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SOME CLASSICAL C^*-ALGEBRAS AND THEIR CLASSIFICATION

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Abstract

We in this thesis studies six classes of C^*-algebras, namely UHF algebras, more general AF algebras along with finite dimensional C^*-algebras, reduced group C^*-algebras, irrational rotation algebras and the Cuntz algebras $\mathcal{O}_n$ for $2 \leq n < \infty$. We present an introductory discussion about each of these classes and develops some basic results. We completely classify UHF algebras and finite dimensional C^*-algebras and compare UHF algebras to general AF algebras. We prove that the reduced group C^*-algebras each have a tracial state and consider a class of projections built from finite subgroups. We prove that irrational rotation algebras are simple and that there are uncountably many pairwise non-isomorphic irrational rotation algebras. We also prove that the Cuntz algebras are simple and purely infinite and therefore differ from the five other classes of C^*-algebras we study.

Contents

Introduction

Mathematicians always strive to understand the objects of their studies to the extreme. It is therefore not always enough to study what the objects look like. The way they behave around and compared to other objects should also be taken into consideration. In many places it is also possible to construct new exiting objects from old ones and then use properties of the old ones to deduce properties of the constructed objects. This is one of the areas where mathematicians shine: they do not settle with just observing. This is also mainly what this thesis is about - the study of C^*-algebras and how to construct new C^*-algebras from a given collection.

The study of C^*-algebras arose around the 1930's from trying to describe observations in quantum mechanics. Since then the area has gone through one of the largest, still incomplete, classification programmes in mathematics as it is a humongous branch. This is another thing mathematicians strive to do: describe and relate objects by certain hopefully simple properties as some mathematical objects defy intuition. Today the classification uses a lot of K-theory, which is beyond the scope of this thesis, to help. This thesis will present a number of the classical constructions made with C^*-algebras and also some of the fruits of the giant classification programme. Some of this work is due to Glimm an Elliotts who together classified UHF algebras. Elliotts later classified AF algebras using K-theory.

There will be two complete classifications presented in this thesis, but we present some tools to at least tell other C^*-algebras apart. Section 2 is about inductive limits of C^*-algebras and two special classes of these, one contained within the other: the AF algebras and UHF algebras. UHF algebras are in particular AF algebras, and we will be able to completely classify UHF algebras. We later in section 2 present some theory on finite dimensional C^*-algebras and will also be able to completely classify these.

Section 3 will be about C^*-algebras constructed from discrete countable groups. We will construct a tracial state on these C^*-algebras and thereby place them in one of the two major classes of C^*-algebras: those with a tracial state and those without.

Section 4 will present two examples of C^*-algebras constructed from relations between generators. These are constructed in a different way than the inductive limits, as we actually build these to satisfy certain properties. We will be studying the irrational rotation algebras, where we among other things prove that there are uncountably many non-isomorphic of these, and the Cuntz algebras. The latter differ a lot from the other examples of C^*-algebras presented in this thesis even though the construction does not differ much from the construction of the irrational rotation algebras.

Section 2.1 on inductive limits of C^*-algebras is based on sections 6.1 and 6.2 of [2]. Section 2.2 on UHF algebras is based on a set of notes written by Mikael Rørdam. Section 2.3 is based on sections 7.1 and 7.2 in [2]. Section 4.2 on irrational rotation algebras is based on chapter VI of [3]. Section 4.3 on Cuntz algebras is based on sections V.4 and V.5 in [3]. In all of the above I have filled in minor details that the authors have left out of their proofs. I have also put in some examples of my own here and there.

A big thank you should go to my advisor, Professor Mikael Rørdam, whose guidance and anecdotes I could not have been without while writing this thesis. It is largely thanks to him that I have become so interested in the area of operator algebras. Another thanks should go to my dear friend Mikkel Munkholm who has taken the time to help me with certain parts of this thesis and has come with many good inputs. He is also a big factor in my interest in operator algebras. Lastly, there are a number of my fellow students who deserve my thanks, no matter if they have discussed with me or helped me in any other way during my writing.

I hope that the reader understands and feels some of my fascination and interest in this vast field as he or she reads this thesis.

1 Preliminaries

1.1 Prerequisites

We assume the reader to be familiar with the fundamentals of operator algebras corresponding to the first 15 chapters of [1]. This of course needs some ring theory. This also includes results as the Gelfand-Naimark-Segal (GNS) construction and the continuous functional calculus. We need the axiom of choice for this, and we shall assume throughout the entire thesis that it holds.

We will mention category theory a few times. The reader is assumed to know the definitions of categories and functors between these. It will be helpful if the reader has heard of K-theory, but it is not needed.

Every vector space, and in particular every algebra, mentioned in this thesis will be over the field of complex numbers. Also every linear span of elements we mention in this thesis will be over the complex numbers unless otherwise stated. Some of the algebras we are going to use are listed below.

- $B(H)$ for some Hilbert space H, the C^*-algebra of bounded linear functions from H to H. This will always be endowed with the operator norm given by

$$\|T\| = \sup\{\|T\xi\| \mid \xi \in H,\ \|\xi\| \leq 1\}.$$

- $C(X)$ for some compact Hausdorff space X, the C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$. This will always be endowed with the supremum-norm given by $\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}$ for every $f \in C(X)$. If X is locally compact but not compact, we shall also use $C_0(X)$, the C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$ satisfying that the set $\overline{\{x \in X \mid f(x) \neq 0\}}$ is compact.

- $M_n(\mathbb{C})$, the set of $n \times n$-matrices for some $n \in \mathbb{N}$ with entries in the complex numbers. This is given a C^*-norm via the GNS construction. We will also use $M_n(A)$, when A is any C^*-algebra, the set of all $n \times n$-matrices with entries in A.

1.2 Notation

We will as usual let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. These and subsets of these may sometimes implicitly be given a group structure or the standard topology when needed. We let $\mathbb{T} \subset \mathbb{C}$ denote the unit circle, that is, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

The focus of this thesis will of course be on C^*-algebras. We will mainly use A, B and C to denote these, although we will use special symbols for particular C^*-algebras. Elements of the C^*-algebras will be denoted typically by a, b, c or x, y, z, although other letters are sometimes used. Letters denoting elements of a C^*-algebra will always be lower-case, with the only exception being the unit (see below).

Fix momentarily a C^*-algebra A.

If B is another C^*-algebra, a *-homomorphism from A to B is a linear and multiplicative map that respects adjoints. We will usually denote these by φ and ψ although other letters, mostly greek, will sometimes be used. If both A and B are unital, a *-homomorphism $\varphi\colon A \to B$ is unital if $\varphi(I_A) = I_B$.

If A is a unital C^*-algebra we shall denote the unit by I or I_A if we want to emphasize the ambient algebra.

If $a - b$ is positive for some $a, b \in A$ we will write $b \leq a$ or $a \geq b$. If $a \neq b$ we will write $a > b$ or $b < a$.

If A is unital, we will for $a \in A$ denote by

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot I_A - a \text{ is not invertible}\}$$

the spectrum of a. This is known to be a compact set with $\sigma(a) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|a\|\}$.

Ideals in A will always be closed two-sided ideals, and we will mostly reserve the symbols $\mathcal{I}$ and $\mathcal{J}$ for ideals. If $\mathcal{I}$ is an ideal in A, we shall write $\mathcal{I} \trianglelefteq A$. If $\mathcal{I} \neq A$, we shall also write $\mathcal{I} \triangleleft A$. We will denote by $\langle a \rangle$ the closed two-sided ideal in A generated by $a \in A$. That is,

$$\langle a \rangle = \overline{\left\{ \sum_{i=1}^{n} x_i a y_i \mid i \in \mathbb{N},\ x_i, y_i \in A \right\}}.$$

Lemma 1.1. *Suppose A is unital and simple and $a \in A$ is non-zero. Then there exist $n \in \mathbb{N}$ and $y_i \in A$ such that $I = \sum_{i=1}^{n} y_i^* a y_i$.*

Proof. As $a \neq 0$, we have $\langle a \rangle = A$. Thus the set $\mathcal{I}(a) = \{\sum_{i=1}^{m} x_i a y_i \ \mid\ m \in \mathbb{N},\ x_i, y_i \in A\}$ is dense in A, so we can find $m \in \mathbb{N}$ and $x_i', y_i' \in A$ such that

$$\left\| I - \sum_{i=1}^{m} x_i' a y_i' \right\| < 1.$$

Hence $\sum_{i=1}^{m} x_i' a y_i'$ is invertible. As $\mathcal{I}(a)$ is easily seen to be a two-sided algebraic ideal, we therefore get $I \in \mathcal{I}(a)$, i.e. $I = \sum_{i=1}^{n} x_i a y_i$ for some $n \in \mathbb{N}$ and $x_i, y_i \in A$.

As $I = I^*$, we see that $(x_i a y_i)^* = x_i a y_i$ for every $i = 1, \dots, n$. This implies that $y_i^* = x_i$, and we therefore get

$$I = \sum_{i=1}^{n} y_i^* a y_i.$$

■

We will write $\mathrm{Proj}(A)$ for the set of projections in A and reserve the letters p and q for projections. We will use the Murray-von Neumann equivalence relation $\sim$ between projections given by $p \sim q$ if and only if there exists some $v \in A$ such that $v^*v = p$ and $vv^* = q$. This will be our only use of the symbol $\sim$.

Definition 1.2. Let $p \in \mathrm{Proj}(A)$. We say that p is *infinite* if there exists $q \in \mathrm{Proj}(A)$ such that $p \sim q < p$, that is, if p is equivalent to a proper subprojection of itself.
We say that p is *properly infinite* if there exist $q_1, q_2 \in \mathrm{Proj}(A)$ such that $p \sim q_1 \sim q_2$ and $q_1 + q_2 \leq p$.
We say p is *finite* if it is not infinite.
A is called *(properly) infinite* if A contains a (properly) infinite projection and is called *finite* otherwise.

When A is unital, we shall write $S(A)$ for the state space of A, that is,

$$S(A) = \{\varphi\colon A \to \mathbb{C} \mid \varphi \text{ is linear, positive and } \varphi(I_A) = 1\},$$

and $T(A)$ for the tracial states, that is,

$$T(A) = \{\varphi \in S(A) \mid \varphi(xy) = \varphi(yx) \text{ for every } x, y \in A\}.$$

The symbol τ will mainly be used for tracial states. A tracial state τ will be called faithful if for every positive $x \in A$ it holds that $\tau(x) = 0$ implies $x = 0$.

We shall multiple times in this thesis use the Kronecker delta defined as follows. If X is a set and $x_1, x_2 \in X$, we define

$$\delta_{x_1 x_2} = \begin{cases} 1, & x_1 = x_2, \\ 0, & \text{else.} \end{cases}$$

1.3 Preliminary Results

We will need some results beyond introductory operator algebras focused around integrals of C^*-valued functions. We will state these results without proof. Let (X, μ) be a measure space and A a C^*-algebra. Denote for now by A^* the set of all linear functionals on A.

Definition 1.3. Define $L^1(X, \mu, A)$ to be the set of all functions $f\colon X \to A$ such that
i) the function $x \mapsto \rho(f(x))$, $x \in X$, is measurable for every $\rho \in A^*$,
ii) the function $x \mapsto \|f(x)\|$, $x \in X$, belong to $L^1(X, \mu)$.

We note that $\rho \circ f \in L^1(X, \mu)$ for every $f \in L^1(X, \mu, A)$ and $\rho \in A^*$.

Proposition 1.4. *For every $f \in L^1(X, \mu, A)$ there is a unique element, denoted by $\int_X f d\mu$, in A^{**} such that*

$$\left(\int_X f d\mu \right)(\rho) = \int_X \rho \circ f d\mu$$

for every $\rho \in A^*$, where the right hand side is a usual Lebesgue integral. Moreover, for every $f, g \in L^1(X, \mu, A)$ and $\alpha, \beta \in \mathbb{C}$ we have
i) $\left\| \int_X f d\mu \right\| \leq \int_X \|f\| d\mu$,
ii) $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$.

This proposition says that integrals of C^*-valued functions fulfill some of the same properties as usual integrals. However, so far the integral is just an element in the double dual of A and not necessarily in A itself. But in the case of the next proposition, which is also the only case that will occur in this thesis, the integral is actually an element in A:

Proposition 1.5. *Assume that X is a compact Hausdorff space and that μ is a finite Borel measure on X. If $f \in C(X, A)$, the algebra of continuous functions from X to A, then $f \in L^1(X, \mu, A)$ and $\int_X f d\mu \in A$. Moreover,*
i) $a \left(\int_X f d\mu \right) b = \int_X afb \, d\mu$ *for all* $a, b \in A$ *and* $f \in C(X, A)$,
ii) if $f \in C(X, A)$ is positive, then $\int_X f d\mu$ is positive,
iii) if $f \in C(X, A)$ is positive and non-zero and $\mu(U) > 0$ for every non-empty open $U \subseteq X$ then $\int_X f d\mu$ is non-zero,
iv) for each $f \in C(X, A)$ we have

$$\frac{1}{\mu(X)} \int_X f d\mu \in \overline{\mathrm{conv}\{f(x) \mid x \in X\}}.$$

2 Inductive Limits of C^*-algebras

If we are given a collection of C^*-algebras, there is a number of ways to create new C^*-algebras from these. One of these ways is the use of inductive limits. The limits inherit certain properties from the collection defining it, if this property is shared among every member of the collection.

Inductive sequences are particularly interesting in the C^*-algebraic setting, as the limits turn out to always exist.

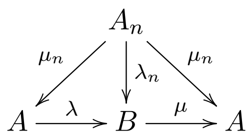
2.1 Definition and Existence

An inductive limit is a notion in category theory. Given a category $\mathcal{C}$, an inductive sequence consists of a sequence $(A_n)_{n\in\mathbb{N}}$ of objects and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of morphisms $\varphi_n\colon A_n \to A_{n+1}$. We represent it as

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots . \tag{2.1}$$

Definition 2.1. An *inductive limit* to the inductive sequence $(A_n, \varphi_n)_{n\in\mathbb{N}}$, denoted by $\varinjlim(A_n, (\varphi_n)_{n\in\mathbb{N}})$, is a system $(A, (\mu_n)_{n\in\mathbb{N}})$, where A is an object in $\mathcal{C}$ and $\mu_n\colon A_n \to A$ is a morphism for each $n \in \mathbb{N}$, such that $\mu_n = \mu_{n+1} \circ \varphi_n$ for all $n \in \mathbb{N}$, and if $(B, (\lambda_n)_{n\in\mathbb{N}})$ is another system satisfying the above, then there is a unique morphism $\lambda\colon A \to B$ such that $\lambda_n = \lambda \circ \mu_n$.

From the definition we see that inductive limits are essentially unique when they exist. Indeed, if $(A, (\mu_n))$ and $(B, (\lambda_n))$ are two limits of the same sequence, then there are unique morphisms $\lambda\colon A \to B$ and $\mu\colon B \to A$ such that the following diagram commutes for every $n \in \mathbb{N}$.



By the uniqueness condition applied to $\mu \circ \lambda$, we must have $\mu \circ \lambda = \mathrm{id}_A$. In the same way, $\lambda \circ \mu = \mathrm{id}_B$. Hence both λ and μ are isomorphisms with $\lambda = \mu^{-1}$.

We can therefore talk about *the* inductive limit of a sequence (2.1) when it exists. Direct limits do not always exist, however; but in the category of C^*-algebras with *-homomorphisms as the morphisms, we shall see shortly that a limit always exists, and we shall only consider this specific category.

We will now begin to develop some tools to show the existence of inductive limits of C^*-algebras. These results are also of some independent interest, though.

Throughout the section we consider some family $\{A_\alpha\}_{\alpha\in\Lambda}$ of C^*-algebras.

Definition 2.2. We define the *product algebra* $\prod_{\alpha\in\Lambda} A_\alpha$ as follows: it is the set of all functions $a\colon I \to \bigsqcup_{i\in I} A_i$, the disjoint union, such that $a(\alpha) \in A_\alpha$ for all $\alpha \in \Lambda$ and

$$\|a\| := \sup_{\alpha\in\Lambda} \|a(\alpha)\|_{A_\alpha} < \infty.$$

We equip it with coordinate-wise operations to make it into an algebra.

We quickly drop the range in the product if this is obvious, and write a_i instead of $a(\alpha)$ for the coordinates in $a \in \prod A_\alpha$. We also drop the subscripts in the norms of the A_α and just write $\|a_\alpha\|$.

Proposition 2.3. *$\prod A_\alpha$ is a C^*-algebra.*

Proof. It is immediate that $\prod A_\alpha$ is a *-algebra. We show that $\prod A_\alpha$ is complete and that the Banach inequality holds. The triangle inequality and C^*-identity follows in a similar way.

To verify the Banach inequality, let $a, b \in \prod A_\alpha$ and notice that $\|a_\alpha\| \leq \|a\|$ and $\|b_\alpha\| \leq \|b\|$ for every $\alpha \in \Lambda$. As each A_i is a Banach algebra, we then get for every $\alpha \in \Lambda$ that

$$\|(ab)_\alpha\| = \|a_\alpha b_\alpha\| \leq \|a_\alpha\| \cdot \|b_\alpha\| \leq \|a\| \cdot \|b\|,$$

and we conclude that the Banach inequality also holds in $\prod A_\alpha$.

To show that $\prod A_i$ is complete, let $\{a^{(n)}\}_{n\in\mathbb{N}} \subset \prod A_\alpha$ be a Cauchy sequence. Then $\{a_\alpha^{(n)}\}_{n\in\mathbb{N}} \subset A_\alpha$ is a Cauchy sequence for each $\alpha \in \Lambda$ as $\|a_\alpha\| \leq \|a\|$ for every $\alpha \in \Lambda$, and hence it has a limit $a_\alpha \in A_\alpha$. Let $a = (a_\alpha) \in \prod A_\alpha$. Let some $\varepsilon > 0$ be given, and find $N \in \mathbb{N}$ such that $\|a^{(n)} - a^{(m)}\| < \varepsilon$ whenever $n, m > N$. Then for each $n > N$ and $\alpha \in \Lambda$ we have

$$\|a_\alpha^{(n)} - a_\alpha\| = \lim_{m\to\infty} \|a_\alpha^{(n)} - a_\alpha^{(m)}\| < \varepsilon,$$

and it follows from the reverse triangle inequality that

$$\|a_\alpha\| \leq \|a_\alpha^{(N)}\| + \varepsilon \leq \|a^{(N)}\| + \varepsilon.$$

Hence $a \in \prod A_\alpha$. Also, it follows from the above that whenever $n > N$,

$$\|a^{(n)} - a\| = \sup \|a_\alpha^{(n)} - a_\alpha\| \leq \varepsilon,$$

so that $a^{(n)} \to a$ as $n \to \infty$. Hence $\prod A_\alpha$ is complete. ■

Definition 2.4. We define the *direct sum algebra* $\sum_{\alpha\in\Lambda} A_\alpha$ as the closure of the set

$$\mathcal{J} = \left\{ a \in \prod A_\alpha \mid a_\alpha \neq 0 \text{ for only finitely many } \alpha \right\}$$

in $\prod A_\alpha$.
If Λ is the finite set $\{\alpha_1, \ldots, \alpha_n\}$ for some $n \in \mathbb{N}$, we shall also write $\sum_{\alpha\in\Lambda} A_\alpha = A_{\alpha_1} \oplus \cdots \oplus A_{\alpha_n}$.

It is immediate that $\mathcal{J}$ is a two sided ideal in $\prod A_\alpha$, and that $\sum A_\alpha$ is a closed two sided ideal. In particular, $\sum A_\alpha$ is a C^*-algebra.

Let $\pi \colon \prod A_\alpha \to \prod A_\alpha / \sum A_\alpha$ denote the quotient mapping. If now $\Lambda = \mathbb{N}$, the following lemma describes what happens in the quotient a bit clearer:

Lemma 2.5. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of C^*-algebras and $a \in \prod A_n$. Then*

$$\|\pi(a)\| = \limsup_{n\to\infty} \|a_n\|$$

In particular, $a \in \sum A_n$ if and only if $\lim_{n\to\infty} \|a_n\| = 0$.

Proof. As $\mathcal{J}$ is dense in $\sum A_n$ we have $\|\pi(a)\| = \inf\{\|a - b\| \mid b \in \mathcal{J}\}$ by continuity of the mapping $b \mapsto \|a - b\|$.

Each $b = (b_n) \in \mathcal{J}$ has the property that $b_n = 0$ eventually, and therefore we get

$$\|a - b\| \geq \limsup_{n\to\infty} \|a_n - b_n\| = \limsup_{n\to\infty} \|a_n\|, \quad b \in \mathcal{J}.$$

This shows that $\|\pi(a)\| \geq \limsup_{n\to\infty} \|a_n\|$.

For each $k \in \mathbb{N}$, let $b^{(k)} = (b_n^{(k)}) \in \mathcal{J}$ be the element given by

$$b_n^{(k)} = \begin{cases} a_n, & n \leq k, \\ 0, & n > k. \end{cases}$$

Then
$$\|\pi(a)\| \leq \inf_{k\in\mathbb{N}} \|a - b^{(k)}\| = \inf_{k\in\mathbb{N}} \sup_{n>k} \|a_n\| = \limsup_{n\to\infty} \|a_n\|.$$

Hence $\|\pi(a)\| = \limsup_{n\to\infty} \|a_n\|$. ■

Example 2.6. The algebra $\prod A_n/\sum A_n$ is often called the limit algebra of $(A_n)_{n\in\mathbb{N}}$, as is justified by this lemma. As an example of this, suppose that $a \in \prod A_n$ is a sequence such that $\|a_n - a_n^*\| \to 0$ as $n \to \infty$. Then $\pi(a) \in \prod A_n/\sum A_n$ is self-adjoint, as

$$\|\pi(a) - \pi(a)^*\| = \|\pi(a - a^*)\| = \limsup_{n\to\infty} \|a_n - a_n^*\| = 0.$$

Thus an approximate property in $\prod A_n$ often turns into an exact property in $\prod A_n/\sum A_n$.

To an inductive sequence (2.1) we define $\varphi_{m,n}\colon A_n \to A_m$ as follows: for $n < m$, define $\varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n$. For $n = m$, define $\varphi_{m,n} = \mathrm{id}_{A_n}$. For $m < n$, define $\varphi_{m,n} = 0$. With this and what we know now, we can prove that inductive limits of C^*-algebras always exists and then really start the study of these.

Theorem 2.7 (Inductive limits of C^*-algebras)**.** *Every inductive sequence of C^*-algebras*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$$

has an inductive limit $(A, \{\mu_n\})$. In addition, the following hold:
i) $A = \overline{\bigcup_{n\in\mathbb{N}} \mu_n(A_n)}$,
ii) $\|\mu_n(a)\| = \lim_{m\to\infty} \|\varphi_{m,n}(a)\|$ *for all* $n \in \mathbb{N}, a \in A_n$,
iii) $\mathrm{Ker}(\mu_n) = \{a \in A_n \mid \lim_{m\to\infty} \|\varphi_{m,n}(a)\| = 0\}$.

Proof. **Existence:** Let π be the quotient mapping as before. For each $n \in \mathbb{N}$, define $\nu_n\colon A_n \to \prod A_n$ by $\nu_n(a) = (\varphi_{m,n}(a))_{m\in\mathbb{N}}$. This is well-defined as each $\varphi_{m,n}$ is a *-homomorphism and hence contractive so that

$$\|\nu_n(a)\| = \sup_{m\in\mathbb{N}} \|\varphi_{m,n}(a)\| \leq \|a\|, \quad a \in A_n.$$

Define also $\mu_n\colon A_n \to \prod A_n/\sum A_n$ by $\mu_n = \pi \circ \nu_n$ for each $n \in \mathbb{N}$. Then ν_n and μ_n are *-homomorphisms for all $n \in \mathbb{N}$. Let now $n \in \mathbb{N}$ and $a \in A_n$; then

$$\nu_n(a) - (\nu_{n+1} \circ \varphi_n)(a) = (\varphi_{m,n}(a))_{m\in\mathbb{N}} - (\varphi_{n+1,m})(a)_{m\in\mathbb{N}} = c \in \sum_{n\in\mathbb{N}} A_n,$$

where $c_n = a$ and $c_m = 0$ for $m \neq n$. Thus,

$$\mu_n(a) - (\mu_{n+1} \circ \varphi_n)(a) = \pi\big(\nu_n(a) - (\nu_{n+1} \circ \varphi_n)(a)\big) = 0.$$

Hence $\mu_{n+1} \circ \varphi_n = \mu_n$. It follows that $(\mu_n(A_n))_{n\in\mathbb{N}}$ is an increasing sequence of C^*-algebras. Hence we get that $A = \overline{\bigcup_{n\in\mathbb{N}} \mu_n(A_n)}$ is a C^*-algebra, the restriction of co-domains of each μ_n is a *-homomorphism from A_n to A, and $(A, \{\mu_n\})$ satisfies the conditions of being an inductive limit; for the uniqueness part, if $(B, \{\lambda_n\})$ is another limit, define $\lambda\colon \bigcup_{n\in\mathbb{N}} \mu_n(A_n) \to B$ by $\lambda(\mu_n(a)) = \lambda_n(a)$ and extend to A by continuity. This will be the unique homomorphism satisfying $\lambda_n = \lambda \circ \mu_n$.
i) This holds by the construction of A above.
ii) Let $a \in A_n$. Use Lemma 2.5 to obtain

$$\|\mu_n(a)\| = \|\pi((\varphi_{m,n}(a))_{m\in\mathbb{N}})\| = \limsup_{n\to\infty} \|\varphi_{m,n}(a)\|.$$

As $(\|\varphi_{m,n}(a)\|)_{n=m}^{\infty}$ is a decreasing positive sequence, because each $\varphi_{m,n}$ is a *-homomorphism, we see that the limit of $\|\varphi_{m,n}(a)\|$ as $n \to \infty$ exists, and therefore

$$\limsup_{n\to\infty} \|\varphi_{m,n}(a)\| = \lim_{n\to\infty} \|\varphi_{m,n}(a)\|,$$

and ii) is proved.
iii) This clearly follows from ii). ■

Example 2.8. Consider the inductive sequence

$$\mathbb{C} \xrightarrow{\varphi_1} M_2(\mathbb{C}) \xrightarrow{\varphi_2} M_3(\mathbb{C}) \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-1}} M_n(\mathbb{C}) \xrightarrow{\varphi_n} \dots,$$

where $\varphi_n\colon M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ is given by

$$\varphi_n(x) = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} & 0 \\ x_{21} & x_{22} & \dots & x_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad x = (x_{ij})_{i,j=1}^{n} \in M_n(\mathbb{C}).$$

By Theorem 2.7 the limit of this sequence exists. Let now H be an infinite-dimensional separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and let $\mathcal{F} \subseteq B(H)$ be the finite rank operators. Define $f_n\colon M_n(\mathbb{C}) \to \mathcal{F}$ for every $n \in \mathbb{N}$ by

$$f_n(x)(h) = \sum_{i,j=1}^{n} x_{ij}\langle h, e_j\rangle e_i, \quad x = (x_{ij})_{i,j=1}^{n} \in M_n(\mathbb{C}), \quad h = (h_1, h_2, \dots) \in H.$$

f_n is seen to be injective for every n. Let I_n be the identity in $M_n(\mathbb{C})$. Then $f_n(I_n) = E_n$ is the projection onto $(e_1, \dots, e_n, 0, \dots)$, and we see that $f_n(M_n(\mathbb{C})) = E_n B(H) E_n$ for every $n \in \mathbb{N}$. It is immediate that the following diagram commutes:

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow[\cong]{f_n} & E_n B(H) E_n \\ \downarrow{\scriptstyle \varphi_n} & & \cap \\ M_{n+1}(\mathbb{C}) & \xrightarrow[\cong]{f_{n+1}} & E_{n+1} B(H) E_{n+1} \end{array}$$

That is, we have $f_{n+1} \circ \varphi_n = f_n$. Thus $\lim M_n(\mathbb{C}) \cong \overline{\bigcup_{n\in\mathbb{N}} E_n B(H) E_n}$. As we know that $\{E_n\}_{n\in\mathbb{N}}$ constitues an approximate identity for $\mathcal{K} \subseteq B(H)$, the compact operators, we have $\overline{\bigcup_{n\in\mathbb{N}} E_n B(H) E_n} = \mathcal{K}$. We have therefore shown that

$$\lim M_n(\mathbb{C}) \cong \mathcal{K}.$$

One of the properties that inductive limits inherit from their defining sequence is simplicity, as this next proposition shows.

Proposition 2.9. *Suppose that A is the inductive limit of a sequence*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

of unital simple C^*-algebras with unital connecting homomorphisms φ_n. Then A is unital and simple.

Proof. As each φ_n is unital, it is non-zero, and as A_n is simple, φ_n is injective for all $n \in \mathbb{N}$. Set $A'_n = \varphi_n(A_n)$. Then

$$I_A \in A'_1 \subseteq A'_2 \subseteq \cdots \subseteq A, \quad A = \overline{\bigcup_{n\in\mathbb{N}} A'_n},$$

and $A'_n \cong A_n$ for all $n \in \mathbb{N}$; in particular, A'_n is a unital simple C^*-algebra for all $n \in \mathbb{N}$.

Let now $\mathcal{I} \trianglelefteq A$, and suppose that $\mathcal{I} \neq A$. Then $I_A \notin \mathcal{I}$. Let $\pi\colon A \to A/\mathcal{I}$ be the quotient mapping, and set $\pi_n = \pi|_{A'_n}$ for each $n \in \mathbb{N}$. The kernel $\mathcal{I}_n$ of π_n is a closed two sided ideal in A'_n, so either $\mathcal{I}_n = \{0\}$ or $\mathcal{I}_n = A'_n$. As $I_A \notin \mathcal{I}$, we get $0 \neq \pi(I_A) = \pi_n(I_A)$, so $\mathcal{I}_n = \{0\}$ for all $n \in \mathbb{N}$, and thus each π_n is injective. Every injective *-homomorphism is isometric, so $\|\pi(x)\| = \|\pi_n(x)\| = \|x\|$ for all $x \in A'_n$. Now, the set $\{x \in A \mid \|\pi(x)\| = \|x\|\}$ is closed and contains the dense subset $\bigcup_{n\in\mathbb{N}} A'_n$ of A, so it must equal A. Hence $\|\pi(x)\| = \|x\|$ for all $x \in A$, and this implies that π is injective. Hence $\mathcal{I} = \operatorname{Ker}(\pi) = \{0\}$, and A is therefore simple. ■

2.2 UHF Algebras

Inductive sequences arise often in the topic of C^*-algebras. One of the more studied areas is the class of UHF algebras.

Definition 2.10. A *UHF algebra* of type $(n_k)_{k\in\mathbb{N}}$, where $n_k \in \mathbb{N}$ for each $k \in \mathbb{N}$, is a C^*-algebra which is (isomorphic to) the inductive limit of a sequence

$$M_{n_1}(\mathbb{C}) \xrightarrow{\varphi_1} M_{n_2}(\mathbb{C}) \xrightarrow{\varphi_2} M_{n_3}(\mathbb{C}) \xrightarrow{\varphi_3} \cdots, \tag{2.2}$$

where each φ_n is unital.

UHF stands for uniformly hyperfinite. Later we shall see that such a sequence exists if and only if $n_k | n_{k+1}$ for all $k \in \mathbb{N}$; we can prove the 'if' part now. Assume therefore that $(n_k)_{k\in\mathbb{N}}$ is a sequence of natural numbers such that $n_k | n_{k+1}$ for all $k \in \mathbb{N}$. Set $m_k = n_{k+1}/n_k$. We note that $M_{n_{k+1}}(\mathbb{C}) = M_{m_k}(M_{n_k}(\mathbb{C}))$. Define $\varphi_k\colon M_{n_k}(\mathbb{C}) \to M_{n_{k+1}}(\mathbb{C})$ by

$$\varphi_k(x) = \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{pmatrix},$$

where x is repeated m_k times down the diagonal. It is now easily seen that φ_k is a unital injective *-homomorphism for all $k \in \mathbb{N}$.

Let now A denote the inductive limit of the above sequence. As each φ_k is injective, there is an increasing sequence of sub-C^*-algebras $(A_k)_{k\in\mathbb{N}}$ of A such that $A_k \cong M_{n_k}(\mathbb{C})$ for all $k \in \mathbb{N}$ by Theorem 2.7 i). Thus Proposition 2.9 shows that A is simple, as each matrix algebra is simple.

Recall that both $S(A)$ and $T(A)$, the state space and the tracial state space of A, respectively, are compact and Hausdorff in the weak-* topology if A is unital.

We know that each matrix algebra admits a unique tracial state, and this is another thing that UHF algebras inherit:

Lemma 2.11. *Let A be a unital C^*-algebra which contains an increasing sequence $I_A \in A_1 \subseteq A_2 \subseteq \dots$ of sub-C^*-algebras, and such that $\overline{\bigcup_{n\in\mathbb{N}} A_n} = A$.*
i) If $T(A_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $T(A) \neq \emptyset$.
ii) Every UHF algebra admits a unique tracial state.

Proof. **i)** Let
$$T_n = \{\varphi \in S(A) \mid \varphi|_{A_n} \in T(A_n)\}.$$
We show that $T_n \neq \emptyset$ for all n; as $T(A_n) \neq \emptyset$, take some $\varphi_0 \in T(A_n)$. By Hahn-Banach, it extends to a linear functional φ on A such that $\|\varphi\| = \|\varphi_0\| = 1$. We have $I_A \in A_n$, so $\varphi(I_A) = \varphi_0(I_A) = 1 = \|\varphi\|$, and thus $\varphi \in S(A)$. As $\varphi|_{A_n} = \varphi_0 \in T(A_n)$, $\varphi \in T_n$. As
$$T_n = \bigcap_{x,y\in A_n} \{\varphi \in S(A) \mid \varphi(xy - yx) = 0\},$$
T_n is a weak-*-closed subset of $S(A)$, and thus T_n is compact. Also, $T_1 \supseteq T_2 \supseteq T_3 \supseteq \dots$, so $T := \bigcap_{n\in\mathbb{N}} T_n$ is non-empty by the finite intersection property. Finally, if $\tau \in T$, then τ is a state that satisfies $\tau(xy) = \tau(yx)$ for all $x, y \in \bigcup_{n\in\mathbb{N}} A_n$. As the latter set is dense in A, we conclude by continuity that $\tau(xy) = \tau(yx)$ for all $x, y \in A$, and hence $T \subseteq T(A)$, so $T(A) \neq \emptyset$.
ii) Let A be a UHF algebra of type (n_k), and by Theorem 2.7 i) find a sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ such that $A_k \cong M_{n_k}(\mathbb{C})$ for all $k \in \mathbb{N}$, and such that $\bigcup_{k\in\mathbb{N}} A_k$ is dense in A. We know that $M_{n_k}(\mathbb{C})$ admits a unique tracial state for every $n \in \mathbb{N}$, so $T(A_k) \neq \emptyset$, and hence by i) we get that $T(A) \neq \emptyset$. This proves existence of a tracial state.

If now τ, ρ are two tracial states on A, then $\tau|_{A_k} = \rho|_{A_k}$ by the uniqueness of the tracial state on $M_{n_k}(\mathbb{C})$, so $\tau(x) = \rho(x)$ for all $x \in \bigcup_{k\in\mathbb{N}} A_k$, and hence by continuity, $\rho = \tau$. ■

Corollary 2.12. *Every UHF algebra is finite.*

Proof. Let A be a UHF algebra with tracial state τ. It is a general result that the set $\{x \in A \mid \tau(x^*x) = 0\}$ constitues a closed left ideal in A, as τ is a state. As τ is further a trace, this is clearly also a right ideal and hence a two sided ideal. As A is simple and $\tau \neq 0$, τ is therefore faithful. Assume now that $p, q \in \operatorname{Proj}(A)$ satisfy $p \sim q \leq p$, and let $v \in A$ witness this equivalence; $v^*v = p$, $vv^* = q$. Then
$$\tau(p - q) = \tau(v^*v) - \tau(vv^*) = \tau(v^*v) - \tau(v^*v) = 0,$$
as τ is a state. As τ is faithful, this implies that $p - q = 0$, so $q = p$ and A is therefore finite. ■

Let now A be a C^*-algebra. If $p \sim q_0 \leq q$ for some projections $p, q, q_0 \in A$, we write $p \preceq q$. It is immediate as in the proof of Corollary 2.12 that if τ is a trace on A and $p \sim q$, then $\tau(p) = \tau(q)$. Moreover, if τ is also positive and $p \preceq q$, then $\tau(p) \leq \tau(q)$.

We conclude this subsection by proving a series of technical lemmas that will help us in the classification of UHF-algebras due to Glimm and Elliotts.

Lemma 2.13. *Let A be a unital C^*-algebra, and suppose $p, q \in \operatorname{Proj}(A)$ satisfy $p = z^{-1}qz$ for some invertible element $z \in A$. Then $p \sim q$.*

Proof. Write $z = u|z|$ for some unitary u. From $zp = qz$ we deduce that $pz^* = z^*q$, and thus
$$z^*zp = z^*qz = pz^*z,$$
and so $z^*z \in \Lambda(p)\{x \in A \mid xp = px\}$. As $\Lambda(p)$ is known to be a C^*-algebra, we conclude that $|z|^{-1} \in \Lambda(p)$, whereby $|z|^{-1}p = p|z|^{-1}$. Thus we get that
$$up = z|z|^{-1}p = zp|z|^{-1} = qz|z|^{-1} = qu.$$
Put now $v = up$. Then $v^*v = pu^*up = p^2 = p$ and $vv^* = up^2u^* = upu^* = q$ using the calculation above. Thus $p \sim q$. ■

Lemma 2.14. *Let A be a unital C^*-algebra, and suppose that $p, q \in \text{Proj}(A)$ satisfy $\|p - q\| < \frac{1}{2}$. Then $p \sim q$.*

Proof. Put $z = pq + (I_A - p)(I_A - q)$. Then $pz = pq = zq$. Also,

$$\begin{aligned} \|z - I_A\| &= \|(pq + (I_A - p)(I_A - q)) - p^2 - (I_A - p^2)\| \\ &= \|p(q - p) + (I_A - p)(p - q)\| \leq \|p(q - p)\| + \|(I_A - p)(p - q)\| \\ &\leq 2\|p - q\| < 1, \end{aligned}$$

and hence z is invertible. Lemma 2.13 now gives us that $p \sim q$. ■

Lemma 2.15. *Let A be a unital C^*-algebra, let $p \in \text{Proj}(A)$, and let $a \in A$ be self adjoint. Let also $\delta > 0$.*
i) If $\|p - a\| \leq \delta$, then $\sigma(a) \subseteq [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$.
ii) If $\sigma(a) \subseteq [\delta, \delta] \cup [1 - \delta] \cup [1 + \delta]$, and if $\delta < 1/2$, then $C^*(I_A, a)$ contains a projection q which satisfies $\|a - q\| \leq \delta$.

Proof. **i)** As a is self adjoint, $\sigma(a) \subseteq \mathbb{R}$. Suppose now that $\lambda \in \mathbb{R} \setminus ([-\delta, \delta] \cup [1 - \delta, 1 + \delta])$. Then by the spectral mapping theorem, the continuous function calculus and the spectral radius theorem for normal elements, we get

$$\|(\lambda \cdot I_A - p)^{-1}\| \leq \max\{|\lambda|^{-1}, |1 - \lambda|^{-1}\} < \delta^{-1},$$

and so $\|(\lambda \cdot I_A - p)^{-1}\|^{-1} > \delta$. We then get

$$\|(\lambda \cdot I_A - a) - (\lambda \cdot I_A - p)\| = \|p - a\| \leq \delta < \|(\lambda \cdot I_A - p)^{-1}\|^{-1}.$$

Now, if $x \in A$ is invertible, then the set $\{y \in A \mid \|y - x\| < \|x^{-1}\|^{-1}\}$ is contained within the invertible elements of A. Indeed, given $y \in A$ such that $\|y - x\| < \|x^{-1}\|^{-1}$, we have

$$\|I_A - x^{-1}y\| = \|x^{-1} \cdot (x - y)\| \leq \|x^{-1}\| \cdot \|x - y\| < 1,$$

and hence $x^{-1}y$ is invertible, so y is also invertible.

As we know that $\sigma(p) \subseteq \{0, 1\}$, we now see that $\lambda \cdot I_A - a$ is invertible, so that $\lambda \notin \sigma(a)$.
ii) Set $X = [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$. As $\delta < 1/2$, $[-\delta, \delta] \cap [1 - \delta, 1 + \delta] = \emptyset$. Therefore we can define the continuous function $\tilde{f} \colon X \to \mathbb{C}$ by

$$\tilde{f}(t) = \begin{cases} 0, & t \in [-\delta, \delta] \\ 1, & t \in [1 - \delta, 1 + \delta]. \end{cases}$$

Set $f = \tilde{f}|_{\sigma(a)}$ and $q = f(a) \in C^*(I_A, a)$. Then $q = q^* = q^2$, as $f = \overline{f} = f^2$, so q is a projection. Moreover,

$$\|a - q\| = \|(I_A - f)(a)\| = \|(\text{id} - f)|_{\sigma(a)}\|_\infty \leq \|(\text{id} - f)|_X\|_\infty = \delta.$$

■

2.2.1 Dimension Range and Classification

We introduce an invariant designed for unital C^*-algebras with an abundance of projections that admits a trace. The invariance will be explained in Corollary 2.19. This invariant will help us multiple times in this thesis.

Definition 2.16. Let A be a unital C^*-algebra, and suppose that A admits a tracial state τ. The *dimension range of A with respect to τ* is then defined as

$$D_\tau(A) = \{\tau(p) \mid p \in \text{Proj}(A)\}.$$

If τ is the only tracial state on A, we write $D(A) = D_\tau(A)$.

We first develop some properties of the dimension range:

Proposition 2.17. *Let A be a unital C^*-algebra, and let $\tau \in T(A)$.*
i) $D_\tau(A) \subseteq [0,1]$.
ii) If A is separable, then $D_\tau(A)$ is countable.
iii) If B is another unital C^*-algebra, $\varphi\colon A \to B$ is a unital *-homomorphism and $\rho \in T(B)$, then $\rho \circ \varphi \in T(A)$ and $D_{\rho\circ\varphi}(A) \subseteq D_\rho(B)$.

Proof. **i)** If $p \in \mathrm{Proj}(A)$, then $0 \le p$ and $\|p\| \le 1$. Hence $\tau(p) \ge 0$ as τ is positive, and $|\tau(p)| \le 1$ as $\|\tau\| = 1$.

ii) Let $\{a_n\}_{n\in\mathbb{N}}$ be a dense subset of A. Then $A = \bigcup_{n\in\mathbb{N}} B(a_n, 1/4)$, where $B(x,r)$ is the open ball of radius $r > 0$ around $x \in A$, so

$$D_\tau(A) = \bigcup_{n\in\mathbb{N}} \{\tau(p) \mid p \in \mathrm{Proj}(A) \cap B(a_n, 1/4)\}.$$

If now $p, q \in \mathrm{Proj}(A) \cap B(a_n, 1/4)$, then $\|p - q\| < 1/2$, so Lemma 2.13 gives us that $p \sim q$, and hence $\tau(p) = \tau(q)$. Thus each of the sets on the right hand side above contain at most one point, so $D_\tau(A)$ is countable.

iii) As φ is a unital *-homomorphism, $\rho \circ \varphi$ is a tracial state on A; first off, $\|\rho \circ \varphi\| \le \|\rho\|\|\varphi\| \le 1$, but $\rho \circ \varphi(I_A) = \rho(I_B) = 1$, so $\|\rho \circ \varphi\| = 1$. Also, $\rho \circ \varphi$ is positive as φ is a *-homomorphism. Lastly, $(\rho \circ \varphi)(xy) = (\rho \circ \varphi)(yx)$. If now $p \in \mathrm{Proj}(A)$, then $\varphi(p) \in \mathrm{Proj}(B)$, so

$$D_{\rho\circ\varphi}(A) = \{\rho(\varphi(p)) \mid p \in \mathrm{Proj}(A)\} \subseteq D_\rho(B).$$

■

The next example is given both to illustrate the dimension range and for future reference.

Example 2.18. Let $n \in \mathbb{N}$. Let τ be the unique trace on $M_n(\mathbb{C})$. If $p \in M_n(\mathbb{C})$ is a projection, it can be diagonalized via a unitary u, that is,

$$upu^* = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where the λ_i's are the eigenvalues of p. As p is a projection, $\lambda_i \in \{0,1\}$ for all i. Of course, as τ is a trace, $\tau(p) = \tau(upu^*) = n^{-1}\sum_{i=1}^n \lambda_i = (\#\{i \mid \lambda_i = 1\})/n$. Thus we see that

$$D(M_n(\mathbb{C})) = \{0, 1/n, \dots, (n-1)/n, 1\} = \frac{1}{n}\mathbb{Z} \cap [0,1].$$

The following corollary follows directly from Proposition 2.17.

Corollary 2.19. *Suppose A and B are unital C^*-algebras each with a unique tracial state.*
i) If there is a unital *-homomorphism $\varphi\colon A \to B$, then $D(A) \subseteq D(B)$.
ii) If A and B are isomorphic, then $D(A) = D(B)$.

So far we will only be able to compute the dimension range of C^*-algebras that have a finite amount of projections. However, we can find a simple way to also calculate the dimension range of C^*-algebras that arise as an inductive limit:

Proposition 2.20. *Let A be a unital C^*-algebra containing an increasing sequence $I_A \in A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ of sub-C^*-algebras such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A. Suppose $\tau \in T(A)$. Then*

$$D_\tau(A) = \bigcup_{n\in\mathbb{N}} D_{\tau|_{A_n}}(A_n).$$

Proof. "$\supseteq$" Taking $\varphi\colon A_n \to A$ to be the inclusion we see by Proposition 2.17 iii) that $D_{\tau|_{A_n}}(A_n) \subseteq D_\tau(A)$, and the desired inclusion follows.

"$\subseteq$" Let $p \in \mathrm{Proj}(A)$. As $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A, find some $n \in \mathbb{N}$ and self adjoint $a \in A_n$ such that $\|p - a\| < 1/4$. By Lemma 2.15 i), $\sigma(a) \subseteq [-1/4, 1/4] \cup [3/4, 5/4]$, and then by Lemma 2.15 ii) there exists a projection $q \in C^*(I_A, a) \subseteq A_n$ such that $\|a - q\| \le 1/4$. As then $\|p - q\| < 1/2$, Lemma 2.14 implies that $p \sim q$, whence $\tau(p) = \tau(q) = \tau|_{A_n}(q)$, so $\tau(p) \in D_{\tau_{A_n}}(A_n)$. ■

The above recipe allows us now to prove that an inductive sequence with unital connecting homomorphisms φ_n exists only if $n_k|n_{k+1}$ for all $k \in \mathbb{N}$ and really begin the classification of UHF algebras.

Proposition 2.21. *Let $n, m \in \mathbb{N}$.*
i) There exists a unital *-homomorphism $M_n(\mathbb{C}) \to M_m(\mathbb{C})$ if and only if $n|m$.
ii) If $n|m$ and if $\varphi, \psi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$ are unital *-homomorphisms, then there exists a unitary $u \in M_m(\mathbb{C})$ such that $\psi = u\varphi u^*$.

Proof. **i)** "$\Leftarrow$" This was proven in the start of this section.

"$\Rightarrow$" From Proposition 2.17 iii) we conclude that $D(M_n(\mathbb{C})) \subseteq D(M_m(\mathbb{C}))$, which by Example 2.18 amounts to

$$\frac{1}{n}\mathbb{Z} \cap [0,1] \subseteq \frac{1}{m}\mathbb{Z} \cap [0,1],$$

and this is possible only if $n|m$.

ii) Let τ and τ' denote the (unique) tracial states on $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$, respectively. By Proposition 2.17 iii), $\tau' \circ \varphi$ and $\tau' \circ \psi$ are tracial states on $M_n(\mathbb{C})$, and hence they are equal (to τ). In particular, $\tau'(\varphi(p)) = \tau'(\psi(p))$ for all $p \in \mathrm{Proj}(M_n(\mathbb{C}))$. It is not hard to see by Example 2.18 that this implies $\varphi(p) \sim \psi(p)$ whenever $p \in \mathrm{Proj}(M_n(\mathbb{C}))$.

Let now $(e_{ij})_{i,j=1}^n$ be a system of matrix units for $M_n(\mathbb{C})$. Then $\varphi(e_{11}) \sim \psi(e_{11})$; let $v \in M_m(\mathbb{C})$ be such that $v^*v = \varphi(e_{11})$ and $vv^* = \psi(e_{11})$. Put

$$u = \sum_{j=1}^n \psi(e_{j1})v\varphi(e_{1j}) \in M_m(\mathbb{C}).$$

As the e_{ij} are matrix units, we get

$$\begin{aligned} u\varphi(e_{ij}) &= \sum_{k=1}^n \psi(e_{k1})v\varphi(e_{1k})\varphi(e_{ij}) = \psi(e_{i1})v\varphi(e_{1i})\varphi(e_{ij}) = \psi(e_{i1})v\varphi(e_{1j}),\\ \psi(e_{ij})u &= \sum_{k=1}^n \psi(e_{ij})\psi(e_{k1})v\varphi(e_{1k}) = \psi(e_{ij})\psi(e_{i1})u\varphi(e_{1i}) = \psi(e_{i1})v\varphi(e_{1j}), \end{aligned}$$

so $u\varphi(e_{ij}) = \psi(e_{ij})u$ for all i, j. As $M_n(\mathbb{C})$ is spanned by these matrix units, we conclude that $u\varphi = \psi u$.

A similar calculation shows that u is unitary, and the proof is done. ■

Taking a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $n_k|n_{k+1}$ for all $k \in \mathbb{N}$ (so that the UHF algebra of type (n_k) exists), we can define a supernatural number $N = N[(n_k)]$ that contains all the for us relevant information of the sequence as follows.

Let $\{p_1, p_2, \dots\}$ be the usual enumeration of the prime numbers. Define

$$\alpha_k = \sup\{\alpha \in \mathbb{N}_0 \cup \{\infty\} \mid \exists j \in \mathbb{N} :\ p_k^\alpha | n_j\}, \quad k \in \mathbb{N}.$$

We then define the formal product

$$N = N[(n_k)] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots .$$

More formally, N is actually the sequence $(\alpha_k)_{k\in\mathbb{N}}$ of prime powers.

Note that $N \in \mathbb{N}$ if and only if $\sum_{k\in\mathbb{N}} \alpha_k < \infty$. Also, if $M = p_1^{\beta_1} p_2^{\beta_2} \cdots$ is another supernatural number, we say that $N|M$ if $\alpha_k \le \beta_k$ for all $k \in \mathbb{N}$. Thus $N[(n_k)]$ constructed above is the smallest supernatural number such that $n_k|N$ for every $k \in \mathbb{N}$.

Further, if we are given a supernatural number N, we can define a subgroup $\mathbb{Q}(N)$ of $\mathbb{Q}$ by

$$\mathbb{Q}(N) = \left\{ \frac{p}{q} \;\Big|\; p \in \mathbb{Z},\ q \in \mathbb{N},\ q|N \right\}.$$

Proposition 2.22. *Let A be a UHF algebra of type (n_k), and let $N = N[(n_k)]$ be the associated supernatural number. Then $D(A) = \mathbb{Q}(N) \cap [0,1]$.*

Proof. From Proposition 2.20 and Example 2.18 it follows that

$$D(A) = \bigcup_{n\in\mathbb{N}} \left(\frac{1}{n_k}\mathbb{Z} \cap [0,1] \right).$$

It is furthermore immediate that the right hand side above is equal to $\mathbb{Q}(N) \cap [0,1]$. ■

We now come to the main theorem of this section: Glimm and Elliott's classification of UHF algebras. In the classification of other classes of C^*-algebras, K-theory is widely used. One can construct the so-called $K_0(A)$ and $K_1(A)$ groups for a C^*-algebra A using projections in the same way that we are doing here; so we are actually somewhat doing K-theory without explicitly saying so!

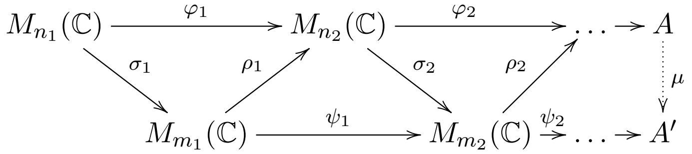
It turns out that the supernatural number $N[(n_k)]$ defined for a UHF algebra of type (n_k) is a complete invariant, and that there is only one UHF algebra of type (n_k) up to isomorphism.

Theorem 2.23 (Classification of UHF algebras)**.** *Let A and A' be UHF algebras of type $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$, respectively. Let N and M be the supernatural numbers associated to the sequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$, respectively. Then the following are equivalent:*
i) $A \cong A'$.
ii) $N = M$.
iii) $\mathbb{Q}(N) = \mathbb{Q}(M)$.
iv) $D(A) = D(A')$.

Proof. "**i) ⇒ iv)**" This is Corollary 2.19 ii).
"**iv) ⇒ iii)**" This follows from Proposition 2.22, as $\mathbb{Q}(N)$ is a countable union of translates of $\mathbb{Q}(N) \cap [0,1]$.
"**iii) ⇒ ii)**" If $\mathbb{Q}(N) = \mathbb{Q}(M)$, then we have the following equivalences for all $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0 \cup \{\infty\}$:

$$p_j^{\alpha}|N \iff p_j^{-\alpha} \in \mathbb{Q}(N) \iff p_j^{-\alpha} \in \mathbb{Q}(M) \iff p_j^{\alpha}|M.$$

This clearly implies that $N = M$.
"**ii) ⇒ i)**" Observe that there are, since $N = M$, subsequences $(n_{k_j})_{j\in\mathbb{N}}$ and $(m_{k'_j})_{j\in\mathbb{N}}$ such that $n_{k_j}|m_{k'_j}$ and $m_{k'_j}|n_{k_{j+1}}$. We can without loss of generality assume for simplicity that $n_j|m_j$ and $m_j|n_{j+1}$; this is because the inductive limit of a subsequence of C^*-algebras is easily checked to be isomorphic to the limit of the whole sequence. We then construct the following commutative diagram:



The top and bottom rows are the inductive sequences defining A and A', respectively. We construct the σ_i's and the ρ_i's as follows:

As $n_1|m_1$, there exists a unital *-homomorphism $\sigma_1 : M_{n_1}(\mathbb{C}) \to M_{m_1}(\mathbb{C})$ by Proposition 2.21 i). Again, using that $m_1|n_2$, there exists a unital *-homomorphism $\rho_1' : M_{m_1}(\mathbb{C}) \to M_{n_2}(\mathbb{C})$. Now, φ_1 and $\rho_1' \circ \sigma_1$ are not necessarily equal, but by Proposition 2.21 ii), there exists a unitary $u \in M_{n_2}(\mathbb{C})$ such that $\varphi_1 = u(\rho_1' \circ \sigma_1)u^*$. Define then $\rho_1 : M_{m_1}(\mathbb{C}) \to M_{n_2}(\mathbb{C})$ by $\rho_1 = u\rho_1' u^*$. Then $\varphi_1 = \rho_1 \circ \sigma_1$. We can now continue this idea to construct the rest of σ_i's and the ρ_i's.

The existence of an isomorphism μ can now be proved; we use the uniqueness of the limit of an inductive sequence. We therefore construct *-homomorphisms f_k from $M_{n_k}(\mathbb{C})$ to A' satisfying $f_{k+1} \circ \varphi_k = f_k$ for all $k \in \mathbb{N}$.

From the diagram we see that the following hold for all $k \in \mathbb{N}$:

$$\begin{aligned} \varphi_k &= \rho_k \circ \sigma_k, \\ \psi_k &= \sigma_{k+1} \circ \rho_k. \end{aligned}$$

As A' is an inductive limit it comes with a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of homomorphisms $\lambda_k : M_{m_k}(\mathbb{C}) \to A'$ for all $k \in \mathbb{N}$. Define $f_k : M_{n_k}(\mathbb{C}) \to A'$ by $f_k = \lambda_k \circ \sigma_k$ for all $k \in \mathbb{N}$. Then every f_k is a *-homomorphism. Also, we see that

$$\begin{aligned} f_{k+1} \circ \varphi_k = \lambda_{k+1} \circ \sigma_{k+1} \circ \varphi_k &= \lambda_{k+1} \circ \sigma_{k+1} \circ \rho_k \circ \sigma_k \\ &= \lambda_{k+1} \circ \psi_k \circ \sigma_k = \lambda_k \circ \sigma_k = f_k. \end{aligned}$$

We therefore get by the uniqueness of the limit that there exists a *-isomorphism $\mu : A \to A'$. ■

Proposition 2.24. *For each supernatural number N, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $n_k|n_{k+1}$ for all $k \in \mathbb{N}$, and whose associated supernatural number is N. In turns there exist a UHF algebra A of type $(n_k)_{k\in\mathbb{N}}$ such that $D(A) = \mathbb{Q}(N) \cap [0,1]$.*

Proof. Write $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$. For each $k \in \mathbb{N}$ define

$$n_k = \prod_{i=1}^{k} p_i^{\max\{\alpha_i, k\}}.$$

Then $(n_k)_{k\in\mathbb{N}}$ is as desired. From Proposition 2.21 we know that the UHF-algebra of type $(n_k)_{k\in\mathbb{N}}$ exists, and then Proposition 2.22 gives us that $D(A) = \mathbb{Q}(N) \cap [0,1]$. ■

Corollary 2.25. *There are uncountably many non-isomorphic UHF algebras.*

Proof. As each supernatural number is uniquely determined by its prime powers, we see that the cardinality of the set of supernatural numbers is the same as the cardinality of all sequences of natural numbers, which is known to be uncountable. Proposition 2.24 and Theorem 2.23 now gives us that there are uncountably many non-isomorphic UHF algebras. ■

If A is the UHF algebra of type (n_k) and N is the associated supernatural number, we shall also say that A is of type N. Moreover, if A is the UHF algebra of type n^∞ for some $n \in \mathbb{N}$, we shall also write $A = M_{n^\infty}(\mathbb{C})$.

Example 2.26. Consider the sequence

$$M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to \cdots \to M_{2^n}(\mathbb{C}) \to \ldots$$

The inductive limit $M_{2^\infty}(\mathbb{C})$ is called the CAR algebra, standing for canonical anticommutative relations. This algebra arises in physics, more precisely in the study of the quantum mechanical studies of fermions, where it is defined in another way. This algebra will come up a few more times in this thesis.

Example 2.27. Consider the sequence

$$M_{p_1}(\mathbb{C}) \longrightarrow M_{p_1^2 p_2}(\mathbb{C}) \longrightarrow M_{p_1^3 p_2^2 p_3}(\mathbb{C}) \longrightarrow \cdots \longrightarrow M_{p_1^n p_2^{n-1} \cdots p_{n-1}^2 p_n}(\mathbb{C}) \longrightarrow \cdots ,$$

where each connecting homomorphism is inclusion. This defines the UHF-algebra of type $N = p_1^\infty p_2^\infty \cdots$, typically denoted $\mathcal{Q}$. It is the universal UHF-algebra in the sense that it contains every other UHF-algebra. It is easily seen using Proposition 2.21 i) that if M is another supernatural number and $M|N$, then the UHF-algebra associated to M lies within $\mathcal{Q}$; and it is clear that $M|N$ for every supernatural number M. This last observation also implies that $\mathbb{Q}(N) = \mathbb{Q}$.

Example 2.28. Taking limits and forming matrix algebras commute; more precisely, if $A = \lim A_k$, then $M_n(A) \cong \lim M_n(A_k)$ for every $n \in \mathbb{N}$.

To see this, let $n \in \mathbb{N}$ and note that for every $k \in \mathbb{N}$ the connecting homomorphism $\varphi_k \colon A_k \to A_{k+1}$ induces a connecting homomorphism $\varphi_k^n \colon M_n(A_k) \to M_n(A_{k+1})$ by

$$\varphi_k^n((x_{ij})_{i,j=1}^n) = (\varphi_k(x_{ij}))_{i,j=1}^n, \quad (x_{ij})_{i,j=1}^n \in M_n(A_k).$$

Thus, if the left diagram below commutes for every $k \in \mathbb{N}$, then the right diagram is easily seen to be commutative for every $k \in \mathbb{N}$. Thus $M_n(A) \cong \lim M_n(A_k)$.

$$\begin{array}{ccc} M_n(A_k) & \xrightarrow{\varphi_k^n} & M_n(A_{k+1}) \\ & \searrow \quad \swarrow & \\ & M_n(A) & \end{array} \qquad \begin{array}{ccc} A_k & \xrightarrow{\varphi_k} & A_{k+1} \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

For the CAR algebra $M_{2^\infty}(\mathbb{C})$ this implies that $M_2(M_{2^\infty}(\mathbb{C})) = M_{2^\infty}(\mathbb{C})$. As noted earlier, we have $M_n(M_m(\mathbb{C})) = M_{nm}(\mathbb{C})$ for every $n, m \in \mathbb{N}$, so as subsequences of $(2^n)_{n\in\mathbb{N}}$ define the same UHF algebra as the whole sequence, we see that

$$M_2(M_{2^\infty}(\mathbb{C})) \cong \lim_{k\to\infty} M_2(M_{2^k}(\mathbb{C})) = \lim_{k\to\infty} M_{2^{k+1}}(\mathbb{C}) = M_{2^\infty}(\mathbb{C}).$$

We by the same argument see that $M_n(\mathcal{Q}) = \mathcal{Q}$ for every $n \in \mathbb{N}$.

2.3 Finite Dimensional Algebras and AF Algebras

We now generalize UHF algebras a bit to get to the so-called AF algebras, which are inductive limits of finite dimensional C^*-algebras. We will at this level not be able to classify AF algebras, as this relies on K-theory, but we present some points where they differ from UHF algebras.

Before we get to the AF algebras, we are going to look in detail at finite dimensional algebras. The aim of this is to completely classify finite dimensional C^*-algebras, and it turns out that this does not actually require a lot of work.

2.3.1 Finite Dimensional C^*-algebras

Given a C^*-algebra of the form

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \dots M_{n_r}(\mathbb{C})$$

for some $r, n_1, \dots, n_r \in \mathbb{N}$, we can describe it by its matrix units $\{e_{ij}^m\}$, where e_{ij}^m is the ij'th standard matrix unit in $M_{n_m}(\mathbb{C})$. The following properties are immediate, and we record it for future reference.

Proposition 2.29. *The above matrix units satisfy the following properties:*
i) $e_{ij}^m e_{jl}^m = e_{il}^m$ *for each* $m = 1, \dots, r$,
ii) $e_{ij}^m e_{kl}^n = 0$ *if* $m \neq n$ *or if* $j \neq k$,
iii) $(e_{ij}^m)^* = e_{ji}^m$,
iv) $A = \operatorname{span}\{e_{ji}^m \mid m \in \{1, \dots, r\},\ i,j \in \{1, \dots, n_m\}\}$.

Note that if B is another C^*-algebra with elements $\{f_{ij}^m \mid m \in \{1, \dots, r\},\ i,j \in \{1, \dots, n_m\}\}$ satisfying i), ii) and iii) of Proposition 2.29, then there is a unique *-homomorphism $\varphi \colon A \to B$ satisfying $\varphi(e_{ij}^m) = f_{ij}^m$ for all m, i, j. Indeed, as A is spanned by the e_{ij}^m's, there is a unique linear map defined by this, and then, using that the elements are matrix units, one easily verifies that the map is a *-homomorphism.

Now, if every one of the f_{ij}^m's are non-zero, then φ is injective, and if $B = \operatorname{span}\{f_{ij}^m\}$, φ is surjective. In particular, if the f_{ij}^m's satisfy i)-iv) in Proposition 2.29 then $A \cong B$.

The next lemma tells us that we can build matrix units and therefore direct sums of matrix algebras if we have "enough" projections.

Lemma 2.30. *Suppose* $\{f_{ii}^m \mid m \in \{1, \dots, r\},\ i \in \{1, \dots, n_m\}\}$ *is a family of mutually orthogonal projections in a* C^**-algebra* B *such that*

$$f_{11}^m \sim f_{22}^m \sim \cdots \sim f_{n_m n_m}^m$$

for every $m = 1, \dots, r$*. Then there exists a system of matrix units* $\{f_{ij}^m\}$ *in* B *that extends the given family* $\{f_{ii}^m\}$*.*

Proof. By assumption there are for each $m \in \{1, \dots, r\}$ partial isometries $\{f_{i1}^m\}$ in B such that

$$(f_{i1}^m)^* f_{i1}^m = f_{11}^m, \quad f_{i1}^m (f_{i1}^m)^* = f_{ii}^m$$

for each $i = 1, \dots, n_m$. Define now

$$f_{ij}^m = f_{i1}^m f_{1j}^m \in B$$

for each $i, j = 1, \dots, n_m$. These are now easily checked to be matrix units for each $m = 1, \dots, r$. For example,

$$f_{ij}^m f_{kl}^m = (f_{i1}^m f_{1j}^m)(f_{k1}^m f_{1l}^m) = \delta_{jk} f_{i1}^m f_{11}^m f_{1l}^m = \delta_{jk} f_{i1}^m f_{1l}^m = \delta_{jk} f_{il},$$

using extensively along the way that $f_{i1}^m f_{1j}^m = 0 = f_{1i}^m f_{j1}^m$ if $i \neq j$; this follows from the f_{ii}^m's being orthogonal. ■

Recall that a sub-C^*-algebra D of a C^*-algebra A is a masa (maximal abelian sub-C^*-algebra) if and only if $\{x \in A \mid xd = dx \text{ for every } d \in D\} = D$, and that masas exist by a standard application of Zorn's lemma. We need the following lemma for proving out main theorem of this section:

Lemma 2.31. *Let* A *be a* C^**-algebra, and let* $D \subseteq A$ *be a masa.*
i) If $a \in A$ *commutes with all elements of* D*, then* $a \in D$*.*
ii) If D *is unital, then* A *is unital, and the unit of* D *is equal to the unit of* A*.*
iii) If $p \in D$ *is a projection such that* $pDp = \mathbb{C}p$*, then also* $pAp = \mathbb{C}p$*.*

Proof. **i)** This follows immediately from the observation before this lemma.
ii) We show that $aI_D = a$ for each $a \in A$; this will imply that I_D is the unit for A, as then also $I_D a = (a^* I_D)^* = a$. Let therefore $a \in A$, and set $z = a - aI_D$. Then for each $d \in D$ we have $zd = 0$. As D is self-adjoint, this implies that $dz^* = 0$ for every $d \in D$. Thus we get for some $d \in D$ that

$$(z^* z)d = 0 = d(z^* z),$$

and thus by i), $z^*z \in D$, and by the above calculation, $(z^*z)(z^*z) = 0$. We then get

$$0 = \|(z^*z)(z^*z)\| = \|z^*z\|^2 = \|z\|^4,$$

and thus $z = 0$ as wanted.
iii) Let $a \in pAp$ and $d \in D$. As p is a projection and commutes with every element of D by i), we get $pd = pdp = \lambda p$ for some $\lambda \in \mathbb{C}$. We then get

$$ad = apd = a\lambda p = \lambda a = \lambda pa = da,$$

and thus by i), $a \in D$, and hence $a = pap \in pDp = \mathbb{C}p$ as wanted. ■

We can now prove the main theorem of this section: that every finite dimensional C^*-algebra is isomorphic to some direct sum of matrix algebras.

Theorem 2.32. *For each finite dimensional algebra A there exist $r, n_1, \ldots, n_r \in \mathbb{N}$ such that*

$$A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C}).$$

Proof. Let A be a finite dimensional C^*-algebra, and choose a masa $D \subseteq A$. As D is commutative, $D \cong C_0(X)$ for some locally compact Hausdorff space X. Now, if X had infinitely many points, D would be infinite dimensional, but this is impossible; thus X has finitely many points $x_1, \ldots, x_N$. In particular, X is compact, $D \cong C(X)$, and D is unital. Hence also A is unital by Lemma 2.31 ii).

Let $\{p_1, \ldots, p_N\} \subseteq D$ be the elements corresponding to the functions $q_1, \ldots, q_N \in C(X)$ satisfying $q_j(x_i) = \delta_{ji}$. Then we see that the q_j's are projections, $\sum_{j=1}^N q_j = 1$, and for some $f \in C(X)$ we have

$$(q_j f q_j)(x_i) = f(x_i)\delta_{ij} = f(x_i)q_j(x_i), \quad x_i \in X, \quad j = 1, \ldots, N,$$

and thus $q_j C(X) q_j = \mathbb{C}q_j$. We therefore get that $p_1, \ldots, p_N$ are projections in D, $\sum_{i=1}^N p_i = I_A$ and $p_j D p_j = \mathbb{C}p_j$ for every j. Thus by Lemma 2.31 iii), $p_j A p_j = \mathbb{C}p_j$ for every j.

Assume now that $p_j A p_i \neq \{0\}$, and choose $p_j b p_i = v \in p_j A p_i$ such that $\|v\| = 1$. Then $v^*v = p_i b^* p_j p_j b p_i$ is a positive element in $p_i A p_i$ of norm 1, so we must have $v^*v = p_i$. In the same way, $vv^* = p_j$. If now $p_j a' p_i = a \in p_j A p_i$, we have $a = ap_i = (av^*)v$. Now, $av^* = p_j a' p_i p_i b^* p_j \in p_j A p_j = \mathbb{C}p_j$, and $p_j v = p_j p_j b p_i = v$, so we conclude that $a \in \mathbb{C}v$.

Therefore, we now have two mutually exclusive cases for $i, j \in \{1, \ldots, N\}$:
1) $p_j A p_i = \{0\}$,
2) $p_i \sim p_j$ in A, and $p_j A p_i = \mathbb{C}v$.

Now partition the set $\{p_1, \ldots, p_N\}$ into equivalence classes; let r be the number of these classes, and let n_k be the number of elements in the k'th equivalence class. Rename the p_i's such that the elements of the k'th equivalence class are $f_{11}^k, f_{22}^k, \ldots, f_{n_k n_k}^k$. We then have

$$\{p_1, \ldots, p_N\} = \{f_{ii}^k \mid k \in \{1, \ldots, r\},\ i \in \{1, \ldots, n_k\}\},$$

$f_{ii}^k A f_{jj}^l = \{0\}$ if $k \neq l$ and $f_{ii}^k \sim f_{jj}^k$ for every i, j, k by the mutually exclusive cases. Also, all the f_{ii}^k's are mutually orthogonal. They can thus by Lemma 2.30 be extended by a system of matrix units $\{f_{ij}^k\}$ in A. By the above we have $f_{ii}^k A f_{jj}^k = \mathbb{C}f_{ij}^k$ for every i, j, k. Now,

$$I_A = \sum_{i=1}^N p_i = \sum_{k=1}^r \sum_{i=1}^{n_k} f_{ii}^k,$$

so for $a \in A$ we see that

$$a = \left(\sum_{i,k} f_{ii}^k\right) a \left(\sum_{i,k} f_{ii}^k\right) = \sum_{k=1}^r \sum_{i,j=1}^{n_k} f_{ii}^k a f_{jj}^k = \sum_{k=1}^r \sum_{i,j=1}^{n_k} \lambda_{ij}^k f_{ij}^k$$

for some finite family $\{\lambda_{ij}^k\} \subset \mathbb{C}$. This proves that

$$A = \operatorname{span}\{f_{ij}^k \mid k \in \{1, \dots, r\},\ i, j \in \{1, \dots, n_k\}\}.$$

All in all, we have shown that the system $\{f_{ij}^k\}$ satisfies i)-iv) of Proposition 2.29, and hence by the discussion after the proposition, we now have

$$A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}),$$

and the proof is finished. ∎

Note that this theorem completely classifies finite dimensional C^*-algebras:

Corollary 2.33 (Classifcation of finite dimensional C^*-algebras)**.** *Up to interchanging the order of the direct summands, two finite dimensional C^*-algebras $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ and $B = M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_s}(\mathbb{C})$ are isomorphic if and only if $s = r$ and $n_i = m_i$ for every $1 \leq i \leq r$.*

Example 2.34. Suppose we are given a 6-dimensional C^*-algebra A. By Theorem 2.32, A is isomorphic to a direct sum of matrix algebras whose dimensions correspond to a partition of 6 into quadratics. There are two of these, namely $1+1+1+1+1+1$ and $1+1+2^2$. Therefore, if A is commutative, then $A \cong \mathbb{C}^6$. If A is non-commutative, we then get $A \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$.

2.3.2 AF Algebras

We now put together finite dimensional C^*-algebras and inductive limits to get the AF algebras. We will not be able to prove deep results about these, but we present an introductory discussion.

Definition 2.35. An *AF algebra* is a C^*-algebra which is (isomorphic to) the inductive limit of an inductive sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$$

where each A_n is a finite dimensional C^*-algebra.

AF stands for approximately finite-dimensional. By Theorem 2.32 and Theorem 2.7 i) every AF algebra is separable as each matrix algebra is. Also, we see immediately that UHF algebras are in particular AF algebras.

If we have a finite dimensional C^*-algebra $A = M_{n_1}(\mathbb{C}) \oplus \dots M_{n_r}(\mathbb{C})$ and we assume $r > 1$, then it is immediate that $M_{n_i}(\mathbb{C})$ is a proper ideal in A for every $1 \leq i \leq r$. Hence AF algebras are not in general simple; we could define an AF algebra by the inductive sequence

$$A \xrightarrow{\text{id}} A \xrightarrow{\text{id}} A \xrightarrow{\text{id}} \dots,$$

and the limit would clearly be A itself. To take a more interesting example, consider the sequence

$$M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \xrightarrow{\varphi_1} M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \xrightarrow{\varphi_2} M_8(\mathbb{C}) \oplus M_8(\mathbb{C}) \xrightarrow{\varphi_3} \dots$$

with the connecting homomorphisms $\varphi_n \colon M_{2^n}(\mathbb{C}) \oplus M_{n^2}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C}) \oplus M_{2^{n+1}}(\mathbb{C})$ given by

$$\varphi_n(x, y) = \left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \right), \quad x, y \in M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}).$$

One can fairly easy check that the limit is isomorphic to $M_{2^\infty}(\mathbb{C}) \oplus M_{2^\infty}(\mathbb{C})$, where each copy of $M_{2^\infty}(\mathbb{C})$ is an ideal, and we see, given a proper ideal $\mathcal{I} \lhd M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$, that we must have

$$\mathcal{I} = (\{0\} \oplus M_{2^\infty}(\mathbb{C})) \cap (M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}))$$

or the other way around because of the simplicity of $M_k(\mathbb{C})$. This is in fact generally true; given an AF algebra A arising from an inductive sequence $A_1 \to A_2 \to A_3 \to \dots$ with unital connecting homomorphisms, then every ideal $\mathcal{I} \trianglelefteq A_n$ arises as $\mathcal{J} \cap A_n$ for some $\mathcal{J} \trianglelefteq A$. The proof of this little fact is the same as the one outlined in the concrete example above.

Another point where AF algebras differ from UHF algebras is that AF algebras need not be unital; we have not made any requirements on the connecting homomorphisms in the sequence defining AF algebras, whereas we require them to be unital to define a UHF algebra. As a classical example of this, let X_n be a set with n points for every $n \in \mathbb{N}$, and equip it with the discrete topology. Let then $A_n = C(X_n)$ with $\varphi_n \colon A_n \to A_{n+1}$ being inclusion. Note that φ_n is then not a unital *-homomorphism. The limit of the inductive sequence defined by this is isomorphic to $C_0(\mathbb{N})$, which is non-unital.

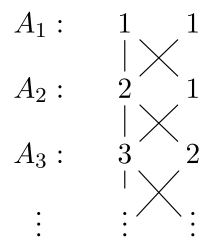
2.3.3 Bratteli Diagrams

A very nice way of describing the inductive sequence of finite dimensional C^*-algebras are the so-called Bratteli diagrams.

One such example is to consider the Fibonacci numbers; $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$. Put $A_n = M_{f_n}(\mathbb{C}) \oplus M_{f_{n-1}}(\mathbb{C})$, and define $\varphi_n \colon A_n \to A_{n+1}$ by

$$\varphi_n(x, y) = \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \right), \quad x \in M_{f_n}(\mathbb{C}), \quad y \in M_{f_{n-1}}(\mathbb{C}).$$

We then have an inductive sequence $A_1 \overset{\varphi_1}{\to} A_2 \overset{\varphi_2}{\to} A_3 \overset{\varphi_3}{\to} \cdots$, which defines some AF-algebra. The relevant information in this sequence can be found in the following Bratteli diagram:

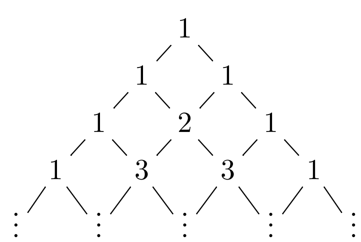


where the numbers indicate the square roots of the dimensions of the matrix algebras in play.

Generally, the n'th row in this Bratteli diagram consist of two vertices, k_1 and k_2, corresponding to the dimensions of the direct summands of A_n. From each vertex we have a number of edges going down to some vertex in the $(n+1)$'th row, indicating where the element from each vertex gets sent by φ_n. The number of edges from one vertex to another is the multiplicity of the *-homomorphism $\pi \circ \varphi_n \circ \iota$, where

$$M_k(\mathbb{C}) \overset{\iota}{\to} A_n \overset{\varphi_n}{\to} A_{n+1} \overset{\pi}{\to} M_l(\mathbb{C}).$$

Another Bratteli diagram defining an AF algebra is the following:



This Pascal's triangle is induced by

$$A_n = \bigoplus_{i=1}^{n} M_{\binom{n}{i}}(\mathbb{C})$$

and $\varphi_n\colon A_n \to A_{n+1}$ defined by

$$\varphi_n(x_1,\dots,x_n) = \left(x_1, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ 0 & x_3 \end{pmatrix}, \dots, \begin{pmatrix} x_{n-2} & 0 \\ 0 & x_{n-3} \end{pmatrix}, \begin{pmatrix} x_{n-1} & 0 \\ 0 & x_n \end{pmatrix}, x_n\right),$$
$$x_i \in M_{\binom{n}{i}}(\mathbb{C}),$$

with $n+1$ coordinates of various matrix sizes.

The last diagram we are going to present here is the following:

$$\begin{array}{c} 1 \\ \| \\ 2 \\ \| \\ 4 \\ \| \\ \vdots \end{array}$$

This one is actually familiar; the connecting homomorphisms $\varphi_k\colon M_{2^k}(\mathbb{C}) \to M_{2^{k+1}}(\mathbb{C})$ is given by

$$\varphi_k(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad x \in M_{2^n}(\mathbb{C}),$$

and we recognize this as the sequence defining the UHF algebra of type 2^∞.

One of the nice things about Bratteli diagrams is that we can read off some of the ideal structure of the AF algebra defined by it. For example, in the Pascal's triangle above, one can pick any vertex and forget everything but the vertices where there is a path downwards from the chosen vertex; the AF algebra defined by this Bratteli diagram is an ideal in the big AF algebra. So the AF algebra defined by the Pascal's triangle has indeed very many ideals.

2.4 Comparison of AF and UHF Algebras

We have noted that there are a number of places where the UHF and AF algebras differ from each other. One more place is the case of tracial states. By Lemma 2.11 ii), each UHF algebra admits a unique tracial state. However, the AF algebras need not even have a tracial state. They can also have uncountably many.

One could be lead to think that AF algebras are only a slight generalization of UHF algebras, but we get a very different class of C^*-algebras.

We now collect the observations we have made regarding differences between UHF and AF algebras.

Property	UHF algebra	AF algebra
Separable	Yes	Yes
$T(A) \neq \emptyset$	Yes	No
Unital	Yes	No
Simple	Yes	No

3 Group C^*-algebras

Throughout this section G will denote a discrete countable group, written multiplicatively with neutral element e. We consider the Hilbert space $H := \ell^2(G)$ with the usual inner product $\langle \xi, \eta \rangle = \sum_{g\in G} \xi(g)\overline{\eta(g)}$. We let $\{\delta_g\}_{g\in G} \subset H$ denote the usual orthonormal basis.

We now define for every $t \in G$ an operator λ_t on H by

$$(\lambda_t\xi)(g) = \xi(t^{-1}g), \quad \xi \in H, \quad g \in G.$$

To see that $\lambda_t\xi \in H$ for every $t \in G$ and $\xi \in H$, we recall from group theory that for every $g \in G$ we have

$$\{gh \mid h \in G\} = \{h \mid h \in G\} = G, \tag{3.1}$$

so that

$$\sum_{g\in G} |(\lambda_t\xi)(g)|^2 = \sum_{g\in G} |\xi(t^{-1}g)|^2 = \sum_{g\in G} |\xi(g)|^2 < \infty, \quad t \in G.$$

This further shows that λ_t is isometric for every $t \in G$. Also, it is immediate that each λ_t is linear, so that we have $\lambda_t \in B(H)$ for every $t \in G$.

We now develop some calculus of these operators:

Proposition 3.1. *Let $s, t, g \in G$. Then*

$$\lambda_t\delta_g = \delta_{tg}, \quad \lambda_{st} = \lambda_s\lambda_t, \quad \lambda_t^* = \lambda_{t^{-1}}, \quad \lambda_e = I.$$

Proof. Let $h \in G$. We see that $(\lambda_t\delta_g)(h) = \delta_g(t^{-1}h) = 1$ if and only if $g = t^{-1}h$ if and only if $h = tg$, and 0 else. Also, $\delta_{tg}(h) = 1$ if and only if $h = tg$ and 0 else. Thus $\lambda_t\delta_g = \delta_{tg}$.

Let now $\xi \in H$. We can for every $g \in G$ write

$$\xi(g) = \sum_{h\in G} f(h)\delta_h(g). \tag{3.2}$$

Using the first property, we get for all $g \in G$ that

$$(\lambda_{st}\xi)(g) = \sum_{h\in G} \xi(h)\delta_{sth}(g) = \lambda_s \sum_{h\in G} \xi(g)\delta_{th}(g) = \lambda_s\lambda_t\xi(g),$$

and thus $\lambda_{st}\xi = \lambda_s\lambda_t\xi$. As $\xi \in H$ was arbitrary we conclude that $\lambda_{st} = \lambda_s\lambda_t$.

It suffices to calculate on the δ_g's to determine λ_t^*. Let therefore $g, h \in G$. We then see that $\langle \lambda_t\delta_g, \delta_h \rangle = \langle \delta_{tg}, \delta_h \rangle = 1$ happens if and only if $g = t^{-1}h$, which occurs if and only if $\langle \delta_g, \delta_{t^{-1}h} \rangle = \langle \delta_g, \lambda_{t^{-1}}\delta_h \rangle = 1$. Doing similar calculations for $\langle \lambda_t\delta_g, \delta_h \rangle = 0$, we conclude that $\lambda_t^* = \lambda_{t^{-1}}$.

Finally, the last statement is trivial by the definition of λ_e. ∎

By these properties, we see that λ is a unitary representation of G on H, meaning that each $\lambda_t \in B(H)$ is a unitary for every $t \in G$ and that $t \mapsto \lambda_t$ is a group homomorphism.

Remark 3.2. One can also define operators ρ_t on H by

$$(\rho_t\xi)(g) = \xi(gt), \quad g \in G.$$

In the same manner as in Proposition 3.1, one can show that $\rho_t\delta_g = \delta_{gt^{-1}}$ for all $g, t \in G$ and that ρ is a unitary representation of G on H. Moreover, it is easily seen that $\rho_s\lambda_t = \lambda_t\rho_s$ for all $s, t \in G$.

We will use $\{\lambda_t \mid t \in G\}$ as a basis for the C^*-algebra we are going to construct. First we need to know that this is possible:

Lemma 3.3. *The set $\{\lambda_t \mid t \in G\}$ consists of linearly independent elements.*

Proof. We need to prove that every finite subset of $\{\lambda_t \mid t \in G\}$ is linearly independent, but we will only prove this for subsets with 2 elements. The general case works in the exact same way.

Let therefore $s, t \in G$, $s \neq t$, and $a, b \in \mathbb{C}$. Assume that $a\lambda_s + b\lambda_t = 0$. It again suffices to calculate on the δ_g's. Let therefore $g \in G$. Then we see that

$$0 = (a\lambda_s + b\lambda_t)\delta_g = a\delta_{sg} + b\delta_{tg}.$$

But as $s \neq t$, δ_{sg} and δ_{tg} are linearly independent, so this implies that $a = b = 0$, so λ_s and λ_t are linearly independent. ■

Definition 3.4. Define

$$C_\lambda(G) = \operatorname{span}\{\lambda_t \mid t \in G\} \subseteq B(H),$$

and let $C^*_\lambda(G)$ be the norm closure of $C_\lambda(G)$. $C^*_\lambda(G)$ is called the *reduced group C^*-algebra of G*.

By Proposition 3.1, $C_\lambda(G)$ is a *-algebra and $I \in C_\lambda(G)$, so $C^*_\lambda(G)$ is indeed a unital C^*-algebra. Note also that $C^*_\lambda(G)$ is commutative if and only if G is abelian by the second property in Proposition 3.1.

From ring theory we know the group algebra $\mathbb{C}G$, which consists of finite $\mathbb{C}$-linear combinations of elements from G. Thus it is easy to see that $\mathbb{C}G \cong C_\lambda(G)$ as algebras under the map $g \mapsto \lambda_g$.

3.1 Traces

We now come to the main theorem of this section; that $C^*_\lambda(G)$ has a faithful tracial state. This will be rather restrictive, as we approach classes of C^*-algebras differently depending on whether they admit a tracial state or not.

Theorem 3.5. *$C^*_\lambda(G)$ admits a faithful tracial state τ.*

Proof. Define $\tau\colon C^*_\lambda(G) \to \mathbb{C}$ by

$$\tau(x) = \langle x\delta_e, \delta_e\rangle, \quad x \in C^*_\lambda(G).$$

It is immediate that τ is linear.

Let $x \in C^*_\lambda(G)$ be positive; $x = y^*y$ for some $y \in C^*_\lambda(G)$. Then we see that

$$\tau(x) = \langle y^*y\delta_e, \delta_e\rangle = \langle y\delta_e, y\delta_e\rangle = \|y\delta_e\|^2 \geq 0,$$

so that τ is positive, and hence τ is bounded with $\|\tau\| = \tau(I)$. As $\tau(I) = \langle\delta_e, \delta_e\rangle = 1$, we get that τ is a state.

To show that τ is a trace, we prove that τ is a trace on $\{\lambda_t \mid t \in G\}$. Let therefore $s, t \in G$. Then

$$\tau(\lambda_s\lambda_t) = \langle\lambda_{st}\delta_e, \delta_e\rangle = \sum_{g\in G} \lambda_{st}\delta_e(g)\overline{\delta_e(g)} = \sum_{g\in G} \delta_{st}(g)\overline{\delta_e(g)} = \delta_{st}(e).$$

In the same way, $\tau(\lambda_t\lambda_s) = \delta_{ts}(e)$. We now see that $\tau(\lambda_s\lambda_t) = \tau(\lambda_t\lambda_s)$ as $st = e$ if and only if $ts = e$, so that τ is a trace on $\{\lambda_t \mid t \in G\}$. By linearity and continuity, τ is therefore also a trace on $C^*_\lambda(G)$.

It follows from the remark after Proposition 3.1 by a standard continuity argument that $\rho_t x = x\rho_t$ for every $t \in G$ and $x \in C^*_\lambda(G)$. Letting $t \in G$ and $\in C^*_\lambda(G)$, we see that

$$\tau(x) = \langle x\delta_e, \delta_e\rangle = \langle x\rho_t\rho_{t^{-1}}\delta_e, \delta_e\rangle = \langle \rho_t x\delta_t, \delta_e\rangle = \langle x\delta_t, \delta_t\rangle.$$

Now, if $x \in C^*_\lambda(G)$ such that $\tau(x^*x) = 0$, we then see that

$$0 = \langle x^*x\delta_t, \delta_t\rangle = \langle x\delta_t, x\delta_t\rangle = \|x\delta_t\|^2, \quad t \in G.$$

This implies that $x\delta_t = 0$ for all $t \in G$, and as these form a basis for H, this implies that $x = 0$. Hence τ is a faithful tracial state on $C^*_\lambda(G)$. ■

As in Corollary 2.12, this result implies that $G^*_\lambda(G)$ is finite.

The existence of a faithful tracial state on $C^*_\lambda(G)$ makes this C^*-algebra rather special. Historically, the existence of a tracial state has made it harder to classify C^*-algebras, as there is one more thing to take into account.

We close this section off by considering a class of projections in $C^*_\lambda(G)$, as projections are of great importance in the rest of this thesis.

Example 3.6. Let H be a finite subgroup of G. Define

$$p := \frac{1}{\#H}\sum_{g\in H}\lambda_g.$$

As H is a subgroup,

$$p^* = \frac{1}{\#H}\sum_{g\in H}\lambda_g^* = \frac{1}{\#H}\sum_{g\in H}\lambda_{g^{-1}} = \frac{1}{\#H}\sum_{g\in H}\lambda_g = p$$

by Proposition 3.1. Further,

$$\begin{aligned} p^2 &= \frac{1}{\#H}\sum_{g\in H}\left(\lambda_g\frac{1}{\#H}\sum_{h\in H}\lambda_h\right) = \frac{1}{(\#H)^2}\sum_{g\in H}\sum_{h\in H}\lambda_{gh} \\ &\overset{(3.1)}{=} \frac{\#H}{(\#H)^2}\sum_{g\in H}\lambda_g = p. \end{aligned}$$

Thus p is a projection in $C^*_\lambda(G)$.

Finally, as H is finite, p is a finite sum, so we see that

$$\tau(p) = \frac{1}{\#H}\sum_{g\in H}\tau(\lambda_g) = \frac{1}{\#H}\sum_{g\in H}\langle\lambda_g\delta_e, \delta_e\rangle = \frac{1}{\#H}\sum_{g\in H}\langle\delta_g, \delta_e\rangle = \frac{1}{\#H}.$$

Kadison conjectured that if a group does not have an element of finite order, and thus not a finite subgroup, then the reduced group C^*-algebra does not have a non-trivial projection. This has still not been verified. There are however examples that support this. One of these is the reduced C^*-algebra of the free group with 2 generators.

3.2 An example

Let $G = \mathbb{Z}$. As $\mathbb{Z}$ is generated by one element, namely $1 \in \mathbb{Z}$, it is easily seen by Proposition 3.1 that $C^*_\lambda(\mathbb{Z}) = C^*(\lambda_1)$. As $\mathbb{Z}$ is abelian and $C^*_\lambda(\mathbb{Z})$ is unital, $C^*(\lambda_1) \cong C(\sigma(\lambda_1))$. We now show that $\sigma(\lambda_1) = \mathbb{T}$. Since λ_1 is unitary, we have $\sigma(\lambda_1) \subseteq \mathbb{T}$. To show the other inclusion, let $z \in \mathbb{T}$ and set

$$\mu_{k,z} := k^{-1/2}\sum_{j=1}^{k}\overline{z}^j\delta_j \in \ell^2(\mathbb{Z})$$

for every $k \in \mathbb{N}$. By the Pythagorean theorem we see that $\|\mu_{k,z}\|^2 = k^{-1}\sum_{j=1}^{k}\|\overline{z}^j\delta_j\|^2 = 1$. Moreover,

$$\lambda_1\mu_{k,z} = k^{-1/2}\sum_{j=1}^{k}\overline{z}^j\delta_{j+1} = k^{-1/2}z\sum_{j=2}^{k+1}\overline{z}^{j-1}\delta_j.$$

This implies, again by the Pythagorean theorem, that

$$\|(z\cdot I - \lambda_1)\mu_{k,z}\| = \left\| k^{-1/2}\sum_{j=1}^{k}\overline{z}^j(z\delta_j - z\delta_{j+1})\right\| = \frac{2}{\sqrt{k}} \to 0 \quad \text{as } k\to\infty.$$

Now recall the following result:

Proposition 3.7. *Let H be a Hilbert space, $T \in B(H)$ and $\lambda \in \mathbb{C}$. Suppose there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ such that $\|x_n\| = 1$ for every n and $\|(\lambda\cdot I - T)x_n\| \to 0$ as $n\to\infty$. Then $\lambda \in \sigma(T)$.*

This, with the computations above, implies that $z \in \sigma(\lambda_1)$. Hence we get $\sigma(\lambda_1) = \mathbb{T}$. We have thus proven now that

$$C^*_\lambda(\mathbb{Z}) = C(\mathbb{T}).$$

4 Universal C^*-algebras

Some C^*-algebras in a given class are more interesting to look at than others. This can for example be because the interesting C^*-algebras contain other C^*-algebras very much like them. One example of this was seen in Example 2.27, where we from each given UHF algebra can find a canonical *-homomorphism into the universal UHF algebra $\mathcal{Q}$, namely the inclusion.

We will examine this concept of universality now. The examples we are going to look at, the irrational rotation algebras and the Cuntz algebras, will be constructed to fulfill a set of relations, and these will also be universal in the sense that for each C^*-algebra generated by elements satisfying the same relations, we can find a canonical *-homomorphism into them.

4.1 Constructing universal C^*-algebras

Suppose we are given a set of elements S that fulfill some relations, R, and we wish to construct a C^*-algebra from this, $C^*\langle S, R\rangle$. If the elements were already in some C^*-algebra, R could be the relations from that given algebra, and we know how to construct the sought C^*-algebra. However, we do not always have an ambient algebra.

We need to work with words in the generators to construct an algebra if we do not already have it. For some $k \in \mathbb{N}$, let $\mathcal{W}_n(S)$ denote the set of words of length k in $S \cup S^*$, where S^* a priori is just the set $\{s^* \mid s \in S\}$, as the adjoint operation has not yet been defined. We define $\mathcal{W}_0(S) = \{1\}$, the "empty word". Set $\mathcal{W}_\infty(S) = \bigcup_{n\in\mathbb{N}_0} \mathcal{W}_n(S)$ and define

$$\mathcal{U}(S) = \operatorname{span}\left(\mathcal{W}_\infty(S)\right),$$

called the universal *-algebra generated by S. It is easy to see that $\mathcal{U}(S)$ is in fact a *-algebra, where we define multiplication by concatenation, and adjoints are defined in the usual way.

We now turn our attention to R. To make the relations into a set we can work with, we construct what is called relators, which are elements in $\mathcal{U}(S)$ that we want to be 0. If for example one relation among the elements $a, b \in S$ is $ab = ba$, the corresponding relator will be $ab - ba \in \mathcal{U}(S)$. We can therefore in $\mathcal{U}(S)$ construct the ideal $\langle R\rangle$ generated by the set of relators. Then define

$$\mathcal{U}\langle S, R\rangle = \mathcal{U}(S)/\langle R\rangle,$$

the universal *-algebra generated by S with relations R.

To next turn $\mathcal{U}\langle S, R\rangle$ into a C^*-algebra, we find all *-representations $\{(\pi_\alpha, H_\alpha)\}_{a\alpha\in\Lambda}$ of $\mathcal{U}\langle S, R\rangle$. We now have two choices: we can let

$$\pi = \bigoplus_{\alpha\in\Lambda} \pi_\alpha \colon \mathcal{U}\langle S, R\rangle \to H = \bigoplus_{\alpha\in\Lambda} H_\alpha$$

be the universal representation and for $x \in \mathcal{U}\langle S, R\rangle$ set $\|x\|_1 = \|\pi(x)\|$, or we can set $\|x\|_2 = \sup_{\alpha\in\Lambda} \|\pi_\alpha(x)\|$.

The question is now, are any of the above norms well-defined? That is, do we have $\|x\|_i < \infty$ for all $x \in \mathcal{U}\langle S, R\rangle$? Not always, but we can make it so.

Proposition 4.1. *If we in the above construction have $\|s\|_1 < \infty$ or $\|s\|_2 < \infty$ for every $s \in S$, that norm is well defined.*

This result is immediate. Of course, in this case we have $\|\cdot\|_1 = \|\cdot\|_2$.

Another question is whether $\|x\| = 0$ implies that $x = 0$. Again, that is not always the case, as there could be some weird *-representations, but if not, we can let $\mathcal{J}$ be the ideal generated by $\{x \in \mathcal{U}\langle S, R\rangle \mid x \neq 0, \|x\| = 0\}$ and work with $\mathcal{U}\langle S, R\rangle/\mathcal{J}$. However, if the answer to both of the above questions are yes, then we let $C^*\langle S, R\rangle$ be the completion of $\mathcal{U}\langle S, R\rangle$ with respect to either one of the above norms. Then $C^*\langle S, R\rangle$ is indeed a C^*-algebra; as the norm is defined via the norm on $B(H)$ for some Hilbert space H, this is a C^*-norm.

Example 4.2. Taking some $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and setting

$$S = \{u, v\}, \qquad R = \{u^*u = I = uu^*, v^*v = I = vv^*, uv = e^{2\pi i\theta}vu\},$$

we in the above construction get the irrational rotation algebra $\mathcal{A}_\theta$. We will work in greater detail with this C^*-algebra and prove everything stated here later. In this case, $\langle R \rangle$ is the ideal generated by the elements $I - u^*u$, $I - uu^*$, $e^{2\pi i\theta}vu - uv$ and so forth. Also, since we require u and v to be unitaries, their norm is finite (and equal to 1), so by Proposition 4.1, the norm on $C^*\langle S, R\rangle$ is well-defined.

Lastly, we will see that $\mathcal{A}_\theta$ is simple. We will also see how this implies that if we have some Hilbert space H and two unitaries $u', v' \in B(H)$ satisfying R, we get $C^*(u', v') \cong \mathcal{A}_\theta$ with the isomorphism sending u' to u and v' to v.

Example 4.3. More generally, given generators S and relations R, if we have some minimal set $S_0 \subseteq B(H)$ for some Hilbert space H that fulfill R (this ensures that $\#S_0 = \#S$), we get a homomorphism from $C^*\langle S, R\rangle$ to $C^*(S_0) \subseteq B(H)$ sending each element in S to the corresponding element in S_0. If moreover $C^*\langle S, R\rangle$ turns out to be simple, this homomorphism is necessarily an isomorphism. This is the concept of universality and the reason we call C^*-algebras generated this way universal; we can find a canonical homomorphism from the given C^*-algebra to other C^*-algebras like it.

The second example justifies how we can sometimes talk about *the* universal C^*-algebra satisfying some relations; in this case we actually talk about the isomorphism class of the concrete C^*-algebra.

4.2 Irrational Rotation Algebras

The first example of constructions of universal C^*-algebras we are going to present is that of irrational rotation algebras. These arise quite fast when looking at L^2-spaces, as one can construct these via $L^2(\mathbb{T})$. We are going to develop some properties of these algebras, but we are going to weakly classify them as the classification of this class of C^*-algebras requires K-theory.

4.2.1 Construction and Uniqueness

For the rest of this section, let θ be a fixed irrational number. Although the construction we are going to do also works for rational values, we shall only in detail consider the irrational case, but we give an example of a rational value at the end of the section.

Consider the circle $\mathbb{T}$, thought of as $\mathbb{R}/\mathbb{Z} = \{t \bmod 1 \mid t \in \mathbb{R}\}$ via the map $z(t) = e^{2\pi i t}$, $t \in \mathbb{R}/\mathbb{Z}$. Consider the following two operators on $H := L^2(\mathbb{T})$: the operator u of multiplication by the function z and the operator v of rotation by θ. Formally,

$$uf(t) = z(t)f(t), \quad vf(t) = f(t - \theta), \quad f \in H, t \in \mathbb{R}/\mathbb{Z}.$$

It is easily seen that $u^*f(t) = \overline{z(t)}f(t)$, $v^*f(t) = f(t + \theta)$, and that u and v are unitary.

Calculating on $f \in H$ and $t \in \mathbb{R}/\mathbb{Z}$ we obtain

$$\begin{aligned} vuf(t) = v(z(t)f(t)) = z(t - \theta)f(t - \theta) &= e^{2\pi i(t-\theta)}vf(t) \\ = e^{-2\pi i\theta}z(t)vf(t) &= e^{-2\pi i\theta}uvf(t), \end{aligned}$$

that is, u and v satisfy the relation

$$uv = e^{2\pi i\theta}vu. \tag{4.1}$$

This is the idea behind our construction: we wish to construct a universal C^*-algebra with $S = \{u, v\}$, and the relations R being $u^*u = I = uu^*$, $v^*v = I = vv^*$, $uv = e^{2\pi i\theta}vu$. We note for future use that $\sigma(u) = \sigma(v) = \mathbb{T}$.

The best idea here is actually not to construct $C^*\langle S, R\rangle$ abstractly like we did in theory; since we have constructed unitaries satisfying (4.1), we can consider the non-empty collection of all pairs $\{(u_\alpha, v_\alpha)\}_{\alpha\in\Lambda}$ of unitaries satisfying (4.1), where $u_\alpha, v_\alpha \in B(H_\alpha)$ for all α, and $\{H_\alpha\}_{\alpha\in\Lambda}$ is a collection of Hilbert spaces. Inside $\bigoplus_{\alpha\in\Lambda} B(H_\alpha)$ form the operators

$$\tilde{u} = \bigoplus_{\alpha\in\Lambda} u_\alpha, \quad \tilde{v} = \bigoplus_{\alpha\in\Lambda} v_\alpha,$$

and set

$$\mathcal{A}_\theta = C^*(\tilde{u}, \tilde{v}) \subseteq \bigoplus_{\alpha\in\Lambda} B(H_\alpha).$$

It is seen that the set $\{\sum_{k,l\in\mathbb{Z}}^{(*)} a_{kl}\tilde{u}^k\tilde{v}^l \mid a_{kl} \in \mathbb{C}\}$, where the $(*)$ in the sum indicates that it is finite, is a dense subset of $\mathcal{A}_\theta$: clearly $a_{kl}\tilde{u}^k\tilde{v}^l \in \mathcal{A}_\theta$, and if we have some element $a_{kl}\tilde{v}^l\tilde{u}^k \in \mathcal{A}_\theta$, we use (4.1) to commute the $\tilde{u}$'s and $\tilde{v}$'s through the cost of multiplying a_{kl} by $e^{\pm 2\pi i\theta}$ a finite number of times. Hence this can be brought to the desired form. Also, the closure of the set is, by definition of $C^*(\tilde{u}, \tilde{v})$, the whole thing.

Let now $A = C^*(u, v)$ be another C^*-algebra generated by unitaries u, v satisfying (4.1). If $\mathcal{A}_\theta$ is our universal C^*-algebra, then there should be a C^*-homomorphism $\varphi\colon \mathcal{A}_\theta \to A$ defined by $\varphi(\tilde{u}) = u, \varphi(\tilde{v}) = v$.

Proposition 4.4. *φ defined as above is a well-defined C^*-homomorphism from $\mathcal{A}_\theta$ to A. Hence $\mathcal{A}_\theta$ is a universal C^*-algebra generated by unitaries $\tilde{u}$ and $\tilde{v}$ satisfying* (4.1).

Proof. It is by the above enough to show that we have $\|p(u, v, u^*, v^*)\| \le \|p(\tilde{u}, \tilde{v}, \tilde{u}^*, \tilde{v}^*)\|$ for all polynomials in four variables, as φ will then be contractive on a dense set of $\mathcal{A}_\theta$, and hence contractive on all of $\mathcal{A}_\theta$ by continuity, so φ will be a *-homomorphism.

Fix now such a polynomial p, and set $a := p(u, v, u^*, v^*)$. Find a faithful representation π of A such that $\|\pi(a)\| = \|a\|$. Consider now $u' = \pi(u), v' = \pi(v)$; this is an irreducible pair of unitaries satisfying (4.1), so by definition of $\tilde{u}, \tilde{v}$, we see that

$$\|p(\tilde{u}, \tilde{v}, \tilde{u}^*, \tilde{v}^*)\| \ge \|p(u', v', u'^*, v'^*)\| = \|a\|.$$

Thus φ is a *-homomorphisms, and $\mathcal{A}_\theta$ is therefore a universal C^*-algebra generated by unitaries satisfying (4.1). ■

We now aim to show that $\mathcal{A}_\theta$ is essentially unique. We do this by proving that $\mathcal{A}_\theta$ is simple. We first need some technicalities:

For any numbers $\lambda, \mu \in \mathbb{T}$, the pair $(\lambda\tilde{u}, \mu\tilde{v})$ is an irreducible pair of unitaries satisfying (4.1). Hence there is, by the universality of $\mathcal{A}_\theta$, an endomorphism $\rho_{\lambda,\mu}$ on $\mathcal{A}_\theta$ such that

$$\rho_{\lambda,\mu}(\tilde{u}) = \lambda\tilde{u}, \quad \text{and} \quad \rho_{\lambda,\mu}(\tilde{v}) = \mu\tilde{v}.$$

Define now $\psi\colon \mathcal{A}_\theta \to \mathcal{A}_\theta$ by $\psi = \rho_{\overline{\lambda},\overline{\mu}} \circ \rho_{\lambda,\mu}$. We then see that $\psi(u) = u, \psi(v) = v$, so $\psi = \mathrm{id}_{\mathcal{A}_\theta}$. Doing the exact same thing for $\psi' = \rho_{\lambda,\mu} \circ \rho_{\overline{\lambda},\overline{\mu}}$, we conclude that $\rho_{\lambda,\mu}$ is actually an automorphism.

Furthermore, the map $f\colon \mathbb{T}^2 \to \mathcal{A}_\theta$ given by $f(\lambda, \mu) = \rho_{\lambda,\mu}(a)$ is norm continuous for each $a \in \mathcal{A}_\theta$; indeed, if $a = p(\tilde{u}, \tilde{v}, \tilde{u}^*, \tilde{v}^*)$ for some polynomial p, then it is seen to be true. As these polynomials are dense in $\mathcal{A}_\theta$, the statement follows.

We next define two maps from $\mathcal{A}_\theta$ into itself, given as integrals:

$$\Phi_1(a) = \int_0^1 \rho_{1,e^{2\pi it}}(a)\ \mathrm{d}t, \quad \text{and} \quad \Phi_2(a) = \int_0^1 \rho_{e^{2\pi it},1}(a)\ \mathrm{d}t, \quad a \in \mathcal{A}_\theta.$$

As the integrands are norm continuous, this makes sense. We need the following technical theorem to continue.

Theorem 4.5. *Φ_1 is a positive, contractive, idempotent and faithful map, and maps $\mathcal{A}_\theta$ onto $C^*(\tilde{u})$. Moreover,*

$$\Phi_1(f(\tilde{u})ag(\tilde{u})) = f(\tilde{u})\Phi_1(a)g(\tilde{u}), \quad a \in \mathcal{A}_\theta,$$

for all $f, g \in C(\mathbb{T})$, and we have

$$\Phi_1\left(\sum_{k,l\in\mathbb{Z}}^{(*)} a_{kl}\tilde{u}^k\tilde{v}^l\right) = \sum_{k}^{(*)} a_{k0}\tilde{u}^k.$$

Lastly, for every $a \in \mathcal{A}_\theta$ we have

$$\Phi_1(a) = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{j=-n}^{n} \tilde{u}^j a\tilde{u}^{-j}. \tag{4.2}$$

Proof. As $\Phi_1(a)$ can be approximated by convex combinations of elements from $\{\rho_{1,e^{2\pi it}}(a) \mid t \in [0,1]\} \subset \mathcal{A}_\theta$, and $\|\rho_{1,e^{2\pi it}}(a)\| = \|a\|$ for every $a \in \mathcal{A}_\theta$, we see that $\|\Phi_1\| \leq 1$. As we have $\Phi_1(I) = I$, $\|\Phi_1\| = 1$.

As $\rho_{1,e^{2\pi it}}(\tilde{u}) = \tilde{u}$ for all $t \in [0,1]$, $\rho_{1,e^{2\pi it}}$ is the identity map on $C^*(\tilde{u})$, which is isomorphic to $C(\mathbb{T})$ as $\sigma(\tilde{u}) = \mathbb{T}$. Since $\rho_{1,e^{2\pi it}}$ is an homomorphism, we get for $f, g \in C(\mathbb{T})$ and $a \in \mathcal{A}_\theta$ that

$$\begin{aligned}\Phi_1(f(\tilde{u})ag(\tilde{u})) &= \int_0^1 \rho_{1,e^{2\pi it}}(f(\tilde{u}))\rho_{1,e^{2\pi it}}(a)\rho_{1,e^{2\pi it}}(g(\tilde{u}))\ \mathrm{d}t \\ &= f(\tilde{u})\left(\int_0^1 \rho_{1,e^{2\pi it}}(a)\ \mathrm{d}t\right)g(\tilde{u}) = f(\tilde{u})\Phi_1(a)g(\tilde{u}).\end{aligned}$$

For a monomial $\tilde{u}^k\tilde{v}^l$ we get

$$\Phi_1(\tilde{u}^k\tilde{v}^l) = \tilde{u}^k\int_0^1 \rho_{1,e^{2\pi it}}(\tilde{v}^l)\ \mathrm{d}t = \tilde{u}^k\int_0^1 e^{2\pi itl}\tilde{v}^l\ \mathrm{d}t = \begin{cases}0, & l \neq 0,\\ \tilde{u}^k, & l = 0.\end{cases}$$

As integrals are $\mathcal{A}_\theta$-linear, the statement about sums follows.

Examining polynomials p in $\tilde{u}, \tilde{u}^*, \tilde{v}, \tilde{v}^*$, we therefore get that Φ_1 maps a dense subset in $\mathcal{A}_\theta$ to a dense subset of $C^*(\tilde{u})$, and hence $\Phi_1(\mathcal{A}_\theta) = C^*(\tilde{u})$. As Φ_1 is the identity here, we get $\Phi_1^2 = \Phi_1$.

If $a \in \mathcal{A}_\theta$ is positive and non-zero, then also $\rho_{1,e^{2\pi it}}(a)$ is positive and non-zero for all $t \in [0,1]$, as $\rho_{1,e^{2\pi it}}$ is an automorphism. It follows that $\Phi_1(a)$ is also positive and non-zero. Hence Φ_1 is positive and faithful.

Lastly, we see for a monomial $\tilde{u}^k\tilde{v}^l$, where we assume $l \neq 0$, that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}\tilde{u}^j(\tilde{u}^k\tilde{v}^l)\tilde{u}^{-j} &\overset{(4.1)}{=} \lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n} e^{2\pi ijl\theta}\tilde{u}^k\tilde{v}^l \\ &\overset{l\neq 0}{=} \lim_{n\to\infty}\frac{1}{2n+1}\left(\frac{\sin\left((2n+1)\pi l\theta\right)}{\sin(\pi l\theta)}\right)\tilde{u}^k\tilde{v}^l = 0 = \Phi_1(\tilde{u}^k\tilde{v}^l).\end{aligned}$$

Similarly, if $l = 0$, then

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n} e^{2\pi ijl\theta}\tilde{u}^k\tilde{v}^l = \tilde{u}^k = \Phi_1(\tilde{u}^k\tilde{v}^l).$$

By linearity and continuity of Φ_1, the formula holds for all $a \in \mathcal{A}_\theta$. ■

With a proof that is completely similar, the corresponding results for Φ_2 and $\tilde{v}$ also hold. We can combine these results to get the following:

Corollary 4.6. *The map $\tau = \Phi_1\Phi_2 = \Phi_2\Phi_1$ is a faithful unital scalar valued trace on $\mathcal{A}_\theta$, and it is the unique trace on $\mathcal{A}_\theta$.*

Proof. Apply first $\Phi_2\Phi_1$ to a monomial $\tilde{u}^k\tilde{v}^l$:

$$\begin{aligned}\Phi_2\Phi_1(\tilde{u}^k\tilde{v}^l) &= \begin{cases}\Phi_2(\tilde{u}^k), & l = 0\\ 0, & l \neq 0\end{cases}\\ &= \begin{cases} I, & k = l = 0,\\ 0, & \text{else.}\end{cases}\end{aligned}$$

Applying $\Phi_1\Phi_2$ to $\tilde{u}^k\tilde{v}^l$ yields the exact same result, so by continuity and linearity, $\Phi_1\Phi_2 = \Phi_2\Phi_1$ on $\mathcal{A}_\theta$. Moreover, we see that $\tau(\mathcal{A}_\theta) \subseteq \mathbb{C} \cdot I$.

As both Φ_1 and Φ_2 are positive, faithful and contractive, so is τ. As we see that $\tau(I) = I$, we have $\|\tau\| = 1$.

We again calculate on monomials to check that τ is a trace:

$$\begin{aligned}\tau((\tilde{u}^k\tilde{v}^l)(\tilde{u}^n\tilde{v}^m)) &\overset{(4.1)}{=} e^{-2\pi i l n\theta}\tau(\tilde{u}^{k+n}\tilde{v}^{l+m})\\ &= \begin{cases} e^{-2\pi i l n\theta}\Phi_2(\tilde{u}^{k+n}), & l+m = 0\\ 0, & \text{else}\end{cases} = \begin{cases} e^{-2\pi i l n\theta}\cdot I, & l+m = k+n = 0,\\ 0, & \text{else.}\end{cases}\\ \tau((\tilde{u}^n\tilde{v}^m)(\tilde{u}^k\tilde{v}^l)) &\overset{(4.1)}{=} e^{-2\pi i k m\theta}\tau(\tilde{u}^{n+k}\tilde{v}^{m+l})\\ &= \begin{cases} e^{-2\pi i k m\theta}\Phi_2(\tilde{u}^{n+k}), & l+m = 0\\ 0, & \text{else}\end{cases} = \begin{cases} e^{-2\pi i k m\theta}\cdot I, & l+m = k+n = 0\\ 0, & \text{else.}\end{cases}\end{aligned}$$

Now, if $l + m = k + n = 0$, we also get $km = ln$. Hence we see that the two calculations above yield the same result. By linearity and continuity, τ is now a trace on $\mathcal{A}_\theta$.

Suppose now that τ' was another trace on $\mathcal{A}_\theta$. Then for all $a \in \mathcal{A}_\theta$ we have $\tau'(a) = \tau'(\tilde{u}^j a\tilde{u}^{-j})$, $j \in \mathbb{Z}$. We then get for all $n \in \mathbb{N}$ that $\tau'(a) = \frac{1}{2n+1}\sum_{j=-n}^{n}\tau'(u^j a u^{-j})$, and hence by the linearity of τ', (4.2) entails that

$$\tau'(a) = \lim_{n\to\infty}\tau'\left(\frac{1}{2n+1}\sum_{j=-n}^{n} u^j a u^{-j}\right) = \tau'(\Phi_1(a)), \quad a \in \mathcal{A}_\theta.$$

In the same way, we get

$$\tau'(a) = \tau'(\Phi_1(a)) = \tau'(\Phi_1\Phi_2(a)) = \tau'(\tau(a)) = \tau(a),$$

where the last equality follows from the fact that $\tau'(I) = I$ and $\tau(a) \in \mathbb{C}\cdot I$. Hence $\tau' = \tau$. ∎

This result, as in Corollary 2.12, implies that $\mathcal{A}_\theta$ is finite.

Now comes the main result of this section: that $\mathcal{A}_\theta$ is simple and therefore essentially the unique C^*-algebra generated by unitaries satisfying (4.1).

Theorem 4.7. *$\mathcal{A}_\theta$ is simple. Thus, if u and v are two unitaries satisfying* (4.1)*, then $C^*(u, v)$ is canonically isomorphic to $\mathcal{A}_\theta$.*

Proof. Suppose $\mathcal{I} \trianglelefteq \mathcal{A}_\theta$ is non-zero; take some non-zero $x \in \mathcal{I}$. Then $\tilde{u}^j x \tilde{u}^{-j} \in \mathcal{I}$ as well, so (4.2) shows that $\Phi_1(x) \in \mathcal{I}$, as $\mathcal{I}$ is closed. The same goes for Φ_2, so $0 \neq \tau(x) \in \mathcal{I}$. But then $\mathcal{I} \cap ((\mathbb{C} \setminus \{0\}) \cdot I) \neq \emptyset$, so $\mathcal{I} = \mathcal{A}_\theta$.

Now, if u and v are two unitaries satisfying (4.1), there is a homomorphism from $\mathcal{A}_\theta$ to $C^*(u, v)$ mapping $\tilde{u}$ to u and $\tilde{v}$ to v. As the kernel of this homomorphism is an ideal in $\mathcal{A}_\theta$, and the homomorphism is non-zero, this is actually an isomorphism. ■

We will by Theorem 4.7 drop the twiddles above u and v when working in $\mathcal{A}_\theta$, as this no longer causes any ambiguity.

4.2.2 Weak Classification

As with every class of mathematical objects, a natural question is, when are two objects isomorphic? We will try to answer this question for irrational rotation algebras now. So, can we find a complete invariant for irrational rotation algebras that classsifies them? As noted before, we will not be able to do this in this project, although it can be done via K-theory. We do, however, prove that there are uncountably many non-isomorphic irrational rotation algebras.

We start this off by noting that when constructing $\mathcal{A}_\theta$, only the value of θ mod 1 matters. Moreover, if (u, v) satisfies (4.1) for the value θ, then (v, u) also satisfies (4.1) for $-\theta$. Therefore, we obtain $\mathcal{A}_\theta \cong \mathcal{A}_{1-\theta}$.

Now, to classify irrational rotation algebras, we try to do the same we did with the UHF algebras: try to calculate something from projections. However, it is not completely clear if $\mathcal{A}_\theta$ contains any proper projections; we know, as $\sigma(u) = \sigma(v) = \mathbb{T}$, that $C^*(u) \cong C(\mathbb{T}) \cong C^*(v)$. As $\mathbb{T}^2$ is connected, $C(\mathbb{T}^2)$ does not contain any proper projections, so as $\mathcal{A}_\theta$ by the above resembles $C(\mathbb{T}^2)$ a bit, perhaps we should expect not to find any proper projections. But there are actually pretty many:

Theorem 4.8 (Rieffel)**.** *For every $\alpha \in (\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$, there is some projection $p \in \mathcal{A}_\theta$ such that $\tau(p) = \alpha$.*

Proof. The proof will be very constructive.

We may assume by the above that $0 < \theta < 1/2$. We realize $\mathcal{A}_\theta$ concretely on $L_2(\mathbb{T})$ with $u = M_z$ and v equal to rotation by θ as in the start of section 4.2.1. For $f \in C(\mathbb{T})$, write f_θ for the translated function, again thinking of $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$. We can also realize $C^*(u)$ as $C^*(\{M_f \mid f \in C(\mathbb{T})\})$. We note that

$$vM_f h(t) = f(t - \theta)h(t - \theta) = M_{f_\theta} v h(t)$$

for all $f \in C(\mathbb{T}), h \in L_2(\mathbb{T})$ and $t \in \mathbb{R}/\mathbb{Z}$.

Now, as before, a dense set in $\mathcal{A}_\theta$ can be realized by a finite linear combination of the form $a = \sum_{j\in\mathbb{Z}}^{(*)} a_j M_{f_j} v^j$. As we can define each f_j by $M_{f_j} = \Phi_1(av^{-j})$, we see that they are uniquely determined by a. This gives us that we may compare elements on the form of finite sums by comparing their coefficients. Also note that on $C^*(M_z)$,

$$\tau(M_f) = \Phi_1(M_f) = \int_0^1 f(t)\ \mathrm{d}t, \quad f \in C(\mathbb{T}).$$

We now look for projections of the special form $p = M_g v + M_f + M_h v^*$ for some $f, g, h \in C(\mathbb{T})$. As we want this to be a projection, we must have $p = p^*$. This gives

$$M_g v + M_f + M_h v^* = v^* M_{\overline{g}} + M_{\overline{f}} + v M_{\overline{h}} = M_{\overline{g}_{-\theta}} v^* + M_{\overline{f}} + M_{\overline{h}_\theta} v.$$

By comparing the coefficients of this, we must have $f = \overline{f}$ and $h(t - \theta) = \overline{g(t)}$ for every $t \in [0, 1]$.

Moreover, we must have $p = p^2$, and thus

$$\begin{aligned} M_g v + M_f + M_h v^* &= M_g v M_g v + M_g v M_f + M_g v M_h v^* + M_{fg} + M_{f^2} \\ &\quad + M_{fh} v^* + M_h v^* M_g v + M_h v^* M_f + M_h v^* M_h v^* \\ &= M_{g g_\theta} v^2 + M_{g(f+f_\theta)} v + M_{g h_\theta + f^2 + h g_{-\theta}} + M_{h(f+f_{-\theta})} v^* + M_{h h_{-\theta}} (v^*)^2. \end{aligned}$$

Again comparing coefficients and using the relations above, we get the following necessary and sufficient conditions for p to be a projection:

$$g g_\theta = 0,$$

$$g(1 - f - f_\theta) = 0,$$

$$f - f^2 = |g|^2 + |g_\theta|^2.$$

We can solve this explicitly. Pick some $0 < \varepsilon < \theta$ such that $\theta + \varepsilon < 1/2$. Define $f \in C(\mathbb{T})$ by

$$f(t) = \begin{cases} t/\varepsilon, & t \in [0, \varepsilon], \\ 1, & t \in [\varepsilon, \theta], \\ (\theta + \varepsilon - t)/\varepsilon, & t \in [\theta, \theta + \varepsilon], \\ 0, & t \in [\theta + \varepsilon, 1), \end{cases}$$

and $g \in C(\mathbb{T})$ by

$$g(t) = \begin{cases} \sqrt{f(t) - f(t)^2}, & t \in [\theta, \theta + \varepsilon], \\ 0, & \text{else.} \end{cases}$$

We then see that the first condition is fulfilled. Secondly, $f(t) + f(t - \theta) = 1$ for $t \in [\varepsilon, 2\theta]$, which includes the support of g, so the second condition is also fulfilled. Thirdly, $f(t) - f(t)^2$ is non-zero for $t \in (0, \varepsilon) \cup (\theta, \theta + \varepsilon)$, where we have constructed f and g such that the third condition is fulfilled. Hence these functions determine a projection $p = M_g v + M_f + M_{\overline{g_{-\theta}}} v^*$ in $\mathcal{A}_\theta$. The trace of this projection is now

$$\tau(p) = \tau(M_f) = \int_0^1 f(t)\ \mathrm{d}t = \theta.$$

We also get the projection $I - p$ with $\tau(I - p) = 1 - \theta$.

Now, note that $uv^k = e^{2\pi i k\theta} v^k u$ for all $k \in \mathbb{Z}$, and hence $\mathcal{A}_\theta$ contains a copy of $\mathcal{A}_{k\theta}$ for all $k \in \mathbb{Z}$. If we replace v by v^k and θ by $\alpha := k\theta \bmod 1$ we get a projection in $\mathcal{A}_{k\theta}$ with trace α. Thus we can aquire every value in $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$ as the trace of a projection in $\mathcal{A}_\theta$. ■

Now, one could think that the above result is a product of our construction of the projections, and there could be a lot more projections. However, we have the following result, which we will not prove here:

Theorem 4.9. *If $p \in \mathcal{A}_\theta$ is a projection, then $\tau(p) \in (\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$, and thus*

$$\{\tau(p) \mid p \in \mathcal{A}_\theta,\ p \text{ is a projection}\} = (\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1].$$

We can, using Theorem 4.8, weakly classify irrational rotation algebras. Recall Proposition 2.17 ii): if A is a separable C^*-algebra and τ is a trace on A, then $\{\tau(p) \mid p \in A \text{ is a projection}\}$ is countable. Combined with the fact that only the value of θ mod 1 matters, we get the following result, which will count as our sought weak classification of irrational rotation algebras.

Corollary 4.10. *There are uncountably many isomorphism classes among irrational rotation algebras.*

Proof. Assume to reach a contradiction that the set of isomorphism classes was countable. As earlier, denote the dimension range of $\mathcal{A}_\theta$ by $D(\mathcal{A}_\theta)$. By Proposition 2.17, $D(\mathcal{A}_\theta)$ is countable, and $\theta \in D(\mathcal{A}_\theta)$ by Theorem 4.8.

Recall that if $\mathcal{A}_\theta \cong \mathcal{A}_\eta$ for irrational numbers $\theta, \eta \in (0,1)$, then $D(\mathcal{A}_\theta) = D(\mathcal{A}_\eta)$ by Corollary 2.19. Thus, if $D(\mathcal{A}_\theta) \neq D(\mathcal{A}_\eta)$, then $\mathcal{A}_\theta \ncong \mathcal{A}_\eta$, and hence there are, by assumption, at most countably many different $D(\mathcal{A}_\theta)$'s.

Since $\theta \in D(\mathcal{A}_\theta)$, we have that

$$(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \subseteq \bigcup_{\theta \in \mathbb{R}\setminus\mathbb{Q}} D(\mathcal{A}_\theta) \subseteq [0,1].$$

Find now a countable set $T = \{\theta_1, \theta_2, \dots\} \subseteq \mathbb{R} \setminus \mathbb{Q}$ such that $\bigcup_{\theta \in \mathbb{R}\setminus\mathbb{Q}} D(\mathcal{A}_\theta) = \bigcup_{\theta \in T} D(\mathcal{A}_\theta)$, which is then countable, being a countable union of countable sets. This is our contradiction, as there are uncountably many irrational numbers in $[0,1]$. Thus there are uncountably many isomorphism classes. ■

4.2.3 Why not Rational values?

If we took $\theta \in \mathbb{Q}$, we could of course still form $\mathcal{A}_\theta$, but this turns out to be a construction which is not interesting in the scope of this thesis. We give here an example as to why we have chosen to only consider irrational values of θ.

Example 4.11. Consider the case $\theta = 0$. $\mathcal{A}_0$ is then the universal C^*-algebra generated by two commuting unitaries u and v. Define

$$\widehat{\mathcal{A}_0} := \{\rho \colon \mathcal{A}_0 \to \mathbb{C} \mid \rho \text{ is a non-zero multiplicative linear functional}\}.$$

We endow $\widehat{\mathcal{A}_0}$ with the weak-* topology. We know that $\mathcal{A}_0 \cong C(\widehat{\mathcal{A}_0})$ as $\mathcal{A}_0$ is commutative. We show that $\widehat{\mathcal{A}_0} \cong \mathbb{T}^2$ as topological spaces as is done multiple times in chapter 6 of [1].

Define $\Phi \colon \widehat{\mathcal{A}_0} \to \mathbb{T}^2$ by

$$\Phi(\rho) = (\rho(u), \rho(v)), \quad \rho \in \widehat{\mathcal{A}_0}.$$

Recall that if A is a commutative Banach algebra and $x \in A$, then $\sigma(x) = \{\rho(x) \mid \rho \in \widehat{A}\}$. By this, we do indeed have $\Phi(\rho) \in \mathbb{T}^2$ for every $\rho \in \widehat{\mathcal{A}_0}$. Also, Φ is continuous as each coordinate function is known to be.

For two functionals $\rho, \rho' \in \widehat{\mathcal{A}_0}$ we have that if $\rho(u) = \rho'(u)$ and $\rho(v) = \rho'(v)$, then $\rho = \rho'$ as ρ and ρ' are both linear and multiplicative. Thus Φ is injective.

Define now $u_0, v_0 \in C(\mathbb{T}^2)$ by

$$u_0(z_1, z_2) = z_1, \quad v_0(z_1, z_2) = z_2, \quad z_1, z_2 \in \mathbb{T}.$$

One can then use the Stone-Weierstraß approximation theorem to see that we have $C(\mathbb{T}^2) = C^*(u_0, v_0)$. Thus by universality there is a unique homomorphism $\psi \colon \mathcal{A}_0 \to C(\mathbb{T}^2)$ such that $\psi(u) = u_0$ and $\psi(v) = v_0$. For each $z_1, z_2 \in \mathbb{T}$ we can define $\mathrm{ev}_{(z_1,z_2)} \colon C(\mathbb{T}^2) \to \mathbb{C}$ by $\mathrm{ev}_{(z_1,z_2)}(f) = f(z_1, z_2)$. We then get a multiplicative linear functional

$$\mathcal{A}_0 \xrightarrow{\psi} C(\mathbb{T}^2) \xrightarrow{\mathrm{ev}_{(z_1,z_2)}} \mathbb{C}$$

for every $z_1, z_2 \in \mathbb{T}^2$, where we clearly have $\mathrm{ev}_{(z_1,z_2)} \circ \psi(u) = z_1$ and $\mathrm{ev}_{(z_1,z_2)} \circ \psi(v) = z_2$. Thus Φ is also surjective, and hence Φ is a homeomorphism.

We have now proven that $\mathcal{A}_0 \cong C(\mathbb{T}^2)$. For every proper closed subset $U \subseteq \mathbb{T}^2$, the set $\{f \in C(\mathbb{T}^2) \mid f(z) = 0 \text{ for every } z \in U\}$ constitutes a proper ideal in $C(\mathbb{T}^2)$ and hence $\mathcal{A}_0$

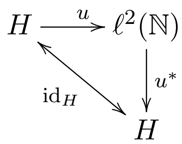
is non-simple. Also as $\mathbb{T}^2$ is connected, $\mathcal{A}_0$ does not contain any proper projections. This is therefore a completely different situation from what we get in Theorems 4.7 and 4.8 when θ is irrational.

In the general case with $\theta \in \mathbb{Q}$ it takes quite some work to find out what $\mathcal{A}_\theta$ is, but in every case it turns out that $\mathcal{A}_\theta$ is non-simple and does not have a unique tracial state.

4.3 Cuntz algebras

We will in this section give another example of universal C^*-algebras: the so-called Cuntz algebras, named after Joachim Cuntz. These turn out to be simple purely infinite C^*-algebras, a notion that turns out to be extremely restrictive, as we shall see later on. We first give a motivating example for our coming construction:

Example 4.12. Consider $\ell^2(\mathbb{N})$, and let $H := \ell^2(\mathbb{N}) \oplus \cdots \oplus \ell^2(\mathbb{N})$, n times for some $n \geq 2$. As H is also a seperable Hilbert space, there exists a unitary $u\colon H \to \ell^2(\mathbb{N})$ such that $u^*u = \mathrm{id}_H$.



As we can write $u = u_1 + \cdots + u_n$ with $u_i := uP_i$, where $P_i\colon H \to H$ is the coordinate projection $P_i(\xi_1, \ldots, \xi_n) = (0, \ldots, \xi_i, \ldots, 0)$, $\xi = (\xi_1, \ldots, \xi_n) \in H$, we have that $\mathrm{id}_H = u_1u_1^* + \cdots + u_nu_n^*$; all in all, we have the following:

$$u_i^*u_i = \mathrm{id}_{\ell^2(\mathbb{N})}, \quad i = 1, \ldots, n.$$
$$\sum_{i=1}^n u_iu_i^* = \mathrm{id}_H.$$

Suppose now that we are given n isometries $s_1, \ldots, s_n$ for some $n \geq 2$ such that

$$\sum_{i=1}^n s_is_i^* = I. \tag{4.3}$$

It follows that $s_j^*s_i = 0$ if $i \neq j$: we see that $s_is_i^*$ is a projection for every i, and they can only sum to I if the ranges are orthogonal. Thus, if $i \neq j$, we have

$$s_js_j^*s_is_i^* = 0 \Rightarrow s_j^*s_i = 0,$$

by multiplying from the left by s_j^* and from the right by s_i.

We now want to define the Cuntz algebra $\mathcal{O}_n$ as the universal C^*-algebra generated by these isometries. To construct it we do the same as with the irrational rotation algebras. As we know by Example 4.12 that such isometries exist, we can consider the collection of all tuples $(s_{1,\alpha}, \ldots, s_{n,\alpha})_{\alpha\in\Lambda}$ satisfying (4.3), form the operators

$$\tilde{s}_i = \bigoplus_{\alpha\in\Lambda} s_{i,\alpha}, \quad i = 1, \ldots, n,$$

and define $\mathcal{O}_n := C^*(\tilde{s}_1, \ldots, \tilde{s}_n) \subseteq \bigoplus_{\alpha\in\Lambda} B(H_\alpha)$. By the same argument as with the irrational rotation algebras, taking polynomials in $2n$ variables, we see that $\mathcal{O}_n$ is a universal C^*-algebra generated by elements satisfying (4.3). We will from here on abuse notation a bit and write $s_i = \tilde{s}_i$. This abuse will be justified by Theorem 4.18.

We note for future use that as $\mathcal{O}_n$ is generated by a finite set of elements, it is separable. It is furthermore immediate that $\mathcal{O}_n$ is unital.

4.3.1 Simplicity

To work in $\mathcal{O}_n$ we use the words introduced when constructing universal C^*-algebras. Let $\mathbf{n}$ denote the set $\{1, \ldots, n\}$ and let $\mu = (i_1, \ldots, i_k) \in \mathbf{n}^k$. Then we define $s_\mu = s_{i_1} \cdots s_{i_k} \in \mathcal{O}_n$. Let $|\mu| = k$ denote the length of the word.

Let $\mu = (i_1, \ldots, i_k)$ and $\nu = (j_1, \ldots, j_l)$ be words. As $s_j^* s_i = 0$ if $j \neq i$, we see that if $0 \neq s_\mu^* s_\nu$, then $i_m = j_m$ for all $1 \leq m \leq \min\{|\mu|, |\nu|\}$. For reference, we collect these observations:

Lemma 4.13. *Suppose $|\mu| = k, |\nu| = l$ and $s_\mu^* s_\nu \neq 0$.*
i) If $k = l$, then $\mu = \nu$, and $s_\mu^* s_\nu = I$,
ii) If $k < l$, then there is a word $\nu' \in \mathbf{n}^{l-k}$ such that $\nu = \mu\nu'$ and $s_\mu^* s_\nu = s_{\nu'}$,
iii) If $k > l$, then there is a word $\mu' \in \mathbf{n}^{k-l}$ such that $\mu = \nu\mu'$ and $s_\mu^* s_\nu = s_{\mu'}^*$.

By this lemma, we see that every non-zero word in the s_i's and the s_i^*'s has a unique reduced expression of the form $s_\mu s_\nu^*$. Hence whenever we work with these words, we may assume they are of this form.

Let now W_n^k denote the set of words in $\mathbf{n}^k$ for some $k \in \mathbb{N}_0$, where $\mathbf{n}^0$ consists of the empty word. Set $W_n = \bigcup_{k \in \mathbb{N}_0} W_n^k$. Define also

$$F_n^k := \operatorname{span}\{s_\mu s_\nu^* \mid \mu, \nu \in W_n^k\}, \quad F_n := \overline{\bigcup_{k \in \mathbb{N}_0} F_n^k}.$$

We are now able to see that a dense subset of $\mathcal{O}_n$ is $\operatorname{span}\{s_\mu s_\nu^* \mid \mu, \nu \in W_n\}$. We can pick a countable dense set by taking the span over $\mathbb{Q} + i\mathbb{Q}$ instead of $\mathbb{C}$.

Proposition 4.14. *For each $k, n \in \mathbb{N}$, F_n^k is isomorphic to $M_{n^k}(\mathbb{C})$. In particular, F_n is the UHF algebra of type n^∞.*

Proof. By the calculation

$$(s_\mu s_\nu^*)(s_{\mu'} s_{\nu'}^*) = s_\mu (s_\nu^* s_{\mu'}) s_{\nu'}^* = \delta_{\nu,\mu'} s_\mu s_{\nu'}^*,$$

we see that the set $\{s_\mu s_\nu^* \mid \mu, \nu \in W_n^k\}$, which spans F_n^k, forms a set of matrix units for $M_{n^k}(\mathbb{C})$. Thus we get $F_n^k \cong M_{n^k}(\mathbb{C})$.

Now, we have $F_n^k \subseteq F_n^{k+1}$: for $\mu, \nu \in \mathcal{W}_n^k$, we calculate

$$F_n^{k+1} \ni \sum_{i=1}^n s_{\mu i} s_{\nu i}^* = s_\mu \left(\sum_{i=1}^n s_i s_i^* \right) s_\nu^* \overset{(4.3)}{=} s_\mu s_\nu^*,$$

and thus the matrix units for F_n^{k+1} are compatible with the ones for F_n^k.

As we have $F_n = \overline{\bigcup_{k \in \mathbb{N}} \mathcal{F}_n^k}$, it is now immediate that F_n is the UHF algebra of type n^∞. ■

It is clear that $F_n \subseteq \mathcal{O}_n$. It actually plays a big part in calculating in $\mathcal{O}_n$, as we shall see a little later. Before we can get to this result, we as with the irrational rotation algebras need some technicalities.

For any $\lambda \in \mathbb{T}$, the isometries λs_i, $i = 1, \ldots, n$, satisfy (4.3) if the s_i's do, so it follows, as we did with the irrational rotation algebras, that there is an automorphism ρ_λ of $\mathcal{O}_n$ such that $\rho_\lambda(s_i) = \lambda s_i$ for all i. A quick calculation shows that $\rho_\lambda(s_i^*) = \overline{\lambda} s_i^*$ and $\rho_\lambda(s_\mu s_\nu^*) = \lambda^{|\mu|-|\nu|} s_\mu s_\nu^*$ for any $\mu, \nu \in W_n$. Hence for a fixed $x \in \operatorname{span}\{s_\mu s_\nu^* \mid \mu, \nu \in W_n\}$, the function $f_x : \mathbb{T} \to \mathcal{O}_n$ given by $f_x(\lambda) = \rho_\lambda(x)$ is norm continuous. As this span is dense in $\mathcal{O}_n$, and automorphisms are isometric, f_x is continuous for all $x \in \mathcal{O}_n$.

Theorem 4.15. *There is a positive contractive unital idempotent and faithful map Φ_0 of $\mathcal{O}_n$ onto F_n.*

Proof. Define $\Phi_0 \colon \mathcal{O}_n \to \mathcal{O}_n$ by

$$\Phi_0(x) := \int_0^1 f_x(e^{2\pi it})\ \mathrm{d}t = \int_0^1 \rho_{e^{2\pi it}}(x)\ \mathrm{d}t, \quad x \in \mathcal{O}_n,$$

which makes sense as the integrand is norm continuous. This is easily seen to be a linear unital positive and contractive map on $\mathcal{O}_n$. For instance,

$$\Phi_0(I) = \int_0^1 \rho_{e^{2\pi it}}(I)\ \mathrm{d}t = \int_0^1 \rho_{e^{2\pi it}}(s_1^*s_1)\ \mathrm{d}t = \int_0^1 s_1^*s_1\ \mathrm{d}t = I.$$

We calculate on $s_\mu s_\nu^*$ for some $\mu, \nu \in W_n$:

$$\Phi_0(s_\mu s_\nu^*) = \left(\int_0^1 e^{2\pi it(|\mu|-|\nu|)}\ \mathrm{d}t\right) \cdot s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^*, & |\mu| = |\nu|, \\ 0, & \text{else.} \end{cases} \tag{4.4}$$

Hence we see that Φ_0 maps the algebraic span of words in the s_i's and s_i^*'s onto a dense subset of F_n, where it is also the identity. Thus Φ_0 is a contractive projection of $\mathcal{O}_n$ onto F_n.

Lastly, if x is positive and non-zero, $\rho_{e^{2\pi it}}(x)$ is positive and non-zero for all $t \in [0,1]$, and hence $\Phi_0(x)$ is positive and non-zero. Thus Φ_0 is faithful. ∎

The following two lemmas together give us a more algebraic way to actually calculate Φ_0 in terms of W_n and F_n.

Lemma 4.16. *Suppose $\mu, \nu \in W_n$ are such that $|\mu| \neq |\nu|$ and $\max\{|\mu|, |\nu|\} \leq k \in \mathbb{N}$. Set $s_\gamma = s_1^k s_2$. Then $s_\gamma^*(s_\mu s_\nu^*)s_\gamma = 0$.*

Proof. From Lemma 4.13 it follows that $s_\gamma^* s_\mu = 0$ unless $s_\mu = s_1^{|\mu|}$. In the latter case, $s_\gamma^* s_\mu = s_2^* s_1^{*(k-|\mu|)}$. In the same way, $s_\nu^* s_\gamma = 0$ unless $s_\nu = s_1^{|\nu|}$, in which case $s_\nu^* s_\gamma = s_1^{k-|\nu|} s_2$. So, even assuming that these are both non-zero, we get

$$(s_\gamma^* s_\mu)(s_\nu^* s_\gamma) = s_2^* s_1^{*(k-|\mu|)} s_1^{k-|\nu|} s_2 \overset{|\mu| \neq |\nu|}{=} 0.$$

We in this make sense of s_1^m as $s_1^{*|m|}$ if $m < 0$. ∎

Lemma 4.17. *For each $m \in \mathbb{N}$ there is an isometry $w \in \mathcal{O}_n$ commuting with F_n^m such that $\Phi_0(y) = w^* y w$ for all $y \in \operatorname{span}\{s_\mu s_\nu^* \mid \max\{|\mu|, |\nu|\} \leq m\}$.*

Proof. Let $m \in \mathbb{N}$. Set $s_\gamma := s_1^{2m} s_2$ and define $w = \sum_{|\delta|=m} s_\delta s_\gamma s_\delta^*$. Then we see by Lemma 4.13 that

$$w^*w = \sum_{|\delta|=m} \sum_{|\varepsilon|=m} s_\delta s_\gamma^* (s_\delta^* s_\varepsilon) s_\gamma s_\varepsilon^* = \sum_{|\delta|=m} s_\delta s_\gamma^* s_\gamma s_\delta^* = \sum_{|\delta|=m} s_\delta s_\delta^* = I.$$

The last equality follows from an easy induction argument. Thus w is an isometry. Further, if $|\mu| = |\nu| = m$, then we have

$$w s_\mu = \sum_{|\delta|=m} s_\delta s_\gamma (s_\delta^* s_\mu) = s_\mu s_\gamma.$$

Likewise, $s_\nu^* w = s_\gamma s_\nu^*$. Thus we have for a matrix unit $s_\mu s_\nu^* \in F_n^m$ that

$$w s_\mu s_\nu^* = s_\mu s_\gamma s_\nu^* = s_\mu s_\nu^* w,$$

and hence w commutes with all of F_n^m. This implies that $w^* x w = x$ for every $x \in F_n^m$. Now, if $|\mu| \neq |\nu|$ and $\max\{|\mu|, |\nu|\} \leq m$, then Lemma 4.16 shows that

$$s_\gamma(s_\delta^* s_\mu s_\nu^* s_\delta) s_\gamma^* = 0 \quad \text{for every } |\delta| = m.$$

This further implies that $w^* s_\mu s_\nu^* w = 0$. Comparing with (4.4), we see that $w^* y w = \Phi_0(y)$ for every $y \in \operatorname{span}\{s_\mu s_\nu^* \mid \max\{|\mu|, |\nu|\} \leq m\}$. ∎

The next result shows that $\mathcal{O}_n$ is simple for all $n \geq 2$, but it is actually a rather strong kind of simplicity. This implies that $\mathcal{O}_n$ is essentially the unique C^*-algebra generated by isometries satisfying (4.3).

Theorem 4.18 (Cuntz). *If $0 \neq x \in \mathcal{O}_n$, then there exist $a, b \in \mathcal{O}_n$ such that $axb = I$. In particular, $\mathcal{O}_n$ is simple, and hence if $t_1, \dots, t_n$ are isometries satisfying (4.3), then $C^*(t_1, \dots, t_n)$ is canonically isomorphic to $\mathcal{O}_n$.*

Proof. As $x \neq 0$ and Φ_0 is faithful, $\Phi_0(x^*x) \neq 0$ holds. We can thus multiply x by a scalar such that $\|\Phi_0(x^*x)\| = 1$. Pick now by density a self adjoint element $y \in \operatorname{span}\{s_\mu s_\nu^* \mid \mu, \nu \in W_n\}$ such that $\|x^*x - y\| < 1/4$. Then by the reverse triangle inequality we get

$$\begin{aligned}\frac{3}{4} &< 1 - \|x^*x - y\| \leq \|\Phi_0(x^*x)\| - \|\Phi_0(x^*x - y)\| \\ &\leq \|\Phi_0(x^*x) - \Phi_0(x^*x - y)\| = \|\Phi_0(y)\|,\end{aligned}$$

where we in the second inequality used that Φ_0 is contractive. Thus $\|\Phi_0(y)\| > 3/4$.

Let m be the maximum length of the words involved in the expansion of y. By Lemma 4.17 there is an isometry $w \in \mathcal{O}_n$ such that $w^*yw = \Phi_0(y) \in F_n^m$. As F_n^m is a matrix algebra, we can diagonalize $\Phi_0(y)$ and find a matrix unit e such that

$$e\Phi_0(y) = \Phi_0(y)e = \|\Phi_0(y)\|e > \frac{3}{4}e.$$

Choose now a unitary $u \in F_n^m$ such that $ueu^* = e_{11} = s_1^m s_1^{*m}$, where e_{11} is the standard matrix unit in entrance 1,1 in F_n^m, and set

$$z := \|\Phi_0(y)\|^{-1/2} s_1^{*m} uew^*.$$

We then have

$$\|z\| = \frac{\|s_1^{*m}uew^*\|}{\|\Phi_0(y)\|^{1/2}} < \frac{\|s_1^{*m}\| \cdot \|u\| \cdot \|e\| \cdot \|w^*\|}{(3/4)^{1/2}} = \frac{2}{\sqrt{3}}.$$

The last equality uses that all the elements s_1^{*m}, u, e, w have norm 1; s_1^* and w are isometries, u is a unitary, and e is an idempotent element different from 0. The last observation also implies that $e^* = e$. We further get

$$\begin{aligned}zyz^* &= \frac{1}{\|\Phi_0(y)\|} s_1^{*m} ue(w^*yw)eu^* s_1^m = \frac{1}{\|\Phi_0(y)\|} s_1^{*m} ue\|\Phi_0(y)\|eu^* s_1^m \\ &= s_1^{*m} ueu^* s_1^m = s_1^{*m} s_1^m s_1^{*m} s_1^m = I.\end{aligned}$$

Next we estimate

$$\|I - zx^*xz^*\| \leq \|z^2\|\|y - x^*x\| < \frac{4}{3}\frac{1}{4} = \frac{1}{3} < 1,$$

and thus zx^*xz^* is invertible. Set $b = z^*(zx^*xz^*)^{-1/2}$. We then get

$$(b^*x^*)xb = (zx^*xz^*)^{-1/2}(zx^*xz^*)(zx^*xz^*)^{-1/2} = I.$$

Letting $a = b^*x^*$ yields the theorem.

To now see that $\mathcal{O}_n$ is simple, take a non-zero ideal $\mathcal{I} \trianglelefteq \mathcal{O}_n$, and let $x \in \mathcal{I}$ be non-zero. Then with a, b as above we get $I = axb \in \mathcal{I}$. It follows that $\mathcal{I} = \mathcal{O}_n$. ∎

4.3.2 Simple Infinite C^*-algebras

We now aim to prove two results about the behaviour of the Cuntz algebras. First off, each Cuntz algebra is the quotient of a subalgebra of every simple infinite C^*-algebra, and secondly, the Cuntz algebras themselves are purely infinite. Actually these two results will be corollaries to what we are going to prove; the results are also very interesting on their own.

Theorem 4.19. *If A is a simple C^*-algebra and $q \in A$ is an infinite projection, then A contains partial isometries $\{t_i\}_{i\in\mathbb{N}}$ such that $t_i^* t_i = q > \sum_{i=1}^n t_i t_i^*$ for every $n \in \mathbb{N}$. In particular, A is properly infinite.*

Proof. Let s be a partial isometry such that $p = ss^* < s^*s = q$. We may assume without loss of generality that A is unital and that $q = I$; if not, we pass to the subalgebra $B = qAq$ using the next claim.

Claim: B is simple.

Proof of claim: We show that if $J \lhd B$, then there exists $K \lhd A$ such that $J = qKq$. This will clearly imply that B is simple. We show the contrapositive. Assume therefore that there does not exists such an ideal K. We have $J = qJq$, so J is not an ideal in A by assumption. Find therefore $a \in A$ and $j = qjq \in J$ such that $aj \notin J$ (or $ja \notin J$). Then $qaq \in B$. Now, as q is the identity in B, we get

$$qaqqjq = qajq \notin qJq = J,$$

and thus J is not an ideal in B. The claim is now proved. Moreover, B is infinite as q is infinite by assumption and $q = q^3 \in B$.

As A is simple, find by Lemma 1.1 $k \in \mathbb{N}$ and $b_i \in A$ such that $I = \sum_{i=1}^k b_i^*(I-p)b_i$. We see that if $i > j > 0$, then

$$(s^j(I-p))^*(s^i(I-p)) = (I-p)s^{j*}s^i(I-p) = (I-p)p^j s^{i-j}(I-p) = 0,$$

and likewise

$$(s^i(I-p))(s^j(I-p))^* = s^i(s^{j*} - ps^{j*}) = s^{i-j}p - s^i(ss^*)s^{j*} = 0.$$

Therefore the elements of $\{s^i(I-p)\}_{i\in\mathbb{N}}$ have pairwise orthogonal ranges. Define now $t_1 := \sum_{i=1}^k s^i(I-p)b_i$. We then see that

$$\begin{aligned} t_1^* t_1 &= \sum_{i=1}^k \sum_{j=1}^k b_i^*(I-p)s^{i*}s^j(I-p)b_j \\ &= \sum_{i=1}^k b_i^*(I-p)b_i = I. \end{aligned}$$

As $I = I^2$, we have $b_i^*(I-p)b_i b_i^*(I-p)b_i = b_i^*(I-p)b_i$ for every i; this implies that $b_i^* b_i$ commutes with $I-p$. As $b_i^*(I-p)b_i$ is a projection, b_i must be a partial isometry for every i. This implies that $b_i^* b_i \leq I - p$. This in turn implies that

$$\begin{aligned} t_1 t_1^* &= \sum_{i=1}^k \sum_{j=1}^k s^i(I-p)b_i b_j^*(I-p)s^{j*} = \sum_{i=1}^k s^i(I-p)b_i b_i^*(I-p)s^{i*} \\ &\leq \sum_{i=1}^k s^i(I-p)s^{i*} = I - s^k s^{k*} \end{aligned}$$

For $n \geq 2$, define $t_n := s^{k(n-1)}t_1$. We then get

$$\begin{aligned} t_n t_n^* &= s^{k(n-1)}t_1 t_1^* s^{k(n-1)*} \leq s^{k(n-1)}(I - s^k s^{k*})s^{k(n-1)*} \\ &= s^{k(n-1)}s^{k(n-1)*} - s^{kn}s^{kn*} \end{aligned}$$

This implies that $t_n t_n^*$ are pairwise orthogonal projections: if $n \neq m$, then

$$\begin{aligned} &\left\| \left(s^{k(n-1)}s^{*k(n-1)} - s^{kn}s^{*kn}\right)\left(s^{k(m-1)}s^{*k(m-1)} - s^{km}s^{*km}\right) \right\| \\ \leq &\left\| \left(I - s^k s^{*k}\right)\left(s^{k(m+n)}s^{*k(m+n)} - s^{k(m+n+1)}s^{*k(m+n+1)}\right) \right\| = 0, \end{aligned}$$

where the last equality uses that

$$\begin{aligned} (s^k s^{*k})(s^{k(m+n)}s^{*k(m+n)}) &= s^{k(m+n)}s^{*k(m+n)}, \\ (s^k s^{*k})(s^{k(m+n+1)}s^{*k(m+n+1)}) &= s^{k(m+n+1)}s^{*k(m+n+1)}. \end{aligned}$$

It is furthermore easily seen that $t_n t_n^*$ is a projection and that $t_n^* t_n = I$ for every $n \in \mathbb{N}$, so therefore we get that A is properly infinite: as each $t_n t_n^*$ is less than the identity and $t_n t_n^*$ and $t_m t_m^*$ have orthogonal ranges if $n \neq m$, their sum is also less than the identity. ■

The next result will tell us that $\mathcal{O}_n$ exists in quite a few places.

Proposition 4.20. *Let $n \in \mathbb{N}\backslash\{1\}$ and $\mathcal{E}_n$ be a C^*-algebra generated by isometries $s_1, \dots, s_n$ such that $\sum_{i=1}^n s_i s_i^* = p < I$. Then $\langle p^\perp \rangle$ is isomorphic to the compact operators $\mathcal{K}$ on a separable infinite dimensional Hilbert space and $\mathcal{E}_n / \langle p^\perp \rangle \cong \mathcal{O}_n$.*

Proof. We see that

$$0 = pp^\perp = \sum_{i=1}^n s_i s_i^* p^\perp,$$

and using that $s_i^* s_j = 0$ if $i \neq j$, we deduce that $s_i^* p^\perp = 0$, and by a similar argument, $p^\perp s_i = 0$. As $p^\perp$ is a projection, we can from Lemma 4.13 see that

$$\langle p^\perp \rangle = \overline{\operatorname{span}\{s_\mu p^\perp s_\nu^* \mid |\mu|, |\nu| < \infty\}}.$$

Moreover, Lemma 4.13 entails that if $\mu, \nu, \alpha, \beta \in W_n$ we have

$$(s_\mu p^\perp s_\nu^*)(s_\alpha p^\perp s_\beta^*) = \delta_{\nu\alpha} s_\mu p^\perp s_\beta^*.$$

$\{s_\mu p^\perp s_\nu^* \mid |\mu|, |\nu| < \infty\}$ therefore forms a set of matrix units for $\langle p^\perp \rangle$. We thus have an isomorphism

$$\operatorname{span}\{s_\mu p^\perp s_\nu^* \mid |\mu|, |\nu| < \infty\} \cong \mathcal{F},$$

the finite rank operators on a separable Hilbert space. Therefore $\langle p^\perp \rangle \cong \mathcal{K}$.

Let $\pi \colon \mathcal{E}_n \to \mathcal{E}_n / \langle p^\perp \rangle$ be the quotient mapping. Then we see that $\pi(p) = \pi(I) - \pi(p^\perp) = \pi(I)$, the unit in the quotient. Therefore, we in $\mathcal{E}_n / \langle p^\perp \rangle$ have

$$\sum_{i=1}^n pi(s_i)\pi(s_i^*) = \pi(p) = \pi(I).$$

As $\mathcal{E}_n / \langle p^\perp \rangle$ is generated by $\pi(s_1), \dots, \pi(s_n)$, we get by Theorem 4.18 that $\mathcal{E}_n / \langle p^\perp \rangle \cong \mathcal{O}_n$. ■

Combining the last two results we see that if A is a simple infinite C^*-algebra and $n \in \mathbb{N}\backslash\{1\}$ then $\mathcal{O}_n$ is a quotient of a sub-C^*-algebra of A.

To show the next interesting result about the Cuntz algebras, we need the following lemma which is also of some independent interest.

Lemma 4.21. *Let A be a simple C^*-algebra and suppose p and q are projections in A such that p is infinite. Then q is equivalent to a subprojection of p.*

Proof. If $q = 0$, the result is trivial, so assume $q > 0$.

As A is simple, there exists $m \in \mathbb{N}$ and elements x_i, y_i, $i = 1, \dots, m$, such that $\|q - \sum_{i=1}^m x_i p y_i\| < 1/2$. Set $b = \frac{1}{2}(\sum_{i=1}^m x_i p y_i + \sum_{i=1}^m y_i^* p x_i^*)$. Then also $\|q - b\| < 1/2$, and therefore $q - b \leq \frac{1}{2} I$, which implies that $q - qbq \leq \frac{1}{2} q$, so we get

$$q \leq 2qbq = \sum_{i=1}^m q x_i p y_i q + \sum_{i=1}^m q y_i^* p x_i^* q.$$

Now, if $y \geq 0$ in A, we have the general inequality $xyz + z^*yx^* \leq xyx^* + z^*yz$ for every $x, z \in A$. We can use this on the above to see that

$$2qbq \leq \sum_{i=1}^m q x_i p x_i^* q + \sum_{i=1}^m q y_i^* p y_i q =: a \leq cI,$$

where $c := \sum_{i=1}^m \|x_i\|^2 + \|y_i\|^2$. By the continuous functional calculus, if $f\colon (0, \infty) \to (0, \infty)$ is given by $f(x) = x^{-1/2}$, then $f \in C(\sigma(a)) \subseteq [\|q\|, c]$. As $a \in qAq$, we see that

$$q = f(a)af(a) = \sum_{i=1}^m f(a) q x_i p x_i^* q f(a) + \sum_{i=1}^m f(a) q y_i^* p y_i q f(a).$$

By Theorem 4.19 we can for $n = 2m$ find partial isometries $s_i \in A$, $1 \leq i \leq n$, such that $\sum_{i=1}^n s_i s_i^* < p = s_i^* s_i$. Define now for $i = 1, \dots, n$

$$z_i := \begin{cases} f(a) q x_i, & i = 1, \dots, m \\ f(a) q y_{i-m}^*, & i = m+1, \dots, 2m, \end{cases}$$

and use this to define $t := \sum_{i=1}^n z_i p s_i^*$. We then get

$$tt^* = \sum_{i=1}^n \sum_{j=1}^n z_i p s_i^* s_j p z_j^* = \sum_{i=1}^n z_i p z_i^* = q.$$

In particular, t is a partial isometry. Therefore $t^*t = pt^*tp$ is a subprojection of p, and the proof is done. ∎

We now turn our attention a little to the concept of hereditary sub-C^*-algebras of a given C^*-algebra A. A sub-C^*-algebra B of A is hereditary if for every $a \in A$ and $b \in B$ such that $0 \leq a \leq b$, then $a \in B$ as well. We state the following general result without proof.

Lemma 4.22. *Let A be a C^*-algebra and $B \subseteq A$ a sub-C^*-algebra. Then B is hereditary if and only if $bab' \in B$ for every $a \in A$ and $b, b' \in B$.*

By this we see that if B contains an invertible element, then $B = A$. Further, given an element $b \in A$, the sub-C^*-algebra $\overline{bAb}$ is a hereditary sub-C^*-algebra of A.

Definition 4.23. A C^*-algebra A is called *purely infinite* if every hereditary sub-C^*-algebra of A is infinite.

The concept of pure infiniteness plays a rather big role in C^*-theory, especially in classification theory. It is the C^*-edition of being a type III factor von Neumann algebra. The intuition behind pure infiniteness is that if a projection is not 0, it is infinite.

When combined with Theorem 4.18, this next result shows that $\mathcal{O}_n$ is purely infinite for every $n \geq 2$. We show here three equivalent conditions, but there are a lot more.

Theorem 4.24. *If A is a simple unital C^*-algebra of dimension at least 2, the following are equivalent:*
i) A is purely infinite,
ii) For every non-zero element $a \in A$ there are elements $x, y \in A$ such that $xay = I$,
iii) For every non-zero positive element $a \in A$ and every $\varepsilon > 0$ there is an element $x \in A$ with $\|x\| < \|a\|^{-1/2} + \varepsilon$ such that $xax^* = I$.

Proof. "**iii)⇒ii)**" Let $a \neq 0$. Then there is an element $y \in A$ such that $y(a^*a)y^* = I$, so letting $x = ya^*$, we are done.
"**ii)⇒i)**" Let $a \neq 0$ be positive and find elements $x, y \in A$ such that $xa^{1/2}y = I$. We then have

$$I = xa^{1/2}yy^*a^{1/2}x^* \leq \|y\|^2 xax^*,$$

and hence $z := xax^*$ is positive and invertible and $I = (z^{-1/2}x)a(x^*z^{-1/2})$. This proves iii) without the norm estimate.

Suppose now that B is a hereditary sub-C^*-algebra of A and let $b \in B$ be a non-scalar positive element which is not invertible. This is possible as $\dim A \geq 2$ and Lemma 4.22. Then by iii) there is an element $z \in A$ such that $zbz^* = I$. Define $s := b^{1/2}x^*$. Then $s^*s = xbx^* = I$ and s is not invertible, so s is a proper isometry. Further, $p := ss^* = b^{1/2}x^*xb^{1/2}$ belongs to B by Lemma 4.22. This is an infinite projection in B, as $sp = s(s^*s)p = psp \in B$ and $(sp)^*(sp) = p$. We obtain that sps^* is a proper subprojection of p due to the fact that $p = ss^* < I$ implies that $sps^* < ss^* = p$. Thus B is infinite, and we are done.
"**i)⇒iii)**" Let $a \in A$ be positive with $\|a\| = 1$, and let $0 < \varepsilon < 1/2$. Let $c = 1 - \varepsilon$. We define a continuous function $f \colon [0,1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} 0, & 0 \leq t \leq c \\ 1 - \frac{1-t}{\varepsilon}, & c \leq t \leq 1 \end{cases}.$$

Let B be the hereditary sub-C^*-algebra $\overline{f(a)Af(a)}$. By assumption, it contains an infinite projection p.

Claim: $pap \geq (1 - \varepsilon)I$.

Proof of claim: Let $b \in f(a)Af(a)$, $b = f(a)xf(a)$ for some $x \in A$. Considering the function $h \colon [0,1] \to \mathbb{R}$ given by $h(t) = tf(t)^2$, we see that $h \geq (1 - \varepsilon)f^2$. Therefore we get that

$$b^*ab = f(a)x^*f(a)af(a)xf(a) \geq (1 - \varepsilon)\big(f(a)x^*f(a)^2xf(a)\big).$$

One can then use states to prove by continuity that this inequality holds for every $b \in B$. Therefore, taking $b = p \in B$, we get the claim.

By Lemma 4.21, the identity is equivalent to a subprojection of p; therefore, there is an isometry $s \in A$ such that $ss^* \leq p$. We then have $s \in pAp$, as can be seen by multiplying from the left and right by $I - p$. Therefore, we get

$$c := s^*as = s^*paps \geq (1 - \varepsilon)s^*s = (1 - \varepsilon)I,$$

so c is invertible. Further,

$$(c^{-1/2}s^*)a(sc^{-1/2}) = (s^*as)^{-1/2}(s^*as)(s^*as)^{-1/2} = I.$$

Lastly, we see that $\|sc^{-1/2}\| \leq (1 - \varepsilon)^{-1/2} < 1 + \varepsilon$, and the proof is done. ∎

We conclude this section, and thereby the thesis, with a brief further discussion on the Cuntz algebras. This will only be presenting some points and will not prove anything other than small simple arguments.

As each Cuntz algebra is in particular infinite, it can not have a faithful tracial state. Therefore, the Cuntz algebras can not arise as either UHF algebras, group C^*-algebras or irrational rotation algebras. Comparing with the irrational rotation algebras, where we have 2 generators and 3 relations, we have n generators and 2 relations when defining the Cuntz algebra $\mathcal{O}_n$. It certainly looks like these two constructions should be very much alike, but they really differ a lot, although they are both separable simple unital universal C^*-algebras.

The classification of the Cuntz algebras requires K-theory. However, the classification goes as one might expect: the Cuntz algebras are pairwise non-isomorphic. The argument is that the K_0-group of $\mathcal{O}_{n+1}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for every $n \in \mathbb{N}$, and that K_0 is a functor from the category of C^*-algebras to the category of abelian groups. This implies that if A and B are isomorphic C^*-algebras then also $K_0(A)$ and $K_0(B)$ are isomorphic abelian groups, as a functor maps isomorphisms to isomorphisms.

Example 4.25. One can also define the algebra $\mathcal{O}_\infty$; for this, suppose we are given a sequence $(s_n)_{n\in\mathbb{N}}$ of isometries such that

$$\sum_{n=1}^{k} s_n s_n^* \leq I$$

for every $k \in \mathbb{N}$; then $\mathcal{O}_\infty$ is defined as the universal C^*-algebra generated by $\{s_n\}_{n\in\mathbb{N}}$. One can in this case modify the proof of Theorem 4.18 to also include $\mathcal{O}_\infty$.

References

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