Solutions to the exam in Complex Analysis (KomAn), April 2006

Opgave 3

1) Since $\tan z = \sin z / \cos z$, and the zeros of sinus are $z = p\pi$, while the zeros of cosinus is $z = \pi/2 + p\pi$, where in both cases $p \in \mathbb{Z}$, these are respectively the zeros and poles of tan. They are all simple since the zeros mentioned are all simple.

2) Since $\tan z$ and z both have simple zeros at $z = 0$, this point is a removable singularity. The limit is 1 since $\sin z/z$ has limit 1 and $\cos z$ has limit 1.

3) The function $f(z) = \tan z/z^2$ has the poles $z = 0, \pm\pi/2$ in the disc $K(0, \pi)$. The pole 0 has order 1 by 2) and the two others have order 1 by 1). The residues are

$$\mathrm{Res}(f, 0) = 1, \quad \mathrm{Res}(f, \pm\pi/2) = \frac{\sin z}{-z^2 \sin z}|_{z=\pm\pi/2} = -\frac{4}{\pi^2},$$

and by the Cauchy residue theorem we get that the integral in question has the value

$$2\pi i(1 - \frac{8}{\pi^2}).$$

Opgave 4

1) Define $f(z) = z^5$. Then for $|z| = 2$ we have

$$|p(z) - f(z)| \leq 3|z|^3 + e|z| + 1 = 25 + 2e < 31 < |f(z)| = 32,$$

so by Rouché's Theorem p and f have the same number of zeros in $K(0, 2)$, but f has clearly 0 as a zero of order 5, which shows the result.

2) Since the integrand is $p'(z)/p(z)$, the integral is the number of zeros of p in the disc, hence equal to 5.

3) The vertical line $\mathrm{Re}\, z = 2$ is tangent to the circle $K(0, 2)$ at $z = 2$. The zeros $z_1, z_2, \ldots, z_5$ of p lie in the open disc $K(0, 2)$. Let a be the biggest of the real parts $\mathrm{Re}\, z_j, j = 1, \ldots, 5$. Then $a < 2$, so we can choose α with $a < \alpha < 2$. Then the vertical line $\mathrm{Re}\, z = \alpha$ divide the complex plane in two open half-planes. The one to the left contains the zeros of p, the one to the right contains the triangle T. The Cauchy integral theorem can therefore be applied to the function $1/p(z)$ which is holomorphic in the right halfplane, which is convex. Therefore the integral is zero.