A complete characterization of connected Lie groups without the Approximation Property

joint with K happen & devil

- A C*-algebra $A$ has the operator approximation property (OAP) if
  \[ T : \text{id} \to \text{Ask} \text{ bounded operator such that} \]
  \[ \Gamma : \text{discrete group} \quad \text{Ca}^* \text{nucleus} \Rightarrow \text{Ca}^* \text{nucleus (OAP)} \Rightarrow \text{Ca}^* \text{exact} \]
  \[ \text{exact} \Rightarrow \Gamma \text{ has AP} \Rightarrow \Gamma \text{ exact} \]

- Completely bounded (Fourier) multipliers
  \[ G : \text{locally compact group} \]
  \[ \mathcal{L}^1(G) \text{ left-regular representation on } L^2(G) \]
  \[ \mathcal{L}(G) \text{ compact abelian algebra} = \{ \mathcal{I} : G \in G \} \]
  \[ G : C \text{ continuous i.e. completely bounded multiplier} \text{ } C \text{ (Fourier multiplier)} \]
  \[ \text{if } M : \mathcal{I} \rightarrow (G) \mathcal{I} \text{ extends to a completely bounded normal map on } L^1(G). \]
  \[ \mathcal{L}(G) = C \text{-multiplier on } G \]
  \[ \text{with } \| M \|_\text{cb} = \| M \|_G \]
  \[ \mathcal{L}(G) \text{ is a dual Banach space} \]
  \[ \text{if } f \in L^1(G), \text{ new norm } \| f \|_G = \sup \left\{ \| f \|_{\mathcal{L}^1(G)} : \| M \|_G \leq 1 \right\} \]
  \[ \text{then } Q(G) = L^1(G) \text{ satisfies } Q(G)^* \cong \mathcal{L}(G) \]

Definition: (Heoane & Huns '94)

$G$ has the Approximation Property (AP) if

\[ \exists \text{ net } c_i \in \mathcal{L}(G) \text{ such that } c_i \to 1 \text{ weak}^* \]

Theorem: (Heoane & Huns) For $\Gamma$ discrete group, Then:

1. $\Gamma$ has the AP
2. $\text{Ca}^*(\Gamma)$ has the OAP
3. $G$ has the AP (if $\Gamma' \subseteq G$ is a lattice)
Examples of groups with the AP
- compact groups, amenable groups, weakly amenable groups,
  \( Z^2 \times SL(2, \mathbb{Z}), \mathbb{R}^2 \times SL(1, \mathbb{R}) \)
- AP passes to closed subgroups and extension

**Theorem (Lafforgue & de La Salle 2011)**
\( SL(3, \mathbb{R}) \) does not have the AP.
Hence \( C(SL(3, \mathbb{Z})) \) is exact without the OAP.

- **Simple Lie groups** (e.g. \( SL(n, \mathbb{R}) \))
  - after compact subgroup
  - Iwasawa decomposition: \( G = KAN \)
    
    \[ G = SL(2, \mathbb{R}), \quad K = SO(2), \quad A = \{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a > 0 \}, \quad N = \{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \mid z \in \mathbb{R} \} \]
    
    - real rank of \( G = \dim A \)
    - Structure of simple Lie groups:
      \( G \) has real rank at least 2 \( \iff \) \( G \) has closed subgroup \( K \) \( \cong \) \( SL(2, \mathbb{R}) \) or \( Sp(2, \mathbb{R}) \).

**Theorem (Heckey & de Laat 2012)**
\( Sp(2, \mathbb{R}) \) does not have the AP.

**Corollary** A simple Lie group \( G \) with finite center has the AP
\( \iff \) \( G \) has real rank 0 or 1.
Heckey-de Laat (2013) \( Sp(2, \mathbb{R}) \) has no AP.

- **Non-simple Lie groups**
  \( G = \) connected Lie group
  - Levi decomposition: \( G = RS \)

**Question:** \( SL(3, \mathbb{R}) \hookrightarrow G \) is homo.
Does \( G \) fail to have the AP? 2

\( R \): closed normal solvable subgroup
\( S \): semi-simple Lie group
i.e. \( S \cong S_1 \times \ldots \times S_n \) with each \( S_i \) simple locally compact.

\( S \) need not be closed.
An obstruction to the AP

- A mean on $L^\infty(G)$ is a state $m: L^\infty(G) \to C$
- $L^\infty(G)$ admits left-invariant mean $\iff G$ is amenable

However, $B_2(G)$ always admits a left-invariant mean $m: B_2(G) \to C$
Moreover, $m$ is unique!

**Definition** $G$ has property $(T^*)$ if $m: B_2(G) \to C$ is $w^*$-continuous.

**Proposition** If $G$ has the AP and $(T^*)$, then $G$ is compact.

**Proof:** If $G$ is non-compact and $\omega \in B_2(G) \cap C_c(G)$ then $m(\omega) = 0$.
But $m(1) = 1$. $\Box$

**Proposition:** If $\pi: G \to H$ homomorphism and $G$ has $(T^*)$, then $\pi(G)$ has $(T^*)$.

**Theorem:** $SL(3, \mathbb{R})$ and $Sp(2, \mathbb{R})$ and $Sp(2, \mathbb{R})$ have property $(T^*)$.

**Remark:** Property $(T^*) \implies$ Property $(T)$

- So far, no examples of infinite discrete group with $(T^*)$.
- Question: Does $SL(3, \mathbb{Z})$ have $(T^*)$.

**Main Theorem** $G =$ connected Lie group with Levi decomposition $G = RS$

**TFAE:**
1. $G$ has AP
2. $S$ has AP
3. Each $S_i$ has AP
4. Each $S_i$ has real rank 0 or 1.
5. $G$ does not have a Lie subgroup locally isomorphic to $SL(3, \mathbb{R})$ or $Sp(2, \mathbb{R})$
6. $G$ has no non-compact closed subgroup with property $(T^*)$.

**Thm:** A simple Lie group with finite center and real rank at least 2 has property $(T^*)$.

Now $(T)$ implies $(T^*)$:

$B(G) =$ Fourier-Stieltjes algebra $= C^*(G)^*$ $B(G) \subseteq B_2(G)$
$B(G)$ has a unique invariant mean $m: B(G) \to C$

$G$ has property $(T) \iff m$ is weak$^*$ continuous.