15.01.07/CB

Problem set 3. Deadline January 22, 2007

Exc. 1 (i) Prove the power series expansion for $|z| < 1$

$$zF(\tfrac{1}{2}, 1; \tfrac{3}{2}; -z^2) = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots .$$

(ii) Prove that the sum of the series for $-1 < z < 1$ is Arctan z, where Arctan is the inverse of $\tan | \,] - \pi/2, \pi/2[$, and prove Leibniz' formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \cdots .$$

(iv) Prove that Arctan has a holomorphic extension to the cut plane $\mathbb{C} \setminus \{iy | y \in \mathbb{R}, |y| \geq 1\}$.

Exc. 2 Let $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$ Prove that the Gaussian hypergeometric function $F(a, b; c; z)$ satisfies the following formulas

$$\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a + 1, b + 1; c + 1; z)$$

$$(a - b)F(a, b; c; z) = aF(a + 1, b; c; z) - bF(a, b + 1; c; z).$$

Exc. 3 Prove that the arclength L of the ellipse with equation $x^2/a^2 + y^2/b^2 = 1$, where $a \geq b > 0$ and the excentricity $e = \sqrt{1 - (b/a)^2}$, is given by

$$L = 2\pi a F(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2).$$

Exc. 4 With the notation of Chapter 4, prove that the function

$$\varphi_1(z) = \frac{\vartheta_1(z)}{\vartheta_0(z)}$$

is an elliptic function with periods 2 and τ (as claimed on page 14).

Exc. 5 Let f be an elliptic function of order $n \geq 2$ and let p be a polynomial with complex coefficients of degree $d \geq 1$. Prove that the composed function $p(f(z))$ is an elliptic function of order nd.