Modal Logics and Completeness Theorems

Kim Larsen

The purpose of this little article is to introduce the reader to modal logic and to show that for many normal modal logics completeness theorems are essentially just model existence theorems. Some basic knowledge about propositional calculus is probably required.

Modal languages

First we make the definitions needed to build modal logics, via the notion of frames and models.
The languages of modal logics are the language of propositional calculus where sentential operators, the so-called modal operators, have been added.

Definition 1 (Modal similarity type) A modal similarity type is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $\rho : O \to \mathbb{N}$.

The elements of O are called *modal operators.* The function ρ assigns to each operator $\Delta \in O$ a finite *arity*, indicating the number of arguments Δ can be applied to.

Definition 2 (Modal language) A modal language *ML*(τ,Φ) consists of the following:

- A modal similarity type $\tau = (O, \rho)$.
- A set Φ of propositional letters.

The set $Form(\tau, \Phi)$ of modal formulas over τ and Φ is given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \Delta(\phi_1, \ldots, \phi_{\rho(\Delta)})$$

where p ranges over the elements of Φ.

This definiton simply states that a formula is either a proposition letter, the propositonal constant falsum, a negated formula, a disjunction or a set of formulas prefixed by a modal operator. That is, a modal language of a given similarity type is just a propositional language extended with modal operators from the similarity type $\tau = (O, \rho)$.
To each $\Delta \in O$ we associate a dual operator ∇ defined by

$$\nabla(\phi_1, \ldots, \phi_{\rho(\Delta)}) := \neg\Delta(\neg\phi_1, \ldots, \neg\phi_{\rho(\Delta)}).$$

We also use the abbreviations $\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi), \phi \rightarrow \psi := \neg\phi \vee \psi, \phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi), \top := \neg\bot$.

Definition 3 (Uniform Substitution) Given a modal similarity type τ and a set Φ of propositional letters,a substitution is a map $\sigma : \Phi \rightarrow \mathrm{Form}(\tau, \Phi)$. Such a substitution induces a map $(\cdot)^\sigma : \mathrm{Form}(\tau, \Phi) \rightarrow \mathrm{Form}(\tau, \Phi)$ defined recursively as follows:

$$\begin{aligned} \bot^\sigma &= \bot \\ p^\sigma &= \sigma(p) \\ (\neg\psi)^\sigma &= \neg\psi^\sigma \\ (\phi \vee \psi)^\sigma &= \phi^\sigma \vee \psi^\sigma \\ \Delta(\phi_1, \ldots, \phi_n)^\sigma &= \Delta(\psi_1^\sigma, \ldots, \psi_n^\sigma) \end{aligned}$$

This spells out what is meant by performing uniform substitution. We say that χ is a substitution instance of ψ if there is some substitution σ such that $\psi^\sigma = \chi$.

Semantic structures

Now that we have defined the modal languages we would like to be able to form statements, that is, we want to assign meaning to formulas in a modal language. The semantic structure is put on our modal languages by interpreting them in relational structures.

Definition 4 (τ-frame) Let τ be a modal similarity type. A τ-frame is a tuple $\mathfrak{F} = (\mathrm{W}, (R_\Delta)_{\Delta\in\tau})$ such that

1. W is a non-empty set.
2. For each $n \geq 0$ and each n-ary modal operator $\Delta \in \tau$ R_Δ is an $(n+1)$-ary relation on W.

The elements of W will often be referred to as 'states' or 'worlds'.

Definition 5 (τ-model) Given a similarity type τ, a τ-model is a pair $\mathfrak{M} = (\mathfrak{F}$,V) where $\mathfrak{F}$ is a τ-frame and V is a valuation, that is, a map $V : \Phi \to \mathcal{P}(\mathrm{W})$. Here $\mathcal{P}(\mathrm{W})$ denotes the powerset of W.

Definition 6 (Satisfaction) Suppose w is a state in a model $\mathfrak{M} = ((W, (R_\Delta)_{\Delta\in\tau}), V)$. We define the notion of a formula ϕ being satisfied (or 'true') in $\mathfrak{M}$ at w (notation: $\mathfrak{M}, w \Vdash \phi$) by structural induction as follows:

$$\mathfrak{M}, w \Vdash p \quad \text{iff} \quad w \in V(p) \text{ where } p \in \Phi \tag{1}$$

$$\mathfrak{M}, w \Vdash \bot \qquad \text{never} \tag{2}$$

$$\mathfrak{M}, w \Vdash \neg\phi \quad \text{iff} \quad \text{not } \mathfrak{M}, w \Vdash \phi \tag{3}$$

$$\mathfrak{M}, w \Vdash \phi \vee \psi \quad \text{iff} \quad \mathfrak{M}, w \Vdash \phi \text{ or } \mathfrak{M}, w \Vdash \psi \tag{4}$$

$$\mathfrak{M}, w \Vdash \Delta(\phi_1, \ldots, \phi_n) \quad \text{iff} \quad \text{for some } v_1, \ldots, v_n \in W \tag{5}$$

$$\text{with } R_\Delta w v_1 \ldots v_n \text{ we have for each } i \;\; \mathfrak{M}, v_i \Vdash \phi_i$$

If one restricts attention to a single world, w, one should note that $\mathfrak{M}, w \Vdash p$ is analogous to assigning $\top$ to the proposotional variable p in traditional propositional semantics. Continuing this analogy, (2)-(4) likewise corresponds to the semantics for $\vee$ and $\neg$ in traditional propositional logic. Hence, the modal semantics may be considered an extension of the semantics for traditional propositional logic.

Definition 7 (Global truth) A formula ϕ is said to be globally true in a model $\mathfrak{M}$ (notation $\mathfrak{M} \Vdash \phi$) if it is satisfied at all points in $\mathfrak{M}$, that is if $\mathfrak{M}, w \Vdash \phi$, for all $w \in W$. A formula is said to be satisfiable in $\mathfrak{M}$ if there exists a state in $\mathfrak{M}$ at which ϕ is true.

If ϕ is not satisfied at w in $\mathfrak{M}$ we write $\mathfrak{M}, w \nVdash \phi$, and say that ϕ is false or refuted at w. When $\mathfrak{M}$ is clear from the context we simply write $w \Vdash \phi$ for $\mathfrak{M}, w \Vdash \phi$, and $w \nVdash \phi$ for $\mathfrak{M}, w \nVdash \phi$. We say that a set of formulas Σ is true at a state w of our model $\mathfrak{M}$, notation $\mathfrak{M}, w \Vdash \Sigma$, if all members of Σ are true at w.
Finally we extend our valuation V, so that for abitrary formulas ϕ, $V(\phi)$ allways denotes the set of states at which ϕ is true, that is,

$$V(\phi) := \{w \in W \mid \mathfrak{M}, w \Vdash \phi\} \tag{6}$$

Note that this means that for formulas ϕ, ψ

$$\begin{aligned} V(\neg\phi) &= \complement V(\phi) \qquad (7) \\ V(\phi \vee \psi) &= V(\phi) \cup V(\psi) \qquad (8) \end{aligned}$$

where $\complement$ means complement in W.

The basic modal language

The above is the general setup. But from now on we only look at the special case where we only have one unary modal operator. This we denote with by a diamond $\Diamond$, and the dual operator by a box: $\Box\phi := \neg\Diamond\neg\phi$.
A frame for the basic modal modal language is then just a pair $\mathfrak{F} = (W, R)$ where W is a non-empty set and R is a binary relation on W. The requirements regarding satisfaction in Definition 6 (5) then spells out that

$$\mathfrak{M}, w \Vdash \Diamond\phi \quad \text{iff} \quad \text{for some } v \in W \text{ with } Rwv \text{ we have } \mathfrak{M}, v \Vdash \phi \tag{9}$$

To understand the meaning of $\Box$ we consider

$$\mathfrak{M}, w \Vdash \neg\Diamond\neg\phi$$

By (3) and (9) this holds if and only if it is not the case that there is a $v \in W$ such that Rwv and $\mathfrak{M}, v \Vdash \neg\phi$. In other words,

$$\mathfrak{M}, w \Vdash \Box\phi \quad \text{iff} \quad \text{for all } v \in W \text{ with } Rwv \text{ we have } \mathfrak{M}, v \Vdash \phi$$

We see that our definition of a formula being satisfied is intrinsically local.
The satisfaction of a formula ϕ not containing modal operators depend solely upon whether the image of ϕ under the valuation V contains the current state. However, note that since $\Diamond$ is essentially an instruction to examine states related to the current state, the satisfaction of formulas containing $\Diamond$ may depend upon the satisfaction of certain formulas in states related to the current state.

As noted earlier we often refer to states w in a frame $\mathfrak{F} = (W, R)$, as worlds. This notion is derived from the reading of $\Diamond\phi$

as 'it is possibly the case that ϕ' (i.e $\Box\phi$ means 'it is not the case that not ϕ is possible', that is 'necessarily ϕ').
Given this reading of $\Diamond$ the concepts of frames, models and satisfaction, is an attempt to formalize the conception that nessecity means truth in all possible worlds, and that possibility means truth in some possible world.

Validity

So far we have defined the notion of satisfiability via the concept of models. As we saw models are composite entities consisting of a frame and a valuation, where the valuation so to speak is a carrier of contingent information in our model. We often want to ignore the effects of particular valuations and work on the more fundamental level of frames only. The following definition lets us do this.

Definition 8 A formula ϕ is valid at a state w in a frame $\mathfrak{F}$ (notation: $\mathfrak{F}, w \Vdash \phi$) if ϕ is true at w in every model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$. ϕ is valid in a frame $\mathfrak{F}$ (notation; $\mathfrak{F} \Vdash \phi$) if it is valid at every state in $\mathfrak{F}$. A formula ϕ is valid on a class of frames $\mathbf{F}$(notation; $\mathbf{F} \Vdash \phi$) if it is valid on every frame in $\mathfrak{F}$, and it is valid (notation; $\Vdash \phi$) if it is valid on the class of all frames. The set of formulas that are valid in a class of frames $\mathbf{F}$ is called the logic of $\mathbf{F}$ (notation $\Lambda_{\mathbf{F}}$).

We see that the notion of validity abstracts away from the effects of particular interpretations. It should be clear that validity differs from truth in many ways. For example for a formula $\phi \vee \psi$ to be true at a state w means by definition 6 that either ϕ or ψ is true at w. But if $\phi \vee \psi$ is valid on a frame $\mathfrak{F}$, this does not mean that either ϕ or ψ is valid on $\mathfrak{F}$. Consider for example the formula $p \vee \neg p$ which certainly is valid on any frame. However, it is not the case that either p is true in all states or $\neg p$ is true in all states. **Examples**

(i) The formula $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ is valid on the class of all frames, that is it is valid ($\Vdash \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$). To see this, take any frame $\mathfrak{F}$ and state w in $\mathfrak{F}$, and let V be a valuation on $\mathfrak{F}$. We have to show that if $(\mathfrak{F}, V), w \Vdash \Diamond(p \vee q)$, then $(\mathfrak{F}, V), w \Vdash (\Diamond p \vee \Diamond q)$. So assume that $(\mathfrak{F}, V), w \Vdash \Diamond(p \vee q)$. Then by definition 6(5) there is a state v s.t Rwv and $(\mathfrak{F}, V), v \Vdash p \vee q$. But if $v \Vdash p \vee q$ then either $v \Vdash p$ or $v \Vdash q$. Hence either we have $w \Vdash \Diamond p$ or we have $w \Vdash \Diamond q$. Either way $w \Vdash \Diamond p \vee \Diamond q$.

(ii) The formula $\Diamond\Diamond p \to \Diamond p$ is not valid on all frames. To see this let $\mathfrak{F}$ be a three-point frame with universe $\{0, 1, 2\}$ and relation $\{(0, 1), (1, 2)\}$. Let V be any valuation on $\mathfrak{F}$ s.t $V(p) = \{2\}$. Then $(\mathfrak{F}, V), 0 \Vdash \Diamond\Diamond p$, but $(\mathfrak{F}, V), 0 \nVdash \Diamond p$ since 0 is not related to 2.

(iii) We have just seen that the formula $\Diamond\Diamond p \to \Diamond p$ is not valid on the class of all frames. However, it *is* valid if we restrict our attention to the class of transitive frames. To see this take any transitive frame $\mathfrak{F}$ and a state w in $\mathfrak{F}$, and let V be a valuation on $\mathfrak{F}$. We have to show that if $(\mathfrak{F}, V), w \Vdash \Diamond\Diamond p$, then $(\mathfrak{F}, V), w \Vdash \Diamond p$. So assume that $(\mathfrak{F}, V), w \Vdash \Diamond\Diamond p$, then there are states u and v s.t Rwu and Ruv and $(\mathfrak{F}, V), v \Vdash p$. Now, since R is transitive we have that Rwv, hence by definition 6 $(\mathfrak{F}, V), w \Vdash \Diamond p$.

In the same fashion one can easily verify the following:

(iv) $\Box(p \to q) \to (\Box p \to \Box q)$ is valid on all frames.

(v) $\Diamond p \leftrightarrow \neg\Box\neg p$ is valid on the class of all frames.

(vi) All Tautologies are valid.

We have not yet discussed what it means for a modal formula to be a logical consequence of a set of other modal formulas.

Definition 9 (Local semantic consequence) Let τ be a similarity type, and let $\mathbf{S}$ be a class of structures of type τ. Let Σ and ϕ be a set of formulas and a single formula from a language of type τ. We say that ϕ is a local semantic consequence of Σ over $\mathbf{S}$(notation: $\Sigma \Vdash_{\mathbf{S}} \phi$) if for all models $\mathfrak{M}$ from $\mathbf{S}$, and all points w in $\mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Sigma$ then $\mathfrak{M}, w \Vdash \phi$.

Note that by the above definition our inferences will depend upon the class of structures we are working with. Thus, f.ex. an inference may be valid on the class of transitive frames, yet not valid on the class all frames.

Syntactic specification of modal logics

Until now we have only considered the semantic side of modal logics. Now we will develop the syntactic approach and then show how this is linked with the semantics.

Modal logics

From a syntactic viewpoint, modal logics are roughly speaking just obvious extensions of propositional logic taking modal languages into consideration. They are sets of modal formulas subject to certain closure rules. Formally:

Definition 10 (Modal logic) A modal logic Λ is a set of modal formulas that contains all propositional tautologies and is closed under the following rules

- if $\phi \in \Lambda$ and $\phi \to \psi \in \Lambda$, then $\psi \in \Lambda$ ('modus ponens')
- uniform substitution (cf. Definition 3)

If $\phi \in \Lambda$ we say that ϕ is a theorem of Λ and write $\vdash_\Lambda \phi$. If $\phi \notin \Lambda$ we write $\nvdash_\Lambda \phi$.

If the above definition is to make any sense we will expect it to imply that a modal logic contains $\psi_1 \wedge \psi_2$ whenever it contains ψ_1 and ψ_2. Fortunately, this is indeed the case: A modal logic is closed under forming logical conjunctions and disjunctions in a natural way. To see this, assume that $\vdash_\Lambda \psi_1$ and $\vdash_\Lambda \psi_2$. Since $\psi_1 \to (\psi_2 \to (\psi_1 \wedge \psi_2))$ is a tautology we have $\vdash_\Lambda \psi_1 \to (\psi_2 \to (\psi_1 \wedge \psi_2))$. Hence by modus ponens $\vdash_\Lambda \psi_2 \to (\psi_1 \wedge \psi_2)$ and thus, again by modus ponens, $\vdash_\Lambda \psi_1 \wedge \psi_2$. Likewise, whenever $\vdash_\Lambda \psi$ and φ is some modal formula, we have $\vdash_\Lambda \psi \vee \varphi$ by modus ponens since $\psi \to (\psi \vee \varphi)$ is a tautology.
A trivial example of a modal logic is the set of all modal formulas. One should also note, that the fact that modal logics are defined in terms of closures, implies that for any family $(\Lambda_i)_{i\in I}$ of modal logics, $\bigcap_{i\in I} \Lambda_i$ is a modal logic.
A special class of the modal logics is the *normal* modal logics. These are modal logics satisfying some additional requirements.

Definition 11 (Normal modal logic) A modal logic Λ is a normal modal logic if it contains the formulas

- *(K)* $\Box(p \to q) \to (\Box p \to \Box q)$
- *(Dual)* $\Diamond p \leftrightarrow \neg\Box\neg p$

and is closed under

- if $\vdash_\Lambda \phi$ then $\vdash_\Lambda \Box\phi$ ('generalization')

In the above definition, if no other modal formulas than (K) and (Dual) are included as axiomatic formulas, we call the normal modal logic **K**.
When we want to specify a normal modal logic we do it by stating the formulas that generate it, that is, we add modal formulas to **K** as axioms. For example, if we add the formula

- (4) $\Diamond\Diamond p \to \Diamond p$

as an axiom to **K** we get the normal modal logic denoted **K4**.
The following axioms are commonly known by the name given in their corresponding parentheses:

- (4) $\Diamond\Diamond p \to \Diamond p$
- (T) $p \to \Diamond p$
- (B) $p \to \Box\Diamond p$
- (D) $\Box p \to \Diamond p$
- (.3) $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$
- (L) $\Box(\Box p \to p) \to \Box p$

In general, these logics will be named by appending the axiom names to the logic name **K**. Thus, the logic **K4B** is the logic **K** extended by adding the axioms (4) and (B).
As we will later see, many of these normal modal logics nicely connects with the frame based semantics we have already developed. This connection is captured by the concept of *soundness* and its counterpart *completeness*. However, before delving further into this, we need to consider the notion of consistency.

Consistency

An inherent feature of the semantics of our modal languages is that it is impossible for contradictions to occur, that is, for a formula φ to both be satisfied and not be satisfied at a given world. However, from a syntactic point of view there is a priori no guarantee that a modal logic will not contain both a formula ϕ and its negation. We will now investigates this issue.

Definition 12 Let $\psi_1, \psi_2, \ldots, \psi_n, \varphi$ be modal formulas. We say that φ *is deducible in propositional calculus from the assumptions* $\psi_1, \ldots, \psi_n$ if $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$ is a tautology.

Lemma 13 *All modal logics are closed under deduction in propositional calculus.*

Proof. We must show that if $\vdash_\Lambda \psi_1, \ldots, \vdash_\Lambda \psi_n$ then $\vdash_\Lambda \varphi$ whenever φ is deducible in propositional logic from assumptions $\psi_1, \ldots, \psi_n$. Thus assume that $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$ is a tautology, so that $\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$, and that $\vdash_\Lambda \psi_1, \ldots, \vdash_\Lambda \psi_n$. Then $\vdash_\Lambda \psi_1 \wedge \cdots \wedge \psi_n$ and by modus ponens $\vdash_\Lambda \varphi$. □

Definition 14 (Deducibility) Let Λ be a modal logic. If $\Gamma \cup \{\phi\}$ is a set of formulas then ϕ is said to be deducible in Λ from Γ if $\vdash_\Lambda \phi$ or there are formulas $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$\vdash_\Lambda (\gamma_1 \wedge \cdots \wedge \gamma_n) \to \phi$$

If this is the case we write $\Gamma \vdash_\Lambda \phi$ and if it is not the case we denote this by $\Gamma \not\vdash_\Lambda \phi$.

These concepts enable us to express the notion of consistency:

Definition 15 (Consistency) For a modal logic Λ a set of formulas Γ is said to be Λ-consistent if $\Gamma \not\vdash_\Lambda \bot$ and correspondingly Λ-inconsistent if $\Gamma \vdash_\Lambda \bot$.

One can easily show the following

Lemma 16 *For a modal logic Λ and a set of formulas Γ the following are equivalent:*

1. *Γ is Λ-inconsistent*
2. *There exists a formula ϕ such that $\Gamma \vdash_\Lambda \phi \wedge \neg\phi$*
3. *For all ψ, $\Gamma \vdash_\Lambda \psi$*

Lemma 17 *Let Γ be a set of formulas. For a formula φ the following holds:*

$$\Gamma \vdash_\Lambda \varphi \text{ if and only if } \Gamma \cup \{\neg\varphi\} \text{ is inconsistent.}$$

Proof. If Γ is inconsistent then the result is obvious, since $\Gamma \cup \{\neg\varphi\}$ is certainly inconsistent and $\Gamma \vdash_\Lambda \varphi$ by Lemma 3.
Now assume that Γ is consistent.
Assume $\Gamma \vdash_\Lambda \varphi$. Then we must also have $\Gamma \cup \{\neg\varphi\} \vdash_\Lambda \varphi$. Now since obviously $\Gamma \cup \{\neg\varphi\} \vdash_\Lambda \neg\varphi$ we have $\Gamma \cup \{\neg\varphi\} \vdash_\Lambda \varphi \wedge \neg\varphi$, thus $\Gamma \cup \{\neg\varphi\}$ is inconsistent by Lemma 2.
We show the right-to-left direction by contraposition, so assume that $\Gamma \nvdash_\Lambda \varphi$, that is, $\nvdash_\Lambda \varphi$ and there are no γ_i's in Γ so that $\vdash_\Lambda \bigwedge_i \gamma_i \to \varphi$. If $\Gamma \cup \{\neg\varphi\}$ is inconsistent then $\Gamma \cup \{\neg\varphi\} \vdash_\Lambda \varphi$ by Lemma 16. Thus there are $\gamma_i \in \Gamma \cup \{\neg\varphi\}$ s.t. $\vdash_\Lambda \gamma_1 \wedge \cdots \wedge \gamma_r \to \varphi$. It follows that $\gamma_i = \neg\varphi$ for some i, say $\gamma_r = \neg\varphi$. Since the formula $((a \wedge \neg b) \to b) \to (a \to b)$ is a tautology we see that

$$\vdash_\Lambda \gamma_1 \wedge \cdots \wedge \gamma_{r-1} \to \varphi$$

contradicting the assumption that $\Gamma \nvdash_\Lambda \varphi$. Thus $\Gamma \cup \{\neg\varphi\}$ must be consistent. $\square$

Lemma 18 *Modus ponens preserves consistency.*

Proof. Assume that Γ is a Λ-consistent set of formulas and that ϕ, $\phi \to \psi \in \Gamma$. If $\Gamma \cup \{\psi\}$ is inconsistent, then by Lemma 17 $\Gamma \vdash_\Lambda \neg\psi$. But clearly $\Gamma \vdash_\Lambda \psi$ since ϕ, $\phi \to \psi \in \Gamma$, so $\Gamma \vdash_\Lambda \neg\psi \wedge \psi$, i.e. Γ is inconsistent by Lemma 16, a contradiction. $\square$

Soundness

The semantic and the syntactic sides of our modal logics are linked by the notions of *soundness* and its counterpart, *completeness*. We will introduce these concepts now.

Definition 19 (Soundness) Let S be a class of frames (or models). A normal modal logic Λ is sound with respect to S if $\Lambda \subseteq \Lambda_S$.

What this definition says is that Λ is sound with respect to S if for all formulas ϕ and all structures $\mathfrak{S} \in S$, $\vdash_\Lambda \phi$ implies $\mathfrak{S} \Vdash \phi$. In other words, if our logic is sound with respect to a class of structures S, we are assured that any syntactically produced formula is valid on that class.
To show that a given normal modal logic is sound with respect to a class of structures is quite straightforward. The only thing we need to prove is that our inference rules preserve validity over the given class, and that our axiomatic modal formulas are valid on this class.
What we are actually going to show is that our inference rules preserve validity over any class of frames. Notice then that to prove that a given normal modal logic is sound with respect to a given class of frames it suffices to show that the added axiomatic formulas are valid on that specific class.

Modus Ponens To see that modus ponens preserves validity on any class of frames, let $\mathfrak{F}$ be an arbitrary frame, let w be a state in $\mathfrak{F}$ and let V be a valuation on $\mathfrak{F}$. We must show that if

$$(\mathfrak{F}, V), w \Vdash \varphi \qquad \text{and} \qquad (\mathfrak{F}, V), w \Vdash \varphi \to \psi$$

then $(\mathfrak{F}, V), w \Vdash \psi$. But this easily follows since $\varphi \to \psi$ is just short-hand for $\neg\varphi \lor \psi$ and if $(\mathfrak{F}, V), w \Vdash \varphi$ then $(\mathfrak{F}, V), w \nVdash \neg\varphi$. Now it follows from definition 6 (4) that $(\mathfrak{F}, V), w \Vdash \psi$.

Uniform substitution Assume that we are given a formula φ that is valid on some class S of frames, that is, a formula such that $(\mathfrak{F}, V), w \Vdash \varphi$ for all frames $\mathfrak{F} = (W, R) \in S$, all valuations V and all states $w \in W$. By (6) on page 23 this means that $V(\varphi) = W$ for all valuations on all frames $\mathfrak{F} = (W, R) \in S$.

Consider a substitution instance ψ of φ. We may assume that ψ is obtained from φ by applying the induced substitution map $(\cdot)^\sigma : \mathrm{Form}(\Phi) \to \mathrm{Form}(\Phi)$. Now given any valuation V for a frame $\mathfrak{F}$, we have that $V(\psi) = V(\sigma(\varphi)) = (V\sigma)(\varphi) = W$, since $V\sigma : \mathrm{Form}(\Phi) \to \mathrm{Form}(\Phi)$ is a valuation. Thus $\mathfrak{F}, w \Vdash \psi$ for all w.

Generalization Validity is obviously preserved by generalization, since if φ is satisfied at every w in every model based on a given frame then of course φ is also satisfied in every state accessible from a given state in that model.

Now that we have seen that our inference rules preserve validity over the class of all frames Example (iv) and Example (v) give us, that the normal modal logic **K** is sound with respect to the class of all frames. From Example (iii) we get that **K4** is sound with respect to the class of all transitive frames.

Completeness

When considering soundness we were dealing with the question of whether formulas produced in a logic were also valid on a set of structures. Now we consider the reverse situation: for a given set of structures S, are all valid formulas on S also producible in some specific logic Λ? This may also be expressed as a question of whether $\Lambda \supseteq \Lambda_S$. This is formalized in the following definition:

Definition 20 (Completeness) Let S be a class of frames (or models). A logic Λ is strongly complete with respect to S if for any

set of formulas $\Gamma \cup \{\phi\}$, if $\Gamma \Vdash_S \phi$ then $\Gamma \vdash_\Lambda \phi$.
A logic Λ is weakly complete with respect to S if for any formula ϕ, if $\Vdash_S \phi$ then $\vdash_\Lambda \phi$.

Thus, a logic Λ is weakly complete with respect to S if $\Lambda \supseteq \Lambda_S$.

Theorem 21 *A logic Λ is strongly complete with respect to a class of structures S iff every Λ-consistent set of formulas is satisfiable on some $\mathfrak{S} \in S$. Λ is weakly complete with respect to a class of structures S iff every Λ-consistent formula is satisfiable on some $\mathfrak{S} \in S$.*

Proof. The proof is by contraposition in both directions.
Assume that Λ is not strongly complete, that is, there is some set of formulas $\Gamma \cup \{\phi\}$ such that $\Gamma \Vdash_S \phi$ and $\Gamma \not\vdash_\Lambda \phi$. Thus Γ must be Λ-consistent by Lemma 16. Then, what is more, $\Gamma \cup \{\neg\phi\}$ must be Λ-consistent by Lemma 17. However, $\Gamma \cup \{\neg\phi\}$ is certainly not satisfiable on any $\mathfrak{S} \in S$.
Assume on the other hand that there is some Λ-consistent set Γ not satisfiable on any $\mathfrak{S} \in S$. Thus $\Gamma \Vdash_S \bot$. But since Γ is Λ-consistent, it is not the case that $\Gamma \vdash_\Lambda \bot$. Hence, Λ is not strongly complete with respect to S.
The result concerning weak completeness follows from the result for strong completeness. □

Definition 22 (Λ-MCSs) A set of formulas Γ is maximal Λ-consistent if Γ is Λ-consistent and any set of formulas properly containing Γ is Λ-inconsistent. If Γ is maximal Λ-consistent we say that it is Λ-MCS.

It is not difficult to show the following

Theorem 23 *If Λ is a logic the following holds for any Λ-MCS of formulas Γ:*

1. *Γ is closed under modus ponens.*
2. *$\Lambda \subseteq \Gamma$.*
3. *for all formulas ϕ: $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.*
4. *if $\phi, \psi \in \Gamma$ then $\phi \wedge \psi \in \Gamma$.*
5. *for all formulas ϕ, ψ: $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.*

By a pretty standard application of Zorn's lemma we get

Lemma 24 (Lindenbaum's Lemma) *If Σ is a Λ-consistent set of formulas then there is a Λ-MCS such that $\Sigma \subseteq \Sigma^+$.*

It should be noted that if the collection of propositional letters is assumed to be enumerable one may prove the lemma without having to resort to Zorn's Lemma.

Canonical models

It follows from theorem 21 that completeness theorems are essentially model existence theorems. Given a normal modal logic Λ we prove its strong completeness with respect to some structure by showing that every Λ-consistent set of formulas can be satisfied in some suitable model. What we are going to do now is to build models out of maximal consistent sets and we will see that these models do the job.

Definition 25 Let a normal modal logic Λ be given. The canonical model $\mathfrak{M}^\Lambda$ for Λ is the triple $(W^\Lambda, R^\Lambda, V^\Lambda)$ where

(i) W^Λ is the set of all Λ-MCSs.

(ii) R^Λ is the binary relation on W^Λ defined by $R^\Lambda wu$ if for all formulas ψ, $\psi \in u$ implies $\Diamond\psi \in w$. R^Λ is called the canonical relation.

(iii) $V^\Lambda : \Phi \to W^\Lambda$ is given by $V^\Lambda(p) = \{w \in W^\Lambda \mid p \in w\}$. V^Λ is called the canonical valuation.

The pair $\mathfrak{F}^\Lambda = (W^\Lambda, R^\Lambda)$ is called the canonical frame for Λ.

Lemma 26 *For any normal logic Λ, $R^\Lambda wv$ if and only if for all formulas ψ, $\Box\psi \in w$ implies $\psi \in v$.*

Proof. Assume that $R^\Lambda wv$. We prove the implication by contraposition, so we assume that $\psi \notin v$. Since v is a Λ-MCS, by Theorem 23 $\neg\psi \in v$. Since $R^\Lambda wv$ by assumption, $\Diamond\neg\psi \in w$. Hence, again by Theorem 23, $\neg\Diamond\neg\psi \notin w$, that is, $\Box\psi \notin w$ as desired.

On the other hand, assume that $\Box\psi \in w$ implies $\psi \in v$. Now, if $\psi \in v$ then $\neg\psi \notin v$ since v is a Λ-MCS. Thus $\Box\neg\psi \notin w$ and we see that $\neg\Box\neg\psi \in w$, because w is also a Λ-MCS. Hence, $\Diamond\psi \in w$, so $R^\Lambda wv$. $\Box$

For proving the crucial Existence Lemma we will need the following

Sublemma 27 The formula

$$(\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \to \Box(\psi_1 \wedge \cdots \wedge \psi_n)$$

is a theorem of every normal modal logic.

Lemma 28 (Existence Lemma) *For any normal modal logic Λ and any state $w \in W^\Lambda$, if $\Diamond\varphi \in w$ then there is a state $v \in W^\Lambda$ such that $R^\Lambda wv$ and $\varphi \in v$.*

Proof. Suppose $\Diamond\varphi \in w$. Let $A = \{\psi \mid \Box\psi \in w\}$ and v^- be $\{\varphi\} \cup A$. To see that v^- is consistent we assume the opposite and aim for a contradiction. Hence, assume that v^- is inconsistent.

Then there are $\psi_1, \ldots, \psi_n \in A$ such that $\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \neg\varphi$. By generalization $\vdash_\Lambda \Box((\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \neg\varphi)$, hence $\vdash_\Lambda \Box(\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \Box\neg\varphi$ by the K-axiom, uniform substitution and modus ponens. Since $(\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \rightarrow \Box(\psi_1 \wedge \cdots \wedge \psi_n)$ is a theorem of any normal modal logic, as we saw in Lemma 27, we obtain $\vdash_\Lambda (\Box\psi_1 \wedge \cdots \wedge \Box\psi_n) \rightarrow \Box\neg\varphi$ by a familiar deduction rule from propositional calculus. Now, since $\Box\psi_i \in w$ for $1 \leq i \leq n$ and w is a Λ-MCS, $\Box\psi_1 \wedge \cdots \wedge \Box\psi_n \in w$, so by modus ponens $\Box\neg\varphi \in w$. By the Dual-axiom this translates into $\neg\Diamond\varphi \in w$, but since w is a Λ-MCS containing $\Diamond\varphi$, this is impossible by Theorem 23. Thus, v^- is in fact consistent. Now, by Lindenbaum's Lemma, v^- can be extended to some Λ-MCS v. Thus $v \in W^\Lambda$ and by construction of v we have $\varphi \in v$ and that for all formulas ψ, if $\Box\psi \in w$ then $\psi \in v$. That $R^\Lambda wv$ now follows from Lemma 26. $\Box$

Lemma 29 (Truth Lemma) *For any normal modal logic Λ and any formula φ, $\mathfrak{M}^\Lambda, w \Vdash \varphi$ if and only if $\varphi \in w$.*

Proof. The proof is by structural induction on φ. If φ consists solely of a propositional letter the result follows from the definition of V^Λ. If φ is of the form $\psi_1 \vee \psi_2$ then by Theorem 23 $\varphi \in w$ if and only if $\psi_1 \in w$ or $\psi_2 \in w$. This is the case if and only if $w \in V^\Lambda(\psi_1)$ or $w \in V^\Lambda(\psi_2)$, that is, if and only if $\mathfrak{M}^\Lambda, w \Vdash \varphi$. The case where φ is of the form $\neg\psi$ follows similarly. What remains to be considered are the modal cases. We have

$$\begin{array}{lll} \mathfrak{M}^\Lambda, w \Vdash \Diamond\psi & \text{if and only if} & \exists v(R^\Lambda wv \wedge \mathfrak{M}^\Lambda, v \Vdash \psi) \\ & \text{if and only if} & \exists v(R^\Lambda wv \wedge \phi \in v) \\ & \quad\text{only if} & \Diamond\psi \in w \end{array}$$

where the second biimplication follows from the induction hypothesis and the third is by the definition of R^Λ. Thus the left-to-right part of the argument for the modal cases has been taken care of. Now suppose $\Diamond\psi \in w$. By the Existence Lemma there is a state $v \in W^\Lambda$ such that $R^\Lambda wv \wedge \psi \in v$. Hence, the 'only if' of the above argument actually becomes 'if and only if' and the desired result follows. $\square$

Now we are ready for the result that justifies all of the above work on canonical models:

Theorem 30 (Canonical Model Theorem) *A normal logic is strongly complete with respect to its canonical model.*

Proof. Suppose that Σ is a Λ-consistent set of formulas of the normal modal logic Λ. By Lindenbaum's Lemma there is a Λ-MCS Σ^+ extending Σ. Since any formula of Σ belongs to Σ^+, by the Truth Lemma $\mathfrak{M}^\Lambda, \Sigma^+ \Vdash \Sigma$. The result now follows from Theorem 21. $\square$

With the above theorem we are now able to show completeness for several normal modal logics. We give some few examples.

Theorem 31 ***K*** *is strongly complete with respect to the class of all frames.*

Proof. According to Theorem 21 it suffices to show, that for any **K**-consistent set of formulas Γ, there exists a model in which Γ is satisfied. We achieve this by letting $\mathfrak{M}$ be the canonical model for **K**, that is, $\mathfrak{M} = (\mathfrak{F}^K, V^K)$. By the previous lemma Γ will be satisfied at any MCS extendings Γ. $\square$

Theorem 32 *The logic $K4$ is strongly complete with respect to the class of all transitive frames.*

Proof. Given a $K4$-consistent set of formulas Γ, it suffices by Theorem 21 to find a model $(\mathfrak{F}, V)$ such that the frame $\mathfrak{F}$ is transitive and there is a state w in the model such that $(\mathfrak{F}, V), w \Vdash \Gamma$.
Consider the canonical model (W^{K4}, R^{K4}, V^{K4}) for $K4$. By Lindenbaum's Lemma there is some K4-MCS Γ^+ extending Γ. By the Truth Lemma we have $(W^{K4}, R^{K4}, V^{K4}), \Gamma^+ \Vdash \Gamma$. Thus it remains to be seen that (W^{K4}, R^{K4}, V^{K4}) is transitive.
Suppose u, v, w are states in (W^{K4}, R^{K4}, V^{K4}) so that $R^{K4}uv$ and $R^{K4}vw$. If $\varphi \in w$ Since $R^{K4}vw$ we have $\Diamond\varphi \in v$. Thus, since $R^{K4}uv$, $\Diamond\Diamond\varphi \in u$. But u is a MCS, hence contains $\Diamond\Diamond\varphi \to \Diamond\varphi$. By modus ponens it contains $\Diamond\varphi$ whence $R^{K4}uw$. □

In the same manner one can easily show that **KT** and **KB** are strongly complete with respect to the classes of reflexive and of symmetric frames, respectively.
And then furthermore from this we get that **KT4** is strongly complete with respect to the class of reflexive transitive frames, and **KT4B** is strongly complete with respect to the class of frames whose relation is an equivalence relation.

(The article is based on chapter 1 and 4 in the book "Modal Logic" by Patrick Blackburn, Maarten de Rijke and Yde Venema, Cambridge University Press.)