Moment problems and orthogonal polynomials

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Introduction

In 1894 Thomas Jan Stieltjes (1856-1894) published an extremely influential paper: *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse, **8**, 1–122; **9**, 5–47. He introduced what is now known as the Stieltjes integral with respect to an increasing function ϕ, the latter describing a distribution of mass (a measure μ) via the convention that the mass in an interval $]a,b]$ is $\mu(]a,b]) = \phi(b) - \phi(a)$. This integral was used to solve the following problem which he called *the moment problem:*

Given a sequence $s_0, s_1, \ldots$ of real numbers. Find necessary and sufficient conditions for the existence of a measure μ on $[0,\infty[$ so that

$$s_n = \int_0^\infty x^n \, d\mu(x) \quad \textit{for} \quad n = 0, 1, \ldots . \qquad (*)$$

The number s_n is called the n'th moment of μ, and the sequence (s_n) is called the moment sequence of μ.

Stieltjes was led to the Stieltjes moment problem above via a study of continued fractions.

As one can see from the years, Stieltjes died the same year as the paper was published. In fact he did not see the paper appear. He was quite young when he passed away and his career was unusual. He was enthusiastically interested in mathematics form his youth and spent more time reading deep mathematics than studying what he was supposed to at the Technical University of Delft. So after having failed the final examination twice he could not continue, and his father arranged that he got a job at the Observatory of Leiden helping with calculations. In his spare time he continued to study mathematics and he began to publish and began to correspond with the great French mathematician Charles Hermite. Their correspondence has been published in two volumes, [Baillaud and Bourget(1905)] and is worth reading.

In 1883 he applied for a position at the University of Groningen and the scientific committee ranked him as number 1 and asked him if he would take the job. However the Minister of Education chose one with lower ranking probably because Stieltjes did not have the formal qualifications such as a degree. For more details see [van Dijk (1994)].

Stieltjes became so disappointed that he left Holland and moved to Paris in 1885, where he began to prepare a thesis under the guidance of Hermite. The same day as he had defended his thesis Hermite wrote to the French minister of Education proposing Stieltjes for one of the vacant positions in France, and as a consequence of this he was appointed to a position in Toulouse in 1886, where he worked to his death.

For interesting accounts of his life see [Huron (1974)] and [Cassinet (1995)].

It is natural to generalize the problem (*) by removing the condition that μ shall be concentrated on $[0, \infty[$ and to assume simply that μ is a measure on the real line. This apparently simpler problem was solved by Hans Hamburger (1889-1956) in a series of papers of which the first appeared in 1920: *Über eine Erweiterung der Stieltjesschen Momentenproblems*, Math. Ann. **81**, 235-319. The moment problem with no restrictions on the support of the measure is now called the Hamburger moment problem.

While the original solutions of the two moment problems were closely connected to continued fractions, this theory is not needed in the modern approach based on functional analysis. The Hungarian mathematician Marcel Riesz was invited to Stockholm to help Mittag-Leffler with the editing of a special volume of Acta Mathematica. He settled in Sweden and later he got a professorship in Lund. He wrote three notes (1921-23) on the moment problem where he obtained many of Hamburger's results. The characterization of Hamburger moment sequences by positive definiteness of the Hankel matrices was proved by Riesz using an extension of linear functionals. This technique was later given a general framework in the Hahn-Banach THeorem in functional analysis.

The early history of the moment problem is described in [Kjeldsen(1993)].

Methods from complex function theory and from the theory of operators bring new insight into the problems. Nevanlinnna obtained in 1922 a complete description of all the solutions to the indeterminate Hamburger moment problem via a study of holomorphic functions with positive imaginary part in the upper half-plane. The modern exposition is intimately connected to the theory of orthogonal polynomials. To find all solutions of the indeterminate moment problem is equivalent to the study of all self-adjoint extensions of a certain symmetric operator. The spectral theory of self-adjoint operators is closely related to the moment problem and the main results of both theories are essentially equivalent. A treatment of the moment problem based on the theory of self-adjoint operators is given in Stone's famous monograph on operators in Hilbert space [Stone (1932)].

The present notes are largely inspired by the classical books [Akhiezer (1965)] and [Shohat and Tamarkin (1943)]. A recent treatment is given in [Simon (1998)].

The theory of the k-dimensional moment problem is by far less complete than the one-dimensional. One main difference between the moment problems in one and several dimensions is related to the fact that while a nonnegative polynomial in one variable is a sum (of two) squares of polynomials, this is no longer true for two or more variables, an observation going back to Hilbert in 1888. The first concrete example of a non-negative polynomial in two variables which is not a sum of squares appeared only in the 1960'ies.

From an operator point of view the multi-dimensional moment problem is related to the delicate problem of finding commuting self-adjoint extensions of a finite family of symmetric operators commuting on a common domain. Fuglede gave a status report about the multi-dimensional moment problem in [Fuglede(1983)].

I have included results about the multi-dimensional moment problem when it can be done without too much extra work.

The aim of these notes is to give an account of the moment problem in one and several dimensions with emphasis on the development since Akhiezer's monograph.

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