4. Derivable functors.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories. Let $T\colon \mathfrak{A} \to \mathfrak{B}$ be an additive functor.

(4.0) Setup. The additive functor $T\colon \mathfrak{A} \to \mathfrak{B}$ has obvious extensions to functors of complexes,

$$T^\bullet\colon \mathfrak{A}^\bullet \to \mathfrak{B}^\bullet, \qquad T^+\colon \mathfrak{A}^+ \to \mathfrak{B}^+, \quad \text{etc.,}$$

and to triangular functors on the homotopy categories,

$$\mathrm{Hot}(T)\colon \mathrm{Hot}(\mathfrak{A}) \to \mathrm{Hot}(\mathfrak{B}), \qquad \mathrm{Hot}^+(T)\colon \mathrm{Hot}^+(\mathfrak{A}) \to \mathrm{Hot}^+(\mathfrak{B}), \quad \text{etc.,}$$

(4.1) Definition. Recall that the homotopy categories $\mathrm{Hot}(\mathfrak{A})$, $\mathrm{Hot}^+(\mathfrak{A}), \dots$, are triangulated categories with a natural triangular system (Qis) consisting of all quasi-isomorphisms. In particular, the system is a saturated left and right denominator system. The corresponding class of acyclic objects consists of the acyclic (or exact) complexes in the homotopy category. The corresponding localized categories are the *derived categories*, for instance

$$\mathrm{D}(\mathfrak{A}) := (\mathrm{Qis})^{-1}\,\mathrm{Hot}(\mathfrak{A}), \qquad \mathrm{D}^+(\mathfrak{A}) := (\mathrm{Qis})^{-1}\,\mathrm{Hot}^+(\mathfrak{A}).$$

Derivability of functors between homotopy categories is always assumed to be with respect to these denominator systems in the source and target. For instance, a triangular functor $F\colon \mathrm{Hot}^+(\mathfrak{A}) \to \mathrm{Hot}^+(\mathfrak{B})$ is said to *derivable* at a complex $X \in \mathrm{Hot}(\mathfrak{A})$ if the composition, $\mathrm{Hot}(\mathfrak{A}) \xrightarrow{F} \mathrm{Hot}(\mathfrak{B}) \to \mathrm{D}(\mathfrak{B})$ is right derivable at X with respect to the system of quasi-isomorphisms in the source. If $T\colon \mathfrak{A} \to \mathfrak{B}$ is an additive functor and $\mathrm{Hot}(T)$ is derivable at a complex X, then we will write $RT(X)$ for the value,

$$RT\,X := R\,\mathrm{Hot}(T)(X) \in \mathrm{D}(\mathfrak{B}).$$

(4.2) Lemma. *Let $T\colon \mathfrak{A} \to \mathfrak{B}$ be an additive functor, and let X be a right complex of objects of $\mathfrak{A}$. Then the functor $\mathrm{Hot}(T)$ is derivable at X, if and only if the functor $\mathrm{Hot}^+(T)$ is derivable at X. Moreover, if the condition holds, then under the natural inclusion $\mathrm{D}^+(\mathfrak{B}) \subseteq \mathrm{D}(\mathfrak{B})$ we have the equality $R\,\mathrm{Hot}(T)(X) = R\,\mathrm{Hot}^+(T)(X)$; in addition, if X is in $\mathfrak{A}^{\geqslant n}$, that is, if $X^p = 0$ for $p < n$, then $H^p(RT(X)) = 0$ for $p < n$.*

Proof. To be precise, let F and F^+ denote the compositions,

$$F\colon \mathrm{Hot}(\mathfrak{A}) \to \mathrm{Hot}(\mathfrak{B}) \to \mathrm{D}(\mathfrak{B}), \qquad F^+\colon \mathrm{Hot}^+(\mathfrak{A}) \to \mathrm{Hot}^+(\mathfrak{B}) \to \mathrm{D}^+(\mathfrak{B}),$$

and let i be the fully faithful inclusion functor,

$$i\colon \mathrm{D}^+(\mathfrak{B}) \hookrightarrow \mathrm{D}(\mathfrak{B}).$$

Assume that $X \in \mathfrak{A}^{\geqslant n}$. Denote by S, S^+, and $S^{\geqslant n}$ the systems of quasi-isomorphisms in, respectively, $\mathrm{Hot}(\mathfrak{A})$, $\mathrm{Hot}^+(\mathfrak{A})$, and $\mathrm{Hot}^{\geqslant n}(\mathfrak{A})$. Note that the last homotopy category is not a triangulated category. Using the cocycle truncations it is easy to see that the following two inclusions of index categories are final:

$$\Phi^+ \colon X/S^{\geqslant n} \hookrightarrow X/S^+, \qquad \Phi \colon X/S^{\geqslant n} \hookrightarrow X/S.$$

Now, $\mathrm{Hot}^+(T)$ is derivable at X if and only if the inductive system $F^+(X/S^+)$ is essentially constant in ind- $\mathrm{D}^+(\mathfrak{B})$, with a similar assertion for $\mathrm{Hot}(T)$ and $F(X_{S^+})$. The restrictions to the index category X/S^+ defines an isomorphism in the ind-category. Moreover, we have the equality,

$$F(X_S)\Phi = i\, F^+(X_{S^+})\Phi^+,$$

of inductive systems from the index category $X/S^{\geqslant n}$. Hence, if the restricted system $F(S^+)\Phi^+$, is essentially constant, then so is the left side of (=).

Conversely, assume that $F(X_S)\Phi$ is essentially constant. Then there is a complex U in $\mathfrak{B}^\bullet$ and an isomorphism in ind- $\mathrm{D}(\mathfrak{B})$,

$$F(X_S)\Phi \xrightarrow{\sim} U_1, \tag{4.2.1}$$

Apply the functor $H^p \colon \mathrm{D}(\mathfrak{B}) \to \mathfrak{B}$ and take the inductive limit; the result is an isomorphism in $\mathfrak{B}$,

$$\varinjlim_{s \in X/S^{\leqslant n}} H^p(TX_s) \xrightarrow{\sim} H^p(U) = H^p(RF(X)). \tag{4.2.2}$$

For an index $s \in X/S^{\geqslant n}$ we have that X_s is in $\mathrm{Hot}^{\geqslant n}\mathfrak{A})$, and hence $TX_s \in \mathrm{Hot}^{\geqslant n}(\mathfrak{B}$; hence $H^p(TX_s) = 0$ for $p < n$. So, by the isomorphism the p'th cohomology of $U = RFX$ vanishes for $p < n$. It follows that the first part of the Lemma holds, and that U is isomorphic in $\mathrm{D}(\mathfrak{B})$ to a complex $V \in \mathfrak{B}^{\geqslant n}$, or, more precisely, $U \simeq iV$. As $\mathrm{D}^+(\mathfrak{B})$ is a full subcategory of $\mathrm{D}(\mathfrak{B})$, it follows that the isomorphism (4.2.1) in ind- $\mathrm{D}(\mathfrak{B})$ is the image under i of a unique isomorphism in ind- $\mathrm{D}(\mathfrak{B})$,

$$F^+(X_S)\Phi^+ \xrightarrow{\sim} V_1.$$

Hence, $F^+ = (\ )_{S^+}\,\mathrm{Hot}^+(T)$ is derivable at X, and

$$iRF^+(X) = RF(X).$$

Thus the lemma has been proved. ▯

Corollary. *A right complex Q in* $\mathrm{Hot}^+(\mathfrak{A})$ *is* $\mathrm{Hot}(T)$*-unfolded if and only if it is* $\mathrm{Hot}^+(T)$*-unfolded.*

Proof. Immediate. ▯

(4.3) Note. (Rewrite) (1) Assume that the category $\mathfrak{A}$ has enough injectives. Then the class of injectives is unfolding for any additive functor T, and it follows that the class $\mathrm{Hot}^+(\text{injectives}) \subseteq \mathrm{Hot}^+(\mathfrak{A})$ is a quasi-dense class consisting of quasi-injective objects of $\mathrm{Hot}^+(\mathfrak{A})$. Consequently, any functor F defined on $\mathrm{Hot}^+(\mathfrak{A})$ is derivable: to obtain the value $RF(X)$ at a complex $X \in \mathfrak{A}^+$, chose a quasi-isomorphism $s\colon X \to Q$ into a right complex of injectives; then $RF(X) = F(Q)$. (2) Note that the derived functors RT in particular are defined on complexes concentrated in degree 0, that is, they are functors

$$RT\colon \mathfrak{A} \to \mathrm{D}^+(\mathfrak{B}).$$

To obtain the value $R^nT(A)$ for an object $A \in \mathfrak{A}$, chose a resolution,

$$0 \to A \to Q^0 \to Q^1 \to \cdots,$$

with $Q^i \in \mathfrak{Q}$ (this is possible by (i)). Then $R^nT(A) = H^nT(Q)$. Note that there is a coaugmentation $TA \to TQ$, and hence a canonical morphism,

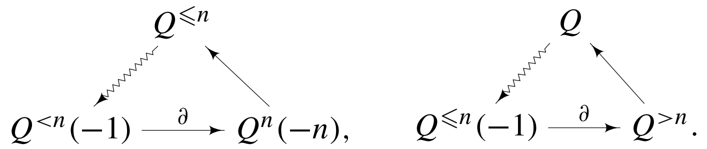
$$TA \to R^0T(A). \tag{4.1.4}$$

The sequence $0 \to A \to Q^0 \to Q^1$ is left exact. Hence (4.1.4) is an isomorphism if and only if T is left exact.

(4.4) Lemma. *Let $T\colon \mathfrak{A} \to \mathfrak{B}$ be an additive functor. Let Q be a complex in $\mathfrak{A}$ such the every Q^i is* $\mathrm{Hot}(T)$*-unfolded. If Q is a finite complex, then Q is* $\mathrm{Hot}(T)$*-unfolded. If Q is a right complex and* $\mathrm{Hot}(T)$ *is derivable at Q, then Q is* $\mathrm{Hot}(T)$*-unfolded.*

Proof. We may assume that Q is a right complex. Recall that Q is unfolded if $\mathrm{Hot}(T)$ is derivable at Q and the canonical morphism is an isomorphism $TQ \xrightarrow{\sim} RTQ$ in $\mathrm{D}(\mathfrak{B})$. The last condition may be checked on cohomology: $H^pTQ \xrightarrow{\sim} H^pRTQ$ for all p.

The chain truncations give rise to the following exact triangles in the homotopy category, see Hot(1.9):



(4.4.1)

The class of $\mathrm{Hot}^+(T)$-unfolded objects in $\mathrm{Hot}^+(\mathfrak{A})$ is a triangular subclass. By assumption it contains every complex of the form $Q^n(0)$, and hence every complex $Q^n(-n)$. Now, $Q^{\leqslant n} = 0$ for $n \ll 0$, because Q is a right complex. Hence, by induction on n, it follows from the first exact diagram that $Q^{\leqslant n}$ is $\mathrm{Hot}(T)$-unfolded for all n. If Q is a finite complex, then $Q = Q^{\leqslant n}$ for $n \gg 0$; hence Q is unfolded. Assume that Q is a right complex. Then the finite complex $Q^{\leqslant n}$ is unfolded for the functor $\mathrm{Hot}(T)$. In particular, the functor is derivable at $Q^{\leqslant n}$. By assumption, the functor is derivable at Q. Therefore, by the second exact triangle, the functor is derivable at the third vertex $Q^{>n}$. Apply the functors $\mathrm{Hot}(T)$ and RT and the transformation $\mathrm{Hot}(T) \to R\,\mathrm{Hot}(T)$ to obtain to tow exact triangles in $\mathrm{D}^+(\mathfrak{B})$ and a morphism of triangles from the first to the second. Consider the corresponding long exact sequences of cohomology. For given p take some $n \geqslant p$. The cohomology objects $H^pTQ^{>n}$ and $H^pR\,\mathrm{Hot}(T)(Q^{>n})$ vanish, the first trivially, the second by Lemma (4.2). cohomology. In degree p, the transformation induces an isomorphism $H^pTQ^{\leqslant n} \xrightarrow{\sim} H^pR\,\mathrm{Hom}(T)(Q^{\leqslant n}$. Therefore it induces an isomorphism $H^pTQ \xrightarrow{\sim} H^pR\,\mathrm{Hom}(T)(Q)$. Hence, $TQ \to R\,\mathrm{Hot}(T)(Q)$ is an isomorphism in $\mathrm{D}^+(\mathfrak{B})$. ∎

(4.5) Proposition. *Let $T : \mathfrak{A} \to \mathfrak{B}$ be an additive functor. Let $\mathfrak{Q} \subseteq \mathfrak{A}$ be an additive subclass. Then the following three conditions are equivalent:*

(i) *The triangular subclass* $\mathrm{Hot}^+(\mathfrak{Q}) \subseteq \mathrm{Hot}^+(\mathfrak{A})$ *is unfolding for the functor* $\mathrm{Hot}^+(T)$, *see* Lim(???).
(ii) *The class $\mathfrak{Q}$ is T-unfolding, see* (1.1).
(iii) *The class $\mathfrak{Q}$ is right dense in $\mathfrak{A}$, every object of $\mathfrak{Q}$ is* $\mathrm{Hot}^+(T)$*-unfolded and* $\mathrm{Hot}^+(T)$ *is derivable at every object in* $\mathrm{Hot}^+(\mathfrak{Q})$.

Proof. Note first that part of the conditions (i) or (ii) is that the class $\mathfrak{Q}$ is right dense in $\mathfrak{A}$. If the latter condition holds for $\mathfrak{Q}$, then, by The Density Theorem, the class $\mathrm{Hot}^+(\mathfrak{Q})$ is quasi-dense in $\mathrm{Hot}^+(\mathfrak{A})$. Conversely, it $\mathrm{Hot}^+(\mathfrak{Q})$ is quasi-dense in $\mathrm{Hot}^+(\mathfrak{A})$, then, in particular, for every object A in $\mathfrak{A}$ there is a complex Q with objects in $\mathfrak{Q}$ and a quasi-isomorphism $A(0) \to Q$. Clearly, then the degree-0 part $A \to Q^0$ is necessarily a monomorphism. Hence $\mathfrak{Q}$ is dense in $\mathfrak{A}$.

By the density Theorem, the class $\mathrm{Hot}^+(\mathfrak{Q})$ is dense in $\mathrm{Hot}^+(\mathfrak{A})$, if and only if $\mathfrak{Q}$ is dense in $\mathfrak{A}$. Using the characterization of unfolding classes Lim(??), the equivalence of (i) and (ii) is a consequence. The conditions imply that T is uniformly derivable, and in particular $Hot^+(T)$ is derivable at every right right complex, and every complex in $\mathrm{Hot}^+(\mathfrak{Q})$ is unfolded.

Conversely, if T is derivable at every complex in $\mathrm{Hot}^+(\mathfrak{A})$ and (iii) holds, then $\mathrm{Hot}^+(\mathfrak{Q})$ is dense in $\mathrm{Hot}^+(\mathfrak{A})$ and every right complex $Q \in \mathrm{Hot}^+(\mathfrak{A})$ is $\mathrm{Hot}^+(\mathfrak{T})$-unfolded by lemma (4); hence condition (i) holds. ∎

Assume that $T\mathfrak{A} \to \mathfrak{B}$ is (uniformly) derivable as defined in (1.1). It follows from the proposition that $\mathrm{Hot}^+(T)\ \mathrm{Hot}\mathfrak{A} \to \mathfrak{B}$ is uniformly derivable. Moreover, if $RT := R\,\mathrm{Hot}(T)$ is the derived functor, as a functor $\mathrm{D}^+() \to \mathrm{D}^+(\mathfrak{B})$, then, by definition of the sequence of derived functors R^pT, we have the equation, for $X \in \mathfrak{A}^+$ and $p \in \mathbb{Z}$,

$$H^p(TX) = R^pT(X),$$

where the left side is the p'th cohomology of $RT(X)$ and the right side is the value of R^pT at X.

(4.6) Definition. Assume that $T : \mathfrak{A} \to \mathfrak{B}$ is uniformly derivable as defined in (1.1). Then it follows from the proposition that the functor $\mathrm{Hot}^+(T)\colon \mathrm{Hot}^+(\mathfrak{A}) \to \mathrm{Hot}^+(\mathfrak{B})$ is uniformly derivable with repect to the systems of quasi-isomorphisms in the source and in the target. Moreover, it follows that the value $R^pT(X)$ of the p'th derived functor at a right complex X is the p'th cohomology of $RT(X)$:

$$R^pT(X) = H^p(RT(X)).$$

In particular, it follows that an object $Q \in \mathfrak{A}$ is T-acyclic if and only if the complex $Q(0)$ is $\mathrm{Hot}(T)$-unfolded.

(4.7) Proposition. *It* $\mathrm{Hot}(T)\colon \mathfrak{A} \to \mathfrak{B}$ *is derivable at every right complex in $\mathfrak{A}^+$, then it is uniformly derivable if and only if the class of T-acyclic objects is dense in $\mathfrak{A}$.*

Proof. The class of all T-acyclic objects in $\mathfrak{A}$ is T-unfolding if it is dense. ∎

(4.8) Note. Assume that $T: \mathfrak{A} \to \mathfrak{B}$ is uniformly derivable. Then R^0T as a functor $\mathfrak{A} \to \mathfrak{B}$ is left exact and $T = R^0T$ on the dense class of T-acyclic objects. Using (4.5)(ii) it follows easily that every T-acyclic object is R^0T-acyclic, and hence

$$RT = R(R^0T).$$

As a consequence arguments related to derivability of functors may often be reduced to the case of left exact functors.

(4.9) Proposition. *Let $T: \mathfrak{A} \to \mathfrak{B}$ be an additive functor. Let $\mathfrak{Q} \subseteq \mathfrak{A}$ be a dense additive $^+$-class. Assume that T is short exact on $\mathfrak{Q}$. Then $\mathfrak{Q}$ is T-unfolding, and, consequently, T is uniformly derivable. Conversely, if T left exact and uniformly derivable, then the class of all T-acyclic objects of $\mathfrak{A}$ is a dense and additive $^+$-class.*

Proof. The assumption on $\mathfrak{Q}$ in the first part is the following: If

$$0 \to Q \to Q' \to Q'' \to 0 \tag{4.9.1}$$

is an exact sequence in $\mathfrak{A}$ with $Q', Q \in \mathfrak{Q}$, then $Q'' \in \mathfrak{Q}$ and the sequence $0 \to TQ \to TQ' \to TQ'' \to 0$ is an exact

To verify Condition (4.5)(ii), split an exact right exact sequence of objects from Q into short exact sequences. By induction, each of these short exact sequences have the objects in $\mathfrak{Q}$, and the image under T of each short exact sequence is short exact. Clearly, then the image under T of the complex is exact.

Assume conversely that T is left exact and uniformly derivable. Consider a short exact sequence (4.9.1) with Q', Q in $\mathfrak{Q}$, and the corresponding long exact sequence,

$$0 \to TQ'TQ \to TQ'' \to R^1TQ \to R^1TQ \to R^1TQ'' \to R^1TQ'' \to \cdots .$$

Since $R^pTQ' = R^pTQ = 0$ for $p > 0$, it follows that $TQ \to TQ''$ is an epimorphism and that $R^pTQ'' = 0$ for $p > 0$. Hence, the image under T of the sequence (4.9.1) is exact, and Q'' is T-acyclic. ∎

(4.10) The Chain Rule. *Let $T: \mathfrak{A} \to \mathfrak{B}$ and $S: \mathfrak{B} \to \mathfrak{C}$ be additive functors between abelian categories. Assume that both functors are derivable at every right complex. In addition, assume that there is a additive class $\mathfrak{Q} \subseteq \mathfrak{A}$ satisfying the following conditions:*

(i) *The class $\mathfrak{Q}$ is dense in $\mathfrak{A}$.*
(ii) *If $Q \in \mathfrak{Q}$, then TQ is S-unfolded.*

Then the composition ST is uniformly derivable, and the canonical transformation is an isomorphism,

$$R(ST) = RS\,RT,$$

of functors $\mathrm{D}^+(\mathfrak{A}) \to \mathrm{D}^+(\mathfrak{C})$.

Moreover, if all objects of $\mathfrak{Q}$ are T-acyclic, then $\mathfrak{Q}$ is unfolding for T and for ST.

Proof. It is easy to see that the class $\mathrm{Hot}^+(\mathfrak{Q})$ satisfies the conditions (1) and (2) in the general Chain Rule for derived functors, see Lim(4??); use Lemma (4.5) ◻

Note. An exact functor $S\colon \mathfrak{B} \to \mathfrak{C}$ between abelian categories is obviously uniformly derivable: The class of all objects in $\mathfrak{B}$ is unfolding for S. The functor $\mathrm{Hot}^+(S)\colon \mathrm{Hot}^+(\mathfrak{B}) \to \mathrm{Hot}^+(\mathfrak{C})$ is local with respect to the systems of quasi-isomorphisms in the source and in the target. The derived functor RS is the extension of $\mathrm{Hot}^+(S)$ to a functor $RS\colon\ \mathrm{D}^+(\mathfrak{B}) \to\ \mathrm{D}^+(\mathfrak{C})$.

Consider the setup of the Chain Rule. If the functor $S\colon \mathfrak{B} \to \mathfrak{C}$ is exact, the conclusion is the trivial isomorphism of functors

$$R(ST) = S\,RT\colon\ \mathrm{D}^+(\mathfrak{A}) \to\ \mathrm{D}^+(\mathfrak{C}).$$

On the other hand, if the functor $T\colon \mathfrak{A} \to \mathfrak{B}$ is exact, and S is, say, uniformly derivable, it does not follow in general that the canonical transformation $R(ST) \to (RS)\,T$ is an isomorphism.

(4.11) Proposition. *Assume that $T\colon \mathfrak{A} \to \mathfrak{B}$ is left exact. Let n be a non-negative integer. Then the following three conditions are equivalent:*

(i) *The functor T is uniformly derivable and for every object $A \in \mathfrak{A}$ we have that $R^pT(A) = 0$ for $p > n$.*
(ii) *The functor T is derivable at any any right complex in $\mathfrak{A}^+$, and the class of T-acyclic objects in $\mathfrak{A}$ is dense in $\mathfrak{A}$ and of right dimension at most n.*
(iii) *There exists a subclass $\mathfrak{Q} \subseteq \mathfrak{A}$ which is dense and of right dimension at most n, and such that T is short exact on $\mathfrak{Q}$.*

Proof. (i) $\Rightarrow$ (ii): It is enough to prove that the class of all T-acyclic objects is of dimension at most n. So, consider an exact sequence,

$$Q_1 \to Q_2 \to \cdots \to Q_n \to Q \to 0, \tag{4.11.1}$$

such that each of $Q_1, \dots, Q_n$ is T-acyclic. Split the sequence into short exact sequences,

$$0 \to Z_i \to Q_i \to Z_{i+1} \to 0, \quad i = 1, \dots, n, \ \text{where } Z_{n+1} = Q.$$

In the corresponding long exact sequences of values of R^pT, we have $R^qT(Q_i) = 0$ for $q > 0$, because Q_i is T-acyclic. Consequently, we obtain isomorphisms, for $p > 0$,

$$R^qT(Q) = R^qT(Z_{n+1}) \approx R^{q+1}T(Z_n) \approx \cdots \approx R^{q+n}T(Z_1).$$

The right hand side vanishes when $q > 0$. Hence $R^qT(Q) = 0$ for $q > 0$. Therefore Q is T-acyclic.

(ii) $\Rightarrow$ (iii): Take as $\mathfrak{Q}$ the class of T-acyclic objects of $\mathfrak{A}$.

(iii) $\Rightarrow$ (i): The class $\mathfrak{Q}$ is necessarily a $^+$class. Therefore, by Proposition (4.9), T is uniformly derivable, and $\mathfrak{Q}$ is T-unfolding. Let A be an object of $\mathfrak{A}$. Then, since $\mathfrak{Q}$ is right dense and of dimension $\leqslant n$, there is a resolution of A of length at most n with objects from $\mathfrak{Q}$. In turn, the resolution defines a quasi-isomorphism $A(0) \to Q$ into a positive complex Q of objects from $\mathfrak{Q}$ of length at most n. Then $RTA = TQ$. In particular, $R^pT(A) = H^pTQ = 0$ for $p > n$. Hence (i) has been proved. ◻

(4.12) Definition. A left exact functor $T:\mathfrak{A}\to\mathfrak{B}$ is said to be of *dimension* $\leqslant n$, if it satisfies the equivalent conditions of (4.11) for the integer n.

(4.13) Proposition. *Let* $T:\mathfrak{A}\to\mathfrak{B}$ *be a left exact functor, and let* $\mathfrak{Q}$ *be a subclass of* $\mathfrak{A}$ *satisfying the condition* (4.11)(iii) *for some integer* n. *Then the triangular subclass* $\operatorname{Hot}(\mathfrak{Q})$ *of* $\operatorname{Hot}(\mathfrak{A})$ *is unfolding for* $\operatorname{Hot}(T)$.

Proof. Since $\mathfrak{Q}$ is dense in $\mathfrak{A}$ and of finite dimension, it follows form the Density Theorem Hot(3.10) that $\operatorname{Hot}(\mathfrak{Q})$ is quasi-dense in $\operatorname{Hot}(\mathfrak{A})$. Hence it suffices to prove for any exact complex Q of objects from $\mathfrak{Q}$ that the complex TQ is exact in $\mathfrak{B}^\bullet$. Split Q into short exact sequences,

$$0\to Z_i\to Q_i\to Z_{i+1}\to 0.$$

It suffices to note that each Z_i belongs to $\mathfrak{Q}$. Clearly, Z_i is the $(n+1)$'st object in the exact sequence,

$$Q^{i-n}\to\cdots\to Q^{i-2}\to Q^{i-1}\to Z_i\to 0.$$

The Q^j belong to $\mathfrak{Q}$ and $\mathfrak{Q}$ has dimension $\leqslant n$. Consequently, Z_i belongs to $\mathfrak{Q}$. □

Corollary. *If* $T:\mathfrak{A}\to\mathfrak{B}$ *is of finite dimension, then the functor* $\operatorname{Hot}(T)\colon\operatorname{Hot}(\mathfrak{A})\to\operatorname{Hot}(\mathfrak{B})$ *is uniformly derivable, and any complex consisting of* T*-acyclic objects is unfolded for* $\operatorname{Hot}(T)$.

Proof. The assertion is an immediate consequence of the Proposition. □

(4.14) The Chain Rule for finite dimensional functors. *Assume for additive functors between abelian categories* $T:\mathfrak{A}\to\mathfrak{B}$ *and* $S:\mathfrak{B}\to\mathfrak{C}$ *that* $\operatorname{Hot}(T)$ *is is derivable at all complexes in* $\mathfrak{A}^\bullet$, *and that* S *is of finite dimension. In addition, assume that there is a subclass* $\mathfrak{Q}$ *of* $\mathfrak{A}$ *such that*

(1) $\mathfrak{Q}$ *is dense and of finite dimension,*

(2) *For every object* $Q\in\mathfrak{Q}$, *the object* TQ *in* $\mathfrak{B}$ *is* S*-acyclic. Then the composition* $\operatorname{Hot}(ST)=\operatorname{Hot}(S)\operatorname{Hot}(T)$ *is uniformly derivable, and the natural transformation is an isomorphism of functors:*

$$R(S\,T)=RS\,RT\,:\,\mathrm{D}(\mathfrak{A})\to\mathrm{D}(\mathfrak{C}).$$

Proof. The arguments are similar to those in the proof of Theorem (4.10). □

(4.15). Let I be a small (index) category. Consider the abelian category $\mathfrak{A}^I$ of all I-systems in $\mathfrak{A}$, that is, functors $\mathcal{X}\colon I\to\mathfrak{A}$. For every object $i\in I$, *evaluation* at i is the functor $\mathcal{X}\mapsto\mathcal{X}_i$. It is an exact functor (see Lim(???)),

$$(\ )_i:\mathfrak{A}^I\to\mathfrak{A}.$$

Note that a functor $T:\mathfrak{A}\to\mathfrak{B}$ has a natural extension to a functor $T:\mathfrak{A}^I\to\mathfrak{B}^I$. It is defined by composition,

$$T\mathcal{X}\colon I\to\mathfrak{A}\to\mathfrak{B}.$$

Proposition. *Let $T: \mathfrak{A} \to \mathfrak{B}$ be a left exact functor. Assume that there is a subclass $\mathfrak{Q}$ of $\mathfrak{A}$ satisfying the following two conditions:*

(1) *$\mathfrak{Q}$ is dense in $\mathfrak{A}$, and $\mathfrak{Q}$ is closed under $\prod_I$'s.*
(2) *Every object Q in $\mathfrak{Q}$ is T-acyclic.*

Then the left exact functor $T^I: \mathfrak{A}^I \to \mathfrak{B}^I$ is uniformly derivable, and an I-system $\mathcal{Q}: I \to \mathfrak{A}$ is T^I-acyclic if and only if its values $\mathcal{Q}_i$ are T-acyclic for all $i \in I$. Moreover, for any $i \in I$ there is an isomorphism of functors,

$$(\,)_i RT^I = RT\,(\,)_i \colon \mathrm{D}^+(\mathfrak{A}^I) \to \mathrm{D}^+(\mathfrak{B}).$$

Proof. Let $\mathfrak{P} \subseteq \mathfrak{A}$ be the class of T-acyclic objects of $\mathfrak{A}$, and denote by $\mathfrak{P}^I$ the subclass of $\mathfrak{A}^I$ consisting of all functors $I \to \mathfrak{A}$ with values in $\mathfrak{P}$. Let $\mathcal{X}$ is an I-system. By Condition (1), there is for every i an object $\mathcal{Q}_i \in \mathfrak{Q}$ and a monomorphism $\mathcal{X}_i \to \mathcal{Q}_i$. As is well known, the family $\mathcal{Q} = \{\mathcal{Q}_i\}$ determines a co-induced I-system $\rho\mathcal{Q}$ given by

$$(\rho\mathcal{Q})_i = \prod_{i \to j} \mathcal{Q}_j,$$

and the family of monomorphisms $\mathcal{X}_i \to \mathcal{Q}_i$ defines a monomorphism of I-systems $\mathcal{X} \to \rho\mathcal{Q}$. It follows from condition (1) that $(\rho\mathcal{Q})_i$ is in $\mathfrak{Q}$. Hence $\rho\mathcal{Q} \in \mathfrak{P}$. Consequently, $\mathfrak{P}$ is dense in $\mathcal{A}^I$. It follows from Condition (2) that $\mathfrak{P}$ is unfolding for the functor T^I. Hence T^I is uniformly derivable. Now apply The Chain Rule (4.10) to the equal compositions $(\,)_i T^i = T^i (\,)_i$. The result is an isomorphism of functors,

$$(\,)_i RT^I = RT\,(\,)_i \colon \mathrm{D}^+(\mathfrak{A}^I) \to \mathrm{D}^+(\mathfrak{B}).$$

Let $\mathcal{X}$ be an I-system, or a right complex of I-systems, in $\mathfrak{A}$. Then, using that the exact functor $(\,)_i$ commutes with the cohomology functors H^p, we obtain an isomorphism in $\mathfrak{B}$,

$$(\,)_i\, R^p(T^I)(\mathcal{X}) = (R^pT)(\mathcal{X}_i).$$

Clearly, the last assertion of the Proposition is a consequence. ∎

(4.16) Corollary. *It the functor T is of dimension at most n, then so is T^I, and there is an isomorphism of functors,*

$$(\,)_i\, R(T^I) = RT\,(\,)_i \colon \mathrm{D}(\mathfrak{A}^I) \to \mathrm{D}(\mathfrak{B}).$$

(4.17) Note. Note that the conditions (1) and (2) in Proposition (4.15) hold when $\mathfrak{A}$ has $\prod_I$'s and enough injectives.

(4.18) Theorem. *Assume that $T : \mathfrak{A} \to \mathfrak{B}$ has an exact resolvent complex $T \to \Pi$:*

$$\overline{\Pi}: \quad 0 \to T@ > \epsilon >> \Pi^0 \to \Pi^1 \longrightarrow \Pi^2 \longrightarrow \cdots . \tag{4.18.1}$$

Then T is uniformly derivable, and for every complex X in $\mathfrak{A}^+$, there is a canonical isomorphism in $D^+(\mathfrak{B})$,

$$RT(X) \xrightarrow{\sim} \operatorname{Tot} \Pi(X).$$

In particular, an object Q of $\mathfrak{A}$ is T-acyclic if and only if the sequence (4.18.1) *is exact.*

Moreover, every complex in $\mathfrak{A}^+$ consisting of objects Q such that the sequence (4.18.1) *is exact, is T-acyclic.*

Proof. The essentials of the proof are presented in Section 1. Let $\mathfrak{Q}^+$ be the class of complexes in $\mathfrak{A}^+$ consisting of objects Q such that the sequence (4.18.1) is exact. The assumptions about $T \to \Pi$ imply that $\mathfrak{Q}$ is right dense in $\mathfrak{A}$ and, moreover, if Q is an exact right complex of objects of $\mathfrak{Q}$, then TQ is exact. Hence $\mathfrak{Q}$ is a T-unfolding class. In particular, T is uniformly derivable.

Let X be a right complex in $\mathfrak{A}^+$. Chose a quasi-isomorphism $X \to Q$ into a right complex of objects from $\mathfrak{Q}$. Use the Row Theorem and the Column Theorem as in the proof of theorem (1.7)It to deduce canonical isomorphisms,

$$RTX = TQ = \operatorname{Tot} \Pi(Q) = \operatorname{Tot} Pi(X),$$

the first by definition of RTX, the second because (4.18.1) is exact when evaluated at on object of $\mathfrak{Q}$, and the last because each functor Π^i is exact. □

(4.19) Silly example. Let $\mathfrak{A} := \mathfrak{B}^\bullet$ be the category of complexes of objects of an abelian category $\mathfrak{B}$. Consider the two functors,

$$I, K : \mathfrak{A} \to \mathfrak{B},$$

given by $I(X) = \operatorname{Im}(X^{-1} \to X^0)$ and $K(X) := \operatorname{Ker}(X^0 \to X^1)$. Let $\Pi(X)$ be the truncated complex $\Pi(X) = X^{\geqslant 0}$, and view Π as the complex of functors $\Pi^n : \mathfrak{A} \to \mathfrak{B}$, given by $\Pi^n(X) = X^n$ for $n = 0, 1, \ldots$. So each functor Π^n is exact. With the obvious co-augmentations given by the inclusions $I(X) \hookrightarrow K(X) \hookrightarrow X^0$, the complex Π is a resolvent complex for both functors I and K. Indeed, every object X of $\mathfrak{A}$ embeds into an acyclic object Q (for instance X embeds into the mapping cone of the identity of X, and the mapping cone is even contractible), and it suffices to note that the sequence (4.18.1), for $T = K$ is exact if Q is acyclic in positive degrees, and for $T = I$ is exact if Q is acyclic in nonnegative degrees.

In particular, it follows that the functors I, K are uniformly derivable and that the values at an object A of $\mathfrak{A}$, that is, the values at a complex $A \in \mathfrak{B}^\bullet$, are the following:

$$RI(X) = RK(X) = X^{\geqslant 0}.$$

(4.20) Remark. An exact resolvent complex for the identity functor 1 of $\mathfrak{A}$,

$$\cdots \to 0 \to 1 \to \Pi^0 \to \Pi^1 \to \cdots \tag{4.20.1}$$

is also called an (exact) *resolution* of the identity, since, for instance by the theorem, the sequence (4.20.1) is necessarily exact. Consider the following condition: Every object A of $\mathfrak{A}$ embeds into an object Q such that the sequence,

$$\cdots \to 0 \to Q \to \Pi^0 Q \to \Pi^1 Q \to \dots,$$

is contractible (in the notions of relative abelian categories, then the sequence (4.20.1) is a *relative* resolvent complex for the identity). Clearly, under this condition, if $T\colon \mathfrak{A} \to \mathfrak{B}$ is any additive functor such that all the functors $T\Pi^n$ are exact, then $T \to T\Pi$ is an exact resolvent complex for T (in fact, a relatively exact resolvent complex).

(4.21) Example. Let B be an object of $\mathfrak{A}$ with a projective resolution $P \to B$,

$$\cdots \to P_2 \to P_1 \to P_0 \to B \to 0 \to \cdots .$$

Consider the functor $T = \mathrm{Hom}(B, -)$. Then there is a T-augmented complex of functors $\mathfrak{A} \to (\mathrm{Ab})$,

$$\cdots \to 0 \to \mathrm{Hom}(B, -) \to \mathrm{Hom}(P_0, -) \to \mathrm{Hom}(P_1, -) \to \cdots ,$$

and each functor $\mathrm{Hom}(P_i, -)$ is exact. The complex is a resolvent complex for $\mathrm{Hom}(B, -)$, if and only if $\mathrm{Hom}(B, -)$ is derivable (for instance, if $\mathfrak{A}$ has enough injectives).

Note that in any case, the cohomology $H^n \operatorname{Tot}\mathrm{Hom}(P, X)$, for any complex $X \in \mathfrak{A}^+$ (in fact, for any complex $X \in \mathfrak{A}^\bullet$ if Tot of a bicomplex of abelian groups is determined by products), is the ext-group,

$$H^n \operatorname{Tot}\mathrm{Hom}(P, X) = \mathrm{Ext}^n(B, X).$$

Indeed, we have the equalities,

$$\begin{aligned} \mathrm{Ext}^n(B, X) = \mathrm{Hom}_D(B, X(n)) &\xrightarrow{\sim} \mathrm{Hom}_D(P, X(n)) \\ = \mathrm{Hom}_{\mathrm{Hot}}(P, X(n)) = H^n \mathrm{Hom}^\bullet_{\mathfrak{A}}(P, X) &= H^n \operatorname{Tot}\mathrm{Hom}_{\mathfrak{A}}(P, X). \end{aligned}$$

The first is the definition of the ext-group as the hom-group in the derived category $D = D(\mathfrak{A})$, the second holds because $P \to B$ is a quasi-isomorphism, the third holds because P is a left complex of projectives, the fourth holds by definition of the homotopy category $\mathrm{Hot} = \mathrm{Hot}(\mathfrak{A})$, and the last holds by the definition of $\mathrm{Hom}^\bullet$ as Tot of a bicomplex.

(4.22) Theorem. *Assume that $T\colon \mathfrak{A} \to \mathfrak{B}$ has an exact resolvent complex. Then T is of dimension $\leqslant n$ if and only if T has an exact resolvent complex $T \to \Pi$ of length at most n, that is, with $\Pi^p = 0$ for $p > n$. Moreover, if $T \to \Pi$ is a finite exact resolvent complex, then for any complex $X \in \mathrm{D}(\mathfrak{A})$, we have the equation in $\mathrm{D}(\mathfrak{B})$:*

$$RT(X) = \operatorname{Tot}\Pi(X).$$

Proof. Immediate. ∎