Examples; applications

1. The Koszul komplex.

(1.1) Setup. Fix a commutative ring A and an r-tupple $\mathbf{f} = (f_1, \dots, f_r)$ of elements of A. Let I_p be the set of all subsets of cardinality p of the the set $\{1, \dots, r\}$. For every A-module M, consider the product,

$$K_f^p(M) := M^{I_p}, \quad \text{the } A\text{-module of all maps } x\colon I_p \to M.$$

Identify the elements of I_p with p-tupples $(i_1, \dots, i_p)$ of integers $i_1 < \cdots < i_p$ in the interval $[1, r]$, and hence the elements of M^{I_p} with functions $x(i_1, \dots, i_p)$. In this notation define $\partial\colon K^p \to K^{p+1}$ by

$$\partial x(i_0, \dots, i_p) := \sum_{\nu=0}^{p} (-1)^\nu f_{i_\nu} x(i_0, \dots, \widehat{i_\nu}, \dots, i_p), \tag{1.1.1}$$

where the "hat" indicates an omitted index. Then,

$$K_{\mathbf{f}}(M): \quad 0 \to K_{\mathbf{f}}^0 \to K_{\mathbf{f}}^1 \to \cdots \to K_{\mathbf{f}}^r \to 0,$$

is a positive complex, the *Koszul* (cochain)*complex* of M. with *Koszul cohomology groups*,

$$\mathrm{H}_{\mathbf{f}}^p(M) = \mathrm{H}^p(K_{\mathbf{f}}(M)).$$

Note that I_0 and I_r are one-point-sets consisting, respectively, of the empty sequence $\emptyset$ and the full sequence $1, 2, \dots, r$. Hence we may identify $K^0 = M$ and $K^r = M$. Clearly, $H^0 \subseteq M$ is the submodule consisting of elements $x \in M$ with $f_\nu x = 0$ for all f_ν, and H^r is the quotient $H^r = M/\mathbf{f}M$.

The dual construction leads to the *Koszul chain complex*: Consider the direct sum $M^{\oplus I_p}$ with canonical embeddings $\iota_{i_1,\dots,i_p}\colon M \to M^{\oplus I_p}$. Then there is a chain complex,

$$K_\bullet^{\mathbf{f}}(M): \quad 0 \to K_r^{\mathbf{f}} \to \cdots \to K_1^{\mathbf{f}} \to K_0^{\mathbf{f}} \to 0, \quad K_p^{\mathbf{f}} := M^{\oplus I_p}.$$

with differential $\partial\colon K_{p+1} \to K_p$ given by the formula,

$$\partial\, \iota_{i_0,\dots,i_p} := \sum_{\nu=0}^{p} (-1)^\nu f_{i_\nu}\, \iota_{i_0,\dots,\widehat{i_\nu},\dots,i_p}\,. \tag{1.1.2}$$

Let us make it a little more concrete: The module $K_{p+1} = M^{\oplus I_{p+1}}$ is the direct sum of identical copies of M, say $M_{i_0,\dots,i_p} = M$, indexed by sequences $(i_0,\dots,i_p) \in I_{p+1}$. So for any $x \in M$ there is an element $x_{i_0,\dots,i_p} \in M_{i_0,\dots,i_p} \in M$, and the elements in K_{p+1} are sums of elements of this form, for varying x and $i_0,\dots,i_p$. To define the differential $\partial K_{p+1} \to K_p$, it suffices to define it on an element of the form $x_{i_0,\dots,i_p}$, for $x \in M$. The formula (1.1.2) yields this:

$$\partial\, x_{i_0,\dots,i_p} := \sum_{\nu=0}^{p} (-1)^\nu f_{i_\nu}\, x_{i_0,\dots,\widehat{i_\nu},\dots,i_p}\,. \tag{1.1.3}$$

It should be emphasized that in spite of the striking similarity between (1.1.1) and (1.1.3), the objects that appear in the formulas are of a very different nature: In the first formula, x is a function $I_p \to M$, in the second, x is an element of M.

The case $M := A$ leads to the chain complex $K^{\mathbf{f}}_\bullet(A)$ and the augmented chain complex,

$$0 \to K_r(A) \to \cdots \to K_1(A) \to K_0(A) \to A/(\mathbf{f}) \to 0. \tag{1.1.4}$$

It is easy to obtain isomorphisms,

$$K^{\mathbf{f}}_\bullet(M) = K^{\mathbf{f}}_\bullet(A) \otimes M, \qquad K_{\mathbf{f}}(M) = \mathrm{Hom}_A(K^{\mathbf{f}}_\bullet(A)), M).$$

The modules $K_p(A) = A^{\oplus I_p}$ er free A-modules of rank $|I_p| = \binom{r}{p}$, and so the differentials are described by matrices of various sizes. For instance, if $r = 4$, the Koszul chain complex has the form

$$\cdots \to 0 \to A \xrightarrow{\partial_4} A^4 \xrightarrow{\partial_3} A^6 \xrightarrow{\partial_2} A^4 \xrightarrow{\partial_1} A \longrightarrow 0 \longrightarrow \cdots$$

The module $K_2 = A^{\binom{4}{2}}$ has basis e_{i_1,i_2} ($= 1_{i_1,i_2}$ in the notation of (1.1.3)), say in the order $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$, and K_1 has basis e_1, e_2, e_3. So $\partial e_{12} = f_1 e_2 - f_2 e_1$, etc. So it it immediate to write up the matrix for ∂_1:

$$\partial_2 = \begin{bmatrix} -f_2 & -f_3 & -f_4 & 0 & 0 & 0 \\ f_1 & 0 & 0 & -f_3 & -f_4 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & -f_4 \\ 0 & 0 & f_1 & 0 & f_2 & f_3 \end{bmatrix}.$$

As we will see, the sequence (1.1.4) is exact, when $\mathbf{f}$ is a regular sequence in A. If (1.1.4) is exact, then the Koszul chain complex $K^{\mathbf{f}}_\bullet(A)$ is a resolution of $A/(\mathbf{f})$, and we obtain isomorphims,

$$\mathrm{H}^i_{\mathbf{f}}(M) = \mathrm{Ext}^i_A(A/\mathbf{f}, M), \quad \mathrm{H}^{\mathbf{f}}_i(M) = \mathrm{Tor}^A_i(A/\mathbf{f}, M).$$

(1.2) The Koszul complex of a complex. It is a terrific excercise to prove the following: *For any sequence $f_0, f_1, \dots, f_r$ of $r+1$ elements of A there is a canonical isomorphism of chain complexes,*

$$K^{f_0,f_1,\dots,f_r}(M) = \mathrm{Con}(f_0, K^{f_1,\dots,f_r}(M)), \tag{1.2.1}$$

where Con(f, X)*, for a complex X and $f \in A$, denotes the mapping cone of multiplication by f on X.*

Hint. Let $\mathbf{f}$ be the sequence $(f_1, \dots, f_r)$ and indicate with a prime objects associated to the extended sequence $\mathbf{f}' = (f_0, \mathbf{f})$, with indices $0, \dots, r$. Hence I'_p consist of alle sequences $i_1, \dots, i_p$ with $0 \leqslant i_1, \dots \leqslant i_p \leqslant r$, etc. In particular, the I'_{p+1} is the disjoint union of two subsets I_p and I_{p+1} consisting, respectively, of sequences $0, i_1, \dots, i_p$ and of sequences $i_0, i_1, \dots, i_r$ with $i_0 > 0$. Accordingly, we may split: $K'_{p+1} = K_p \oplus K_{p+1}$. Now check that the differential ∂'_p under the splitting corresponds to the differential of the mapping cone of $f_0 \colon K \to K$. □

It is natural to take (1.2.1) as the definition of the Koszul complex of a complex X of A-modules. So we define:

$$K^f(X) := \mathrm{Con}(f, X), \qquad K^{f_1,\dots,f_r}(X) := K^{f_1}(K^{f_2,\dots,f_r}(X));$$

for the Koszul co-chain complex we use the cocone:

$$K_f(X) := \mathring{\mathrm{C}}\mathrm{on}(f, X), \qquad K_{f_1,\dots,f_r}(X) := K_{f_1}(K_{f_2,\dots,f_r}(X));$$

(1.3) Observations. (1) *The formation of the Koszul complex $K^{\mathbf{f}}(X) = K^{f_1,\dots,f_r}(X)$ is functorial with respect to X, and defines an additive functor $K^{\mathbf{f}} \colon \mathfrak{Mod}_A^\bullet \to \mathfrak{Mod}_A^\bullet$.*

(2) *The functor $K^{\mathbf{f}}$ is exact.*

(3) *The functor $K^{\mathbf{f}}$ commutes with formation of mapping cones.*

(4) *The functor $K^{\mathbf{f}}$ respects homotopy.*

(5) *The functor $K^{\mathbf{f}}$ respects quasi-isomorphisms. In particular, if X is acyclic, then $K^{\mathbf{f}}(X)$ is acyclic.*

Hints. For all five observations it suffices to treat the case $r = 1$, $f_1 = f$. (1) and (2) are obvious: A morphism $\varphi \colon X \to Y$ commutes with multiplication by f, because it is linear. Hence it induces the diagonale morphism on the cones: $K^f(\varphi) \colon K^f(X) \to K^f(Y)$.

Consider (3). Let Z be the mapping cone of φ. Then there is natural isomorphism of complexes from the mapping cone of $K^f(\varphi)$ to the Koszul complex $K^f(Z)$. Indeed, the two complexes, and their differentials, are the following:

$$\bigoplus\begin{matrix} X(2) \\ X(1) \\ Y(1) \\ Y \end{matrix}, \quad \begin{pmatrix} \partial & 0 & 0 & 0 \\ -f & -\partial & 0 & 0 \\ \varphi & 0 & -\partial & 0 \\ 0 & \varphi & f & \partial \end{pmatrix}, \qquad \bigoplus\begin{matrix} X(2) \\ Y(1) \\ X(1) \\ Y \end{matrix}, \quad \begin{pmatrix} \partial & 0 & 0 & 0 \\ -\varphi & -\partial & 0 & 0 \\ f & 0 & -\partial & 0 \\ 0 & f & \varphi & \partial \end{pmatrix},$$

and an isomorphism from the first to the second is given by the matrix,

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider (4). Assume that $\varphi = \partial s + s\partial$, with a homotopy $s\colon X \to Y(1)$. Use that s is A-linear to prove that

$$K^{\mathbf{f}}(\varphi) = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} = \begin{pmatrix} -s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} -\partial_X & 0 \\ f & \partial_X \end{pmatrix} + \begin{pmatrix} -\partial_Y & 0 \\ f & \partial_Y \end{pmatrix}\begin{pmatrix} -s & 0 \\ 0 & s \end{pmatrix}.$$

Finally, consider (5). A quasi-isomorphism $\varphi\colon X \to Y$ is characterized by the condition that the mapping cone $Z = \operatorname{Con}\varphi$ is acyclic. Therefore, by (3), it suffices to prove the special case. Again, we may assume that $r = 1$. Then, since $H^n X = 0$, it follows from the long exact cohomology sequence associated to the cone of $f\colon X \to X$ that $H^n(K^f(X)) = 0$. Hence $K^f(X)$ is acyclic. □

Corollary 1. *The Koszul complex $K^{f_1,\dots,f_r}(X)$ is, up to canonical isomorphism, invariant under permutation of the f_i.*

Hint. We may assume that $r = 2$, and then the isomorphism $K^f(K^g(X)) = K^g(K^f(X))$ is a special case of (3). □

Corollary 2. *Multiplication by f_i in $K^{\mathbf{f}}(X)$ is homotopic to zero. In particular, the homology modules $H_p^{\mathbf{f}}(X)$ are annihilated by the f_i and hence by all elements in the ideal $(f_1, \dots, f_r)A$.*

Hint. By (4), we may assume that $r = 1$. Now check the equation,

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix} + \begin{pmatrix} -\partial & 0 \\ f & \partial \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

□

(1.4) Definition. The cokernel of multiplication by f on X is denoted X/f. Component for component it is the quotient $X^n/f := X^n/fX^n$. The complex $(X/f)/g$ is, component for component, equal to $X^n/(f,g)X^n$; we denote it $X/(f,g)$, and define $X/(f_1, \dots, f_r)$ inductively. There is a natural morphism,

$$K^{f_1,\dots,f_r}(X) \to X/(f_1, \dots, f_r)$$

defined inductively: $K^f(X)$ is the cone of $f\colon X \to X$, so there is an induced morphism $K^f(X) \to X/f$. If the morphism (1.4.1) is defined for r elements we define it for $r+1$ elements as a composition:

$$K^{f_0,f_1,\dots,f_r}(X) = K^{f_0}(K^{f_1,\dots,f_r}(X)) \to K^{f_0}(X/(f_1, \dots, f_r)) \to X/(f_0, f_1, \dots, f_r).$$

An element $f \in A$ is said to be *regular* on X, if multiplication by f on X is injective. The sequence $(f_1, \dots, f_r)$ is said to be an *X-regular sequence*, if f_1 is regular on X, and f_2 is regular on X/f_1, etc, that is, f_{i+1} is regular on $X/(f_1, \dots, f_i)$ for $0 \leqslant i < r$.

Proposition. *If $(f_1, \dots, f_r)$ is an X-regular sequence, then the canonical morphism is a quasi-isomorphism $K^{f_1,\dots,f_r}(X) \to X/(f_1, \dots, f_r)$. If $(f_1, \dots, f_r)$ is an M-regular sequence, then the Koszul complex is a left resolution of $M/(f_1, \dots, f_r)$,*

$$0 \to K_r(M) \to \cdots K_1(M) \to K_0(M) \to M/(f_1, \dots, f_r) \to 0.$$

(1.5) Excercises.

1. Describe the matrix ∂_3 in the Kozsul complex $K^{f_1,f_2,f_3,f_4}(A)$, cf. (1.1)