General Question

Given \( m \geq 1 \) and \( vN\text{alg } M_1, \ldots, M_m \), does the tensor product \( M_1 \otimes M_2 \otimes \ldots \otimes M_m \) retain the integer \( m \) and each \( M_i \) up to (stable) isom. after permuting indices?

\[ N : \text{(factor)} \text{ is prime if whenever } N = N_1 \otimes N_2 \text{ then } \exists i = 1, 2 \text{ s.t. } N_i \text{ is of type I} \]

Ge : \( L(\mathbb{F}_p) \) [earlier \( \text{Popa } : L(\mathbb{F}_p) \text{ is unctble.} \)] is prime.

Ozawa - use of Akemann - Ostrand property

Class \( C^{(ao)} \) (\( M_k \) : separable)

Def \( M \) satisfies strong condition \( (AO) \) \( M \in IB(H) \) (Standard form)

\( \text{if there exist } C^* \text{-algs. } A \subseteq M \)
\( \cap \)
\( C \subseteq IB(H) \)

\( \text{s.t.} \)
\( \cdot \) \( A \text{ is exact and weakly dense in } M \)
\( \cdot \) \( C \text{ is nuclear} \)
\( \cdot \) \( [C, JAJ] \subseteq I(K(H)) \)

\( \nu : A \otimes_{\text{alg}} JAJ \to \frac{IB(H)}{I(K(H))} \) is min - continuous.

Examples

1) Amenable \( (AFD) \) \( vN\text{algs.} \)

2) \( L(\Gamma) \) \( \Gamma : \text{free groups or non-elementary hyperbolic groups} \)
\( \text{discrete subgroups of simple connected Lie groups} \)
\( \text{of rank 1} \)
\( \Gamma = SL_2(\mathbb{Z}) \times \mathbb{Z}^2 \) (not weakly amenable example)
2) Free quantum groups: $A_u(F)$ or $A_0(F)$.

3) Free Araki-Woods factors. (Shlyakhtenko)

$U: \mathbb{R} \to O(\mathbb{H}_{\mathbb{R}})$

real Hilbert sp. dim $\mathbb{H}_{\mathbb{R}}$ can be finite

free Gaussian functor

$M = \Gamma((U_t)_{t \in \mathbb{R}}, \mathbb{H}_{\mathbb{R}})^{\text{vN alg.}}$

$M$ is a full factor. as soon as dim $\mathbb{H}_{\mathbb{R}} \geq 2$

\[
\begin{cases}
M \text{ is } \text{ II}_1 \iff L(\mathbb{H}_{\text{dim R}}) \iff U \text{ is trivial} \\
M \text{ is } \text{ III}_\lambda \text{ factor } \iff U_t \text{ is } \frac{2\pi}{\log \lambda} \text{ - periodic.} \\
M \text{ is } \text{ III}_1 \text{ factor otherwise.}
\end{cases}
\]

$U = \text{ weakly mix part } \oplus \text{ fin-dim part}$

$\Rightarrow$ no almost periodic state on $M$

$\left( M = \text{ II}_\infty \times \Gamma \right)$

(discrete abelian)

$ECA_0 = \text{ Smallest class of } \text{vN algs} \text{ which contains }$

$vN \text{ algs w/ strong condition (A0) and closed under taking subalgebras with normal faithful conditional expectations.}$

$N \subseteq M$ is with expectation if $\exists$ faithful normal conditional expectation $E:M \to N$.

Thm (Ozawa) (Any nonamenable factor in $ECA_0$ is prime.)
3.

Main Classification Theorem (H - Isono)

Let $m, n \geq 1$. $M_1, \ldots, M_m, N_1, \ldots, N_n$ in the class $\mathcal{C}_{\text{A0}}$.

$M_0, N_0$: AFD factors, (possibly trivial.)

Then $M_0 \overline{\otimes} M_1 \overline{\otimes} M_2 \overline{\otimes} \ldots \overline{\otimes} M_m \cong N_0 \overline{\otimes} N_1 \overline{\otimes} \ldots \overline{\otimes} N_n$

$\iff m = n, \quad M_0 \cong N_0, \quad \exists \sigma \in S_n$

Stably

$M_{\sigma(j)} \cong N_j \quad \forall j \in \{1, \ldots, n\}$

Obata - Popa (03) UPF for $L(C)$ as in 1).

Isono (14) UPF for $L^0(G)$ as in 2)

(1) Haar state $\exists$ discrete decomposition

(2) work in semifinite alg.

New features:

1) We do not appeal to Connes - Takesaki decomposition.

(II$_{\infty} \times \Gamma$)

2) New intertwining theorem $A \succeq M B$

$B \subseteq M$ any VN alg w/ expectation (can be type III)

$A \subseteq M$ any finite VN alg. w/ expectation.

Def: $M$ has a state with large centralizer if $\exists \varphi \in M_+$

faithful state s.t. $(M^\varphi)' \cap M = C_1$.

$M^\varphi = \{x \in M \mid \sigma^\varphi(x) = x, \forall \sigma \in \mathcal{S} \}$

$= \{x \in M \mid x \varphi = \varphi x \}$

$\varphi x (y) = \varphi (xy)$

$M^\varphi$ is finite VN alg.

Ex

0) If $M$: II$_1$, $\varphi = \mathbb{I}$ (II$_{\infty}$: no such $\varphi$.)

1) $M$ type III when $M$: III$_0$, then $M$ has no state w/ large centralizer. $\mathcal{Z}(M^\varphi)$ (Connes thesis) diffuse.
- \( M : \mathbb{II}_1 \), \( 0 < \lambda < 1 \). Then \( M \) has a state \( \varphi \) with large centralizer. \( \varphi = \frac{2\pi}{\log \lambda} \), periodic state.

- \( M \) is \( \mathbb{II}_1 \), then?

**Theorem (Haagerup)** \( M : \mathbb{II}_1 \). TF=AE

1) \( M \) has a state with large centralizer

2) \( M \) has a trivial bicentralizer. \( B(M, \varphi) = C1 \).

\[
B(M, \varphi) = \left\{ a \in M \mid a x_n - x_n a \xrightarrow{n \to \infty} 0 \text{ strongly} \right\}
\]

for any \( (x_n)_{n=1}^\infty \in AC(M, \varphi) \)

\[
AC(M, \varphi) = \left\{ (x_n)_{n=1}^\infty \in C(M) \mid \| x_n \varphi - \varphi x_n \| \xrightarrow{n \to \infty} 0 \right\}
\]

Haagerup: \( B(M, \varphi) = C \), \( M \): injective

+

comes → a classification of injective factors.

\( ( \mathbb{II}_1 \text{ case} ) \)

**Theorem (H-Isomo)**

\( \text{Any nonamenable factor in } \mathcal{E}(AO) \text{ has a state with large centralizer.} \)

\( B(M, \varphi) \): either C or McDuff \( \mathbb{II}_1 \) factor. \( \blacksquare \)