Special local systems are the homotopy groups: Let S^n be the *n-sphere*,

$$S^n = \{ x \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1 \},$$

as a pointed topological space (pointed by the north pole $p = (0, \dots, 0, 1)$. Let $\pi_n(X, b)$ be the set of homotopy classes of maps (of pointed topological spaces) $\varphi\colon (S^n, p) \to (X, b)$. The class in $\pi_n(X, b)$ represented by φ is denoted $[\varphi]$. Clearly, $\pi_0(X, b)$ is the set of path components of X.

Assume that $n \geqslant 1$. Then there is a well defined composition in $\pi_n(X, b)$ determined as follows: Denote by S^n_-, S^n_0 and S^n_+ the subsets of S^n determined, respectively, by the relations $x_1 \leqslant 0$, $x_1 = 0$, and $x_1 \geqslant 0$. By squeezing the equator S^n_0 to a north pole we get a map,

$$*\colon S^n \to S^n \vee S^n,$$

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For maps $\varphi, \psi\colon (S^n, p) \to (X, b)$ we obtain a map $(\varphi, \psi)\colon S^n \vee S^n \to X$ and the composition in $\pi_n(X, b)$ is determined by the formula,

$$[\varphi] * [\psi] := [\varphi * \psi].$$

It is a standard fact that the composition is well defined, and is a group law on $\pi_n(X, b)$, abelian if $n \geqslant 2$. [Note that the obvious identification $\pi_1(X, b) = \pi(X, b)$ is an anti-isomorphism with respect to the group structures as defined here.]

For a morphism $\xi\colon a \to b$ in the path category $\mathfrak{P}$ and an element $z \in \pi_n(X, b)$ there is an element $\xi_* z \in \pi_n(X, a)$ determined similarly the the obvious map $S^n \to S^n \vee I$ squeezing the upper hemisphere to I. The map ξ_* is a group ismomorhism,

$$\xi_*\colon \pi_n(X, b) \to \pi_n(X, a),$$

and the formation of $\pi_n(X, b)$ is an inverse local system on X, denoted $\pi_n(X)$, with values in **Sets** when $n = 0$, in **Gr** when $n = 1$, and in **Ab** when $n \geqslant 2$.

Note that the isomorphism $\pi_1(X, a) \to \pi_1(X, b)$ corresponding to a morhism $\xi\colon a \to b$ is given by the formula,

$$\omega \mapsto \xi_*(\omega) = \xi^{-1} \circ \omega \circ \xi.$$

We will need a few properties of the π_n.

Fact 1. The Homotopy addition Lemma. *20.3*