Type III
Bernoulli crossed products

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Copenhagen, February 25, 2015

* Holds a Ph. D. fellowship of the Research Foundation – Flanders (FWO).
Crossed product von Neumann algebras

All operator algebras in this talk are von Neumann algebras.

Let $G$ be a locally compact group acting by automorphisms $(\alpha_g)_{g \in G}$ on a von Neumann algebra $P$.

The crossed product $M = P \rtimes G$ is the unique von Neumann algebra generated by $P$ and by a group of unitaries $(u_g)_{g \in G}$, such that

- $u_gxu_g^* = \alpha_g(x)$ for all $x \in P$, $g \in G$.

Generic question

Classify families of crossed products $P \rtimes G$ in terms of the group $G$ and the action $G \curvearrowright P$.
Let \((P, \phi)\) be a von Neumann algebra with normal faithful state, and \(\Lambda\) a countable infinite discrete group.

The Bernoulli action of \(\Lambda\) with base algebra \((P, \phi)\) is the action

\[
\Lambda \curvearrowright (P, \phi)^\Lambda = \bigotimes_{k \in \Lambda} (P, \phi)
\]

by shifting indices.

**Question**

Classify the Bernoulli crossed products \((P, \phi)^\Lambda \rtimes \Lambda\) in terms of building data.

- ‘Classical case’: \((P, \phi) = L^\infty(X_0, \mu_0)\). \(\rightsquigarrow\) type \(\text{II}_1\)
- Factor as base algebra: \((P, \phi)\) is a factor with:
  - \(\phi\) a trace, \(\rightsquigarrow\) type \(\text{II}_1\)
  - \(\phi\) a nontracial state, \(\rightsquigarrow\) type \(\text{III}_{\lambda \in (0,1]}\)

Type \(\text{III}\) Bernoulli crossed products
A factor $M$ is of type

- $\text{I}$ if $M = B(H)$ for some $H$,
- $\text{II}_1$ if there exists a finite trace $\text{tr} : M \to \mathbb{C}$, and $M \neq B(H)$,
- $\text{II}_\infty$ if there exists a semifinite trace $\text{Tr} : M^+ \to [0, \infty]$, and $\text{Tr}(1) = \infty$,
- $\text{III}$ if none of the above applies.

Type III factors can be further classified: For any normal faithful weight $\varphi$ on $M$, there is a modular action $\sigma^\varphi : \mathbb{R} \curvearrowright M$ such that

$$M \cong (M \rtimes_{\sigma^\varphi} \mathbb{R}) \rtimes \mathbb{R}_{>0}.$$  

Looking at the period of the action $\mathbb{R}_{>0} \curvearrowright \mathcal{Z}(M \rtimes_{\sigma^\varphi} \mathbb{R})$, we can divide III factors in subclasses $\text{III}_\lambda$ for $\lambda \in [0, 1]$. 

Type III Bernoulli crossed products
Question
Classify the Bernoulli crossed products \((P, \phi)^\Lambda \rtimes \Lambda\) in terms of building data.

Distinguishing between nonisomorphic Bernoulli crossed products

Popa, Vaes ’11, Gaboriau ’01
\(L^\infty(X_0, \mu_0)^{\mathbb{F}_n} \rtimes \mathbb{F}_n\) are nonisomorphic for different \(n\).

Popa, Vaes ’11
For \((P, \text{tr})\) finite amenable factor: \((P, \text{tr})^{\mathbb{F}_n} \rtimes \mathbb{F}_n\) are nonisomorphic for different \(n\).

Isomorphism results

Connes ’75, Haagerup ’85
For \((P, \phi)\) amenable factor, \(\Lambda\) amenable icc group:
\((P, \phi)^\Lambda \rtimes \Lambda\) are fully classified by the type \(\text{II}_1\) or \(\text{III}_{\lambda \in (0,1]}\).

Bowen ’09
For fixed \(n\):
\(\mathbb{F}_n \sim L^\infty(X_0, \mu_0)^{\mathbb{F}_n}, \mathbb{F}_n \sim L^\infty(Y_0, \nu_0)^{\mathbb{F}_n}\)
are orbit equivalent for all nontrivial \((X_0, \mu_0), (Y_0, \nu_0)\).
Main result

Theorem (Vaes – V, 2014)

The set of factors

$$\left\{ (P, \phi)^F_n \rtimes F_n \mid P \text{ is a nontrivial amenable factor with normal faithful almost periodic state } \phi, \text{ and } n \geq 2 \right\}$$

is exactly classified, up to isomorphism, by $n$ and the subgroup $\Gamma(\phi)$ of $\mathbb{R}_{>0}$ generated by the point spectrum of $\Delta_\phi$.

Example:

Put $P = M_k(\mathbb{C})$, $\phi = \text{Tr}(h \cdot)$ with $h = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_k \end{pmatrix}$, $\sum_i \mu_i = 1$.

Then $(M_k(\mathbb{C}), \text{Tr}(h \cdot))^F_n \rtimes F_n$ are completely classified by $n$ and $\Gamma(\phi) = \text{grp}(\mu_i/\mu_j)$.

Isomorphism and non-isomorphism result!

Type III Bernoulli crossed products
1. Almost periodic states
2. Isomorphism result
3. Sd-invariant and nonisomorphism result
4. Beyond free groups
1. Almost periodic states

2. Isomorphism result

3. Sd-invariant and nonisomorphism result

4. Beyond free groups

Type III Bernoulli crossed products
A normal faithful state $\varphi$ on a vNa $M$ is almost periodic if $\Delta_{\varphi}$ is diagonalizable, i.e. the set of eigenvectors has dense linear span in $L^2(M, \varphi)$.

$\Delta_{\varphi} = S^* S$ where $S$ is the closure of $M \to M : x \mapsto x^*$ in $L^2(M, \varphi)$.

**Example:** $P = M_k(\mathbb{C})$, $\phi = \text{Tr}(h \cdot)$ with $h = \text{diag}(\mu_1, \ldots, \mu_k)$, $\sum_i \mu_i = 1$.

Then $\Delta_{\phi}$ is given by $\Delta_{\phi}(e_{ij}) = \frac{\mu_i}{\mu_j} e_{ij}$.

**Example:** $M = \bigotimes_I (M_k(\mathbb{C}), \phi)$ an infinite tensor product, with $\varphi = \phi^I$.

Then $\Delta_{\varphi} = \cdots \otimes \Delta_{\phi} \otimes \Delta_{\phi} \otimes \Delta_{\phi} \otimes \cdots$, again diagonalizable.

Put $\Gamma(\phi)$ the subgroup of $\mathbb{R}_{>0}$ generated by the point spectrum of $\Delta_{\phi}$, i.e. $\text{grp}\{\mu_i/\mu_j\}$. Then $M$ is

- of type $\text{II}_1$ if $\Gamma(\phi) = \{1\}$,
- of type $\text{III}_\lambda$ if $\Gamma(\phi) = \lambda^\mathbb{Z}$,
- of type $\text{III}_1$ if $\Gamma(\phi)$ is dense in $\mathbb{R}_{>0}$.

Type III Bernoulli crossed products
Fix \((M, \varphi)\), with \(\varphi\) an almost periodic normal faithful state. Put\[
\Gamma(\varphi) = \text{grp}\{\text{point spectrum } \Delta\varphi\} \subset \mathbb{R}_{>0}.
\]

Let \(G = \hat{\Gamma}\) be the dual group, then there is a map \(\iota : \mathbb{R} \to G\) given by \(\langle \iota(t), \gamma \rangle = \gamma^it\).

\[
\begin{array}{c}
\Gamma \xrightarrow{\iota} \mathbb{R}_{>0} \\
\downarrow & \updownarrow \\
G & \xleftarrow{\iota} \mathbb{R}
\end{array}
\]

Theorem (Connes ’74)

There exists a unique continuous action \(\sigma : G \curvearrowright M\) such that

- \(\varphi \circ \sigma_g = \varphi\) for \(g \in G\),
- \(\sigma_{\iota(t)} = \sigma^t_{\varphi}\) for \(t \in \mathbb{R}\).

Conversely, if such an action exists, \(\varphi\) is necessarily almost periodic.
Let \((M, \varphi)\) be a factor with a normal faithful almost periodic state \(\varphi\). Put \(\Gamma = \text{grp}\{\text{point spectrum } \Delta \varphi\}\) and \(G = \widehat{\Gamma}\). Then

\[
(M \rtimes G) \rtimes \Gamma \cong M \otimes B(\ell^2\Gamma)
\]

\[
\text{Tr}_{M \rtimes G} \circ E_{M \rtimes G} \leftrightarrow \varphi \otimes \text{Tr}(M_\widehat{\cdot})
\]

where \(\text{Tr}(M_\widehat{x}) = \sum_{\gamma \in \Gamma} \gamma \langle x\delta_\gamma, \delta_\gamma \rangle\).

**Important:** There exists a conditional expectation

\[
(M \rtimes G) \rtimes \Gamma \to M \rtimes G,
\]

and \(B(\ell^2\Gamma)\) has natural minimal projections (in contrast to general decomposition \((M \rtimes \mathbb{R}) \rtimes \mathbb{R}_{>0} \cong M \otimes B(L^2\mathbb{R})\)).
Outline

1. Almost periodic states
2. Isomorphism result
3. Sd-invariant and nonisomorphism result
4. Beyond free groups

Type III Bernoulli crossed products
Isomorphism result: If $\Gamma(\phi) = \Gamma(\psi)$, then the crossed products $(P, \phi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n \cong (Q, \psi)^{\mathbb{F}_n} \rtimes \mathbb{F}_n$ are isomorphic.

Start from amenable factors $(P, \phi)$, $(Q, \psi)$ with n. f. almost periodic states, s. t. the point spectra of $\Delta_\phi$ and $\Delta_\psi$ generate the same subgroup $\Gamma$ of $\mathbb{R}_{>0}$.

Consider the actions

$$
\Gamma \times \mathbb{Z} \curvearrowright P^{\mathbb{Z}} \rtimes G, \quad \Gamma \times \mathbb{Z} \curvearrowright Q^{\mathbb{Z}} \rtimes G.
$$

By Ocneanu ’85, all outer actions of amenable groups on the hyperfinite $\text{II}_\infty$ factor are cocycle conjugate (if they scale the trace in the same way).

Thus the above actions are cocycle conjugate, i.e.

$$
\exists \Phi : P^{\mathbb{Z}} \rtimes G \rightarrow Q^{\mathbb{Z}} \rtimes G, \exists u_{\gamma,n} \in \mathcal{U}(Q^{\mathbb{Z}} \rtimes G) \text{ s.t. } \text{Tr} \circ \Phi = \text{Tr} \text{ and } \Phi \circ \alpha^{1}_{\gamma,n} = \text{Ad} u_{\gamma,n} \circ \alpha^{2}_{\gamma,n} \circ \Phi.
$$

Type III Bernoulli crossed products
Isomorphism result: If $\Gamma(\phi) = \Gamma(\psi)$, then the crossed products $(P, \phi)^{F_n} \rtimes F_n \cong (Q, \psi)^{F_n} \rtimes F_n$ are isomorphic.

Then also

\[ \mathbb{Z} \rtimes (P^\mathbb{Z} \rtimes G) \rtimes \Gamma, \quad \mathbb{Z} \rtimes (Q^\mathbb{Z} \rtimes G) \rtimes \Gamma \]

\[ \cong P^\mathbb{Z} \bar{\otimes} B(\ell^2\Gamma), \quad \cong Q^\mathbb{Z} \bar{\otimes} B(\ell^2\Gamma) \]

are cocycle conjugate in a weight preserving way.

Cutting with a canonical minimal projection in $B(\ell^2\Gamma)$, we get that the actions $\mathbb{Z} \rtimes P^\mathbb{Z}, \mathbb{Z} \rtimes Q^\mathbb{Z}$ are cocycle conjugate (state preserving).

A co-induction argument (Bowen) then gives a cocycle conjugation through state preserving isomorphism between

\[ \mathbb{Z} \ast F_{n-1} \rtimes (P, \phi)^{\mathbb{Z} \ast F_{n-1}} \quad \text{and} \quad \mathbb{Z} \ast F_{n-1} \rtimes (Q, \psi)^{\mathbb{Z} \ast F_{n-1}}. \]

It is essential that the cocycle conjugation is state preserving.

Argument works for all $\Lambda \ast \Sigma$ with $\Lambda$ infinite amenable.

Type III Bernoulli crossed products
Outline

1. Almost periodic states
2. Isomorphism result
3. $Sd$-invariant and nonisomorphism result
4. Beyond free groups

Type III Bernoulli crossed products
Let $M$ be a von Neumann algebra. Then

$$\text{Sd}(M) := \bigcap_{\psi \text{ a.p.}} \text{point spectrum } \Delta_\psi.$$ 

i.e. $\text{Sd}(M)$ is the dual of the “smallest compactification” of $\sigma^\varphi_{t \in \mathbb{R}}$.

**How to compute?**

A von Neumann algebra $M$ is a *full factor* iff

- $\text{Inn}(M) \subseteq \text{Aut}(M)$ is closed
- $M$ has no “central sequences”: $\forall \psi \in M_\times : \|\psi(x_n \cdot) - \psi(\cdot x_n)\| \to 0$

$$\Rightarrow x_n - z_n 1 \to 0 \ \ast\text{-strongly for } z_n \in \mathbb{C}.$$ 

If $M$ is a full factor, $\varphi$ is an almost periodic normal faithful weight on $M$, and $M_\varphi$ is a factor, then $\text{Sd}(M) = \text{point spectrum } \Delta_\varphi$. 

*Type III Bernoulli crossed products*
Suppose \( \Phi : (P, \phi)^F \ltimes F_n \rightarrow (Q, \psi)^F \ltimes F_m \) is an isomorphism.

- \((P, \phi)^F \ltimes F_n\) is a full factor, and the centraliser w.r.t. \(\phi^F\) is a factor, hence

\[
\text{Sd}((P, \phi)^F \ltimes F_n) = \Gamma(\phi)
\]

is the group generated by point spectrum of \(\Delta \phi\). \(\rightsquigarrow \Gamma(\phi) = \Gamma(\psi) = \Gamma\).

- The isomorphism \(\Phi\) extends to an isomorphism between the discrete cores

\[
\Phi : ((P, \phi)^F \ltimes F_n) \ltimes G \rightarrow ((Q, \psi)^F \ltimes F_m) \ltimes G,
\]

reducing the problem to the \(\text{II}_\infty\) setting.

- Popa’s deformation/rigidity in \(\text{II}_1\) setting (cut with \(p \in LG, \text{Tr}(p) < \infty\))

\(\rightsquigarrow \) yields that the groups \(F_n \cong F_m\) are isomorphic.
Outline

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Beyond free groups

Mutatis mutandis, we get the following result.

Theorem (Vaes – V, 2014)

Let \((P, \phi), (Q, \psi)\) be nontrivial amenable factors with n. f. almost periodic states, and let \(\Lambda_1, \Lambda_2\) be icc groups in `a large class of groups` \(\mathcal{C}\). Then \((P, \phi)^{\Lambda_1} \rtimes \Lambda_1 \cong (Q, \psi)^{\Lambda_2} \rtimes \Lambda_2\) are isomorphic iff

- \(\Lambda_1 \cong \Lambda_2\),
- the point spectra of \(\Delta_\phi\) and \(\Delta_\psi\) generate the same subgroup of \(\mathbb{R}_{>0}\),
- and the action \(\Lambda_1 \curvearrowright (P, \phi)^{\Lambda_1}\) is cocycle conjugate to a reduction of the action \(\Lambda_2 \curvearrowright (Q, \psi)^{\Lambda_2}\), in a `state preserving way`.

\(\mathcal{C}\) contains

- all weakly amenable groups \(\Lambda\) with \(\beta_1^{(2)}(\Lambda) \neq 0\) (using Popa – Vaes ’11),
- all weakly amenable, nonamenable, bi-exact groups (using Popa – Vaes ’12),
- all free products \(\Sigma_1 \ast \Sigma_2\) with \(|\Sigma_1| \geq 2, |\Sigma_2| \geq 3\) (using Ioana ’12),

and is closed under extensions and commensurability.
What do we get for Bernoulli crossed products \((P, \phi)^\Lambda \rtimes \Lambda\) of other groups, e.g. \(\Lambda = \mathbb{F}_n \times \mathbb{F}_m\)?

**Theorem (Popa – Vaes, Popa, Bowen, Ornstein)**

The factors

\[
\{ L^\infty(X_0, \mu_0)^{\mathbb{F}_n \times \mathbb{F}_m} \rtimes \mathbb{F}_n \times \mathbb{F}_m \mid (X_0, \mu_0) \text{ nontrivial and } n, m \geq 2 \}
\]

are exactly classified by \(\{n, m\}\) and the entropy \(H(\mu_0)\).

- By Popa – Vaes, classification amounts to classification of the actions \(\mathbb{F}_n \times \mathbb{F}_m \curvearrowright L^\infty(X_0, \mu_0)^{\mathbb{F}_n \times \mathbb{F}_m}\) up to orbit equivalence.
- By Popa’s cocycle superrigidity theorem, this amounts to classification up to conjugacy. \(\sim\) we retrieve \(\{n, m\}\).
- Using Bowen’s sofic entropy, we retrieve \(H(\mu_0)\).
- By Ornstein and co-induction, equal entropy gives conjugate actions.
Beyond free groups

Popa’s cocycle superrigidity theorem also applies in noncommutative case:

**Theorem (Vaes – V, 2014)**

The set of factors

\[
\{(P, \phi)^{F_n \times F_m} \rtimes F_n \times F_m \mid P \text{ is a nontrivial amenable factor with normal faithful almost periodic state, and } n, m \geq 2\}
\]

is exactly classified by \(\{n, m\}\) and a state preserving conjugacy of the actions.

But last two steps remain open problems:

- Connes-Størmer entropy for actions of sofic groups such as \(F_n \times F_m\)?
- A non-commutative Ornstein theorem:
  when are two noncommutative \(\mathbb{Z}\)-Bernoulli shifts conjugate in a state preserving way?
  Is it sufficient that \(\Gamma(\phi)\) and Connes-Størmer entropy coincide?

Type III Bernoulli crossed products