Applications of
Proper Asymptotic
Unitary Equivalence
in KK-theory

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Abstract

The thesis will be concerned with exploring the most relevant discoveries relating to Søren Eilers’ and Marius Dadarlat’s notion of proper asymptotic unitary equivalence in $KK$-theory. The thesis contains an explanatory run-down of the paper in which the notion of proper asymptotic unitary equivalence is introduced, and it is shown that this leads to the same $KK$-theory as that of Kasparov’s $KK$-theory. Having established the equivalence, Hyon Ho Lee took it further and found that the equivalence can find uses in the theory of Fredholm operators and defined the essential co-dimension of a Cuntz pair of projections to be a class of $KK_h(A, B)$, and thus obtained reasonable conditions for lifting a projection of the corona algebra to a projection in the multiplier algebra. Finally we take a look at M. Dadarlat’s definition of a pseudo-metric which looks at how far Cuntz pairs are from being proper asymptotic unitary equivalent, and then shows that the topology induced by that pseudo-metric on $KK(A, B)$ makes $KK(A, B)$ complete and separable.

Specialet omhandler de mest relevante opdagelser, der er kommet af Søren Eilers’ og Marius Dadarlat’s koncept om fuldstændig asymptotisk unitær ækvivalens i $KK$-teori. Specialet indeholder en forklarende gennemgang af artiklen hvor fuldstændig asymptotisk unitær ækvivalens er introduceret, og der bliver vist at dette fører til den samme $KK$-teori som Kasparov’s. Da ækvivalensen var fastsat, tog Hyon Ho Lee det videre og fandt ud af at ækvivalensen kan være brugbar indenfor teorien om Fredholm operatorer, og definerede den essentielle codimension af et Cuntz par af projektorer til at være en klasse i $KK_h(A, B)$, og derved fik nødvendige betingelser for at løfte projektorer i corona algebran til multiplier algebran. Til slut, kigger vi på M. Dadarlat’s definition af en pseudo-metrik som måler hvor langt Cuntz par er fra at være fuldstændig asymptotisk unitært ækvivalente, og vi viser demnærst at topologien induceret af denne pseudo-metrik på $KK(A, B)$ får $KK(A, B)$ til at blive fuldstændigt og separabelt.
1 KK-theory basics

Kasparov’s $KK$-theory was introduced by Gennadi Kasparov in 1980. It was influenced by Atiyah’s concept of Fredholm modules alongside Brown, Douglas and Fillmore’s work on extensions of $C^*$-algebras. $KK$-theory is also one of the major tools in classification theory, for instance of the separable, nuclear, purely infinite, simple $C^*$-algebras.

1.1 Hilbert $C^*$-modules

Obviously Hilbert Modules are a generalization of Hilbert spaces, which is well-known from advanced analysis. It originates from the algebraic notion of modules, so here follow a strictly algebraic notion.

Definition 1.1. Let $R$ be a ring. A right $R$-module is a set $M$ together with

1. a binary operation $+$ on $M$, which makes $M$ into an abelian group and
2. an action of $R$ on $M$, i.e. a map $M \times R \to M$, by $(m, r) \mapsto mr$ for all $r \in R$ and all $m \in M$, such that the action is associative, distributive and

\[(m + n)r = mr + nr, \quad \text{for all } r \in R, m, n \in M\]

In this case our ring $B$ is a $C^*$-algebra with norm $\| \cdot \|$.

Definition 1.2. A pre-Hilbert $B$-module is a right $B$-module $E$ which is also a complex vector space with a map $\langle \cdot, \cdot \rangle : E \times E \to B$, being linear in the second variable, and for all $b \in B$ and $x, y \in E$

1. $\langle x, yb \rangle = \langle x, y \rangle b$
2. $\langle x, y \rangle^* = \langle y, x \rangle$
3. $\langle x, x \rangle \geq 0$
4. $x \neq 0$ implies $\langle x, x \rangle \neq 0$

With the norm of $x$ defined as $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ where the right side is the standard norm, for $x \in E$, then $E$ becomes a normed vector space which satisfies the following inequalities for $x, y \in E$, and $b \in B$

\[\|xb\| \leq \|x\|||b||\]

and

\[\|\langle x, y \rangle\| \leq \|x\|||y||.\]

Definition 1.3. A Hilbert $B$-module is a pre-Hilbert $B$-module $E$, which is complete in the norm

\[\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}} \quad \text{for} \quad x \in E.\]
Note that $B$ is itself a Hilbert $B$-module with inner product $\langle a, b \rangle = a^*b$ for obvious reasons and that any ideal of $B$ also is a Hilbert $B$-module.

As in the case of Hilbert space theory we are able to look at maps from and to the so called Hilbert $B$-modules. However, note the difference in insisting that the space is also a right module. Let $T : E_1 \to E_2$, where $E_i$ for $i = 1, 2$ are Hilbert $B$-modules, then we say that $T$ is an element of the space $\mathcal{L}_B(E_1, E_2)$ if there is another map $T^* : E_2 \to E_1$, such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \text{ for all } x \in E_1, y \in E_2$$

Note that this also yields the fact that the operators are linear, in fact

$$\langle Tx \lambda, y \rangle = \langle T^* y, x \lambda \rangle = \langle T^* y, x \lambda \rangle = \langle T^* y, \lambda x \rangle = \langle T^* y, \lambda^* x \rangle = \langle x, T^* y \rangle \lambda = \langle Tx, y \lambda \rangle$$

and

$$\langle Tx + y, z \rangle = \langle x + y, T^* z \rangle = \langle x, T^* y \rangle + \langle y, T^* z \rangle = \langle Tx, z \rangle + \langle Ty, z \rangle.$$ 

So we denote by $\mathcal{L}_B(E_1, E_2)$ the closed subspace of the Banach space consisting of bounded linear mappings between Hilbert $B$-modules $E_1$ and $E_2$ with the operator norm. We will omit the subindex $B$, when it is apparent that $E$ is a Hilbert $B$-module and we write $\mathcal{L}(E) = \mathcal{L}(E, E)$. Note that $\mathcal{L}(E_1, E_2)$ is also a $C^*$-algebra.

It is possible to define a map $\Theta_{x, y} : E_1 \to E_2$, where $E_i$ are Hilbert $B$-modules, by $\Theta_{x, y}(z) = x \langle y, z \rangle$ for $z \in E_1$. It is an easy exercise to check that $\Theta_{x, y} \in \mathcal{L}(E_1, E_2)$, and that $\Theta_{x, y}^* = \Theta_{y, x}$. Now, define

$$\mathcal{K}_B(E_1, E_2) = \overline{\text{span}\{\Theta_{x, y} : x \in E_2, y \in E_1\}}$$

We will call these operators “compact”, since $\mathcal{K}_C(H) = \mathbb{K}(H)$ for some Hilbert space $H$. The subindex $B$ will again be omitted if there is no doubt what is meant.

When defining the $KK$-groups, the Hilbert $C^*$-modules are of great importance. This is because the $KK$-groups is defined by a set of maps, which are closely related to Hilbert $C^*$-modules, and endow these with equivalences, all of which will be defined below. What is also important when studying the Kasparov version of $KK$-theory is that all $C^*$ algebras are $\sigma$-unital, i.e., contains a strictly positive element. In fact it is enough that our $C^*$-algebra contains a countable approximate unit. Because of the following lemmas. The first lemma is needed for the next.

**Lemma 1.4** ([GKP79], 3.1.4). An element $\varphi \in A^*$ is positive if and only if $\lim \varphi(u) = ||\varphi||$ for some approximate unit $\{u_\lambda\}$ in $A$.

**Lemma 1.5** ([GKP79], 3.10.5). Let $A$ be a $C^*$-algebra. If there is a countable approximate unit of $A$ then there is a strictly positive element $h \in A$. 

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Proof. Assume that \( \{u_n\}_{n \in \mathbb{N}} \) is a countable approximate unit of \( A \), put \( h = \sum 2^{-n}u_n \). Let \( \varphi \) be a positive linear functional with \( \varphi(h) = 0 \) if \( \varphi = 0 \), then by Lemma 1.4 we get \( \varphi(u_n) = 0 \) for all \( n \). So \( h \) is strictly positive. \( \square \)

In fact the reverse implication is also true.

1.2 The non-graded Kasparov groups \( KK_h(A, B) \) and the trivially graded Kasparov group \( KK^0(A, B) \) as in [DE01]

1.2.1 The non-graded Kasparov groups \( KK_h(A, B) \)

Assume in the next section that all \( C^* \)-algebras are \( \sigma \)-unital and consider \( A, B \) as arbitrary \( C^* \)-algebras. Recall the definition of the strict topology which is defined by seminorms \( l_a \) and \( r_a \) where \( l_a(x) = ||ax|| \) and \( r_a(x) = ||xa|| \) for all \( a \) in some \( C^* \)-algebra.

We denote \( H_B = H \otimes B \).

Definition 1.6. A Cuntz pair (or \( KK_h(A, B) \)-cycle) is a pair of \( \ast \)-homomorphisms or rather representations \( (\varphi_0, \varphi_1) : A \to L(H_B) \) such that

\[
\varphi_0(a) - \varphi_1(a) \in \mathcal{K}(H) \otimes B \quad \text{for} \quad a \in A.
\]

The groups of \( KK \)-theory generally depend on a set with some equivalence relation, or rather \textit{exactly on the relation}. In the case of \( KK_h(A, B) \) the set we are looking for is the one consisting of all Cuntz pairs, whilst the relation we are looking for is that of homotopy. The set of Cuntz pairs will be denoted by \( \mathbb{E}_h(A, B) \).

Definition 1.7. Two Cuntz pairs \( (\varphi_0, \varphi_1), (\varphi'_0, \varphi'_1) \in \mathbb{E}_h(A, B) \) are said to be \textit{homotopic}, written as \( (\varphi_0, \varphi_1) \sim_{\text{hom}} (\varphi'_0, \varphi'_1) \) when there is a path \( h^t \), which is itself a Cuntz pair, i.e. there is \( h^t = (h^t_0, h^t_1) \in \mathbb{E}_h(A, B) \) for \( t \in [0, 1] \) such that

- \( t \mapsto h^t_i(a) \) for \( i = 0, 1 \) from \( [0, 1] \to L(H_B) \) are strictly continuous for all \( a \in A \)
- \( t \mapsto h^t_0(a) - h^t_1(a) \) from \( [0, 1] \) to \( \mathcal{K}(H) \otimes B \) is norm-continuous for all \( a \in A \)
- \( h^0 = (\varphi_0, \varphi_1) \), and \( h^1 = (\varphi'_0, \varphi'_1) \)

We define \( KK_h(A, B) \) to be \( \mathbb{E}_h(A, B)/\sim_{\text{hom}} \). We will call \( KK_h(A, B) \) the Cuntz picture since it depends on Cuntz pairs. It is possible to show that it is an abelian group with \( 0 \)-element \( (0, 0) \), and \( -(\varphi_0, \varphi_1) = (\varphi_1, \varphi_0) \) for some Cuntz pair. The composition on \( KK_h(A, B) \) requires another map namely an inner \( \ast \)-isomorphism.
Definition 1.8. Let $B$ be a stable $C^*$-algebra. An inner $\ast$-isomorphism
$\Theta : \text{Mat}_n(B) \to B$ is a homomorphism such that there are isometries
$w_1, \ldots, w_n \in M(B)$, such that $w_i^*w_j = 0$, when $i \neq j$ and $\sum_{i=1}^n w_i w_i^* = 1$ for
$i, j \in [1, \ldots, n]$ and

$$\Theta(\{b_{ij}\}) = \sum_{i,j} w_i b_{ij} w_j^* \text{ for } \{b_{ij}\} \in \text{Mat}_n(B).$$

In our case we require that $\Theta : \text{Mat}_2(\mathcal{L}(H_B)) \to \mathcal{L}(H_B)$ is an inner
$\ast$-isomorphism, however this is an extension of the $\Theta$ in the definition, so no
distinction will be made. We should just require that $KK(A,B)$ be stable,
but since $KK \cong K$ this follows, so there is nothing to hinder in referring
to inner $\ast$-isomorphisms. This gives us the composition $+$ in $KK(A,B)$ we
require. So for $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in E_h(A,b)$, the composition in $KK(A,B)$ is given by

$$[\varphi_0, \varphi_1] + [\psi_0, \psi_1] = \begin{bmatrix} \Theta \circ \begin{pmatrix} \varphi_0 & 0 \\ 0 & \psi_0 \end{pmatrix}, \Theta \circ \begin{pmatrix} \varphi_1 & 0 \\ 0 & \psi_1 \end{pmatrix} \end{bmatrix} = (\varphi_0 \oplus \psi_0, \varphi_1 \oplus \psi_1).$$

Note that the sum does not depend on the choice of $\Theta$ since we get homotopic
sums in fact if $s_1, s_2, t_1, t_2 \in \mathcal{L}(H_B)$ are isometries such that

$$s_1 s_1^* + s_2 s_2^* = t_1 t_1^* + t_2 t_2^* = 1.$$ 

Then we define two different inner $\ast$-isomorphisms as above, $\Theta(\{b_{ij}\})$ with
$s_i$ and $\Theta'(\{b_{ij}\})$ with $t_i$ as above. Then $u = t_1 s_1^* + t_2 s_2^* \in \mathcal{L}(H_B)$ is a unitary
such that

$$u^* \Theta'(\{b_{ij}\}) u = \Theta(\{b_{ij}\}).$$

Then since the unitary group $U(\mathcal{L}(H_B))$ is contractible then there is a norm-
continuous path $(u_t)_{t \in [0,1]}$, such that $u_0 = 1$ and $u_1 = u$. So if $(\varphi_0, \varphi_1)$ and
$(\psi_0, \psi_1)$ are Cuntz pairs then

$$\left( u_t \Theta' \left( \begin{pmatrix} \varphi_0 & 0 \\ 0 & \psi_0 \end{pmatrix} \right) u_t^*, u_t \Theta' \left( \begin{pmatrix} \varphi_1 & 0 \\ 0 & \psi_1 \end{pmatrix} \right) u_t^* \right)$$

is a homotopy of Cuntz pairs from

$$\left( \Theta' \circ \begin{pmatrix} \varphi_0 & 0 \\ 0 & \psi_0 \end{pmatrix}, \Theta' \circ \begin{pmatrix} \varphi_1 & 0 \\ 0 & \psi_1 \end{pmatrix} \right)$$

to

$$\left( \Theta \circ \begin{pmatrix} \varphi_0 & 0 \\ 0 & \psi_0 \end{pmatrix}, \Theta \circ \begin{pmatrix} \varphi_1 & 0 \\ 0 & \psi_1 \end{pmatrix} \right)$$
1.2.2 The trivially graded Kasparov group $KK^0(A, B)$ as in [DE01]

For the $KK^0(A, B)$ groups, we will require some different set, namely the set of $KK$-cycles.

**Definition 1.9.** Let $E_i$ be countably generated Hilbert $C^*$-modules. A $KK$-cycle, or merely cycle, consists of the following $^*$-homomorphisms

$$
\varphi^i : A \to \mathcal{L}(E_i) \quad \text{for } i = 0, 1
$$

and a map $F \in \mathcal{L}(E_0, E_1)$ which satisfies

- $F\varphi^0(a) - \varphi^1(a)F \in \mathcal{K}(E_0, E_1)$.
- $\varphi^0(a)(F^*F - 1) \in \mathcal{K}(E_0)$.
- $\varphi^1(a)(FF^* - 1) \in \mathcal{K}(E_1)$.

We will denote by $E(A, B)$ the set consisting of $KK$-cycles, which are denoted $(\varphi^0, \varphi^1, F)$.

As it turns out, $KK^0(A, B)$ is defined by not only one equivalence but in fact by several. These are listed below.

**Definition 1.10.** Let $\pi : A \to \mathcal{L}(E)$ and $\sigma : A \to \mathcal{L}(E)$ be representations, where $A$ is a $C^*$-algebra, then $\pi$ is said to be *approximately unitarily equivalent* to $\sigma$, and we write $\pi \sim \sigma$, if there is a sequence of unitaries $U_n$ such that

$$
\lim_{n \to \infty} ||U_n\sigma(a)U_n^* - \pi(a)|| = 0, \quad \text{for all } a \in A
$$

and

$$
U_n\sigma(a)U_n^* - \pi(a) \in \mathcal{K}(E)
$$

This gives a notion of approximate unitary equivalence of $KK$-cycles simply by equivalence of the representations in question, in the definition of the $KK$-cycles. Besides unitary equivalence, there is that of adding, what is known as, degenerated $KK$-cycles.

**Definition 1.11.** A cycle $(\varphi^0, \varphi^1, F) \in E(A, B)$ is said to be *degenerate* if

$$
F\varphi^0(a) - \varphi^1(a)F, \varphi^0(a)(F^*F - 1), \varphi^1(a)(FF^* - 1)
$$

are all zero.

Note that if $F = 1$, then it merely depends on whether the $^*$-homomorphisms are equal, for a cycle to be degenerated. Finally there is the equivalence of operatorial homotopy.
**Definition 1.12.** Two $KK$-cycles $(\varphi^0, \varphi^1, F), (\psi^0, \psi^1, F')$ are said to be **operatorially homotopic** if $\varphi^0 = \psi^0$ and $\varphi^1 = \psi^1$ and there is a homotopy $H_t : [0, 1] \times E_0 \to E_1$ such that $H_t(0, e) = F(e)$ and $H_t(1, e) = F'(e)$ and such that the map $t \mapsto H_t$ is norm-continuous. We write it as $(\varphi^0, \varphi^1, F) \sim_{oh} (\psi^0, \psi^1, F')$.

The definition of the $KK$-groups, now follows as before, as being the set $E(A, B)$ where you mod out all of the following mentioned equivalences, namely those of of unitary equivalence, adding degenerated cycles and that of operatorial homotopy. As before we can define a composition in the group which is

$$[(\varphi^0, \varphi^1, F) \oplus (\psi^0, \psi^1, F')] = [\varphi^0, \varphi^1, F] + [\psi^0, \psi^1, F']$$

Define by $\sigma_\infty : A \to \mathcal{L}(E_\infty)$ a sum of representations $\sigma : A \to \mathcal{L}(E)$, by $\sigma_\infty = \sigma \oplus \sigma \oplus \sigma \oplus \ldots$ and such that $E_\infty = E \oplus E \oplus \ldots$

The sum in $KK(A, B)$ leads on to what is to be expected of degenerated elements.

**Lemma 1.13.** If $(\varphi^0, \varphi^1, F)$ is degenerated then $[\varphi^0, \varphi^1, F] = 0$.

**Proof.** Take $E_\infty = E \oplus E \oplus E \ldots$ and representations $\varphi^i_\infty : A \to \mathcal{L}(E_\infty)$ such that $\varphi^i_\infty = \varphi^i \oplus \varphi^i \oplus \ldots$ for $i = 0, 1$. Then $[\varphi^0, \varphi^1, F]_\infty$ is still degenerated since it is merely a sum of 0's. Finally, since

$$(\varphi^0, \varphi^1, F)_\infty \oplus (\psi^0, \psi^1, F) \sim (\varphi^0, \varphi^1, F)_\infty$$

then

$$[\varphi^0, \varphi^1, F]_\infty + [\psi^0, \varphi^1, F] = [(\varphi^0, \varphi^1, F)_\infty \oplus (\psi^0, \varphi^1, F)] = [\varphi^0, \varphi^1, F]_\infty$$

so $[\varphi^0, \varphi^1, F] = 0$. \qed

As it turns out there is a homomorphism from $KK_h(A, B) \to KK^0(A, B)$ given by $[\varphi_0, \varphi_1] \mapsto [\varphi_0, \varphi_1, 1]$, which is actually also an isomorphism, and in general it will be very convenient to freely alternate between the two notions of $KK$-Theory, so we state the theorem without proof. A proof can be found in [JTS9].

**Theorem 1.14.** The map $\Gamma : KK_h(A, B) \to KK^0(A, B)$ defined by $[\varphi_0, \varphi_1] \mapsto [\varphi_0, \varphi_1, 1]$ is an isomorphism.
2 Derivations, automorphisms and absorbing representations

2.1 Automorphisms and derivations

We need two results from [GKP79] to complete one of our major results, given without proof. However, we start of with some definitions, but not before noting that from now on $\mathcal{B}(A)$ will be the Banach algebra of bounded linear operators on $A$, where $A$ is a separable unital $C^*$-algebra.

**Definition 2.1.** Let $A$ be a unital separable $C^*$-algebra and let $\text{Aut}(A)$ be the subset of $\mathcal{B}(A)$ of $*$-automorphisms. Let $u$ be a unitary then we define an *inner $*$-automorphism $\text{Ad}(u) : A \to A$ by

$$\text{Ad}(u)(a) = uau^*$$

We denote by $\text{Der}(A)$ the subset of $\mathcal{B}(A)$ of $*$-derivations. Let $ih$ be a self-adjoint then we define an *inner $*$-derivation $\text{ad}(ih) : A \to A$ by

$$\text{ad}(ih)(a) = i(ha - ah)$$

A derivation is simply a map $\delta : A \to A$ s.t. $\delta(xy) = x\delta(y) + \delta(x)y$, whilst a $*$-derivation is a derivation which maps self-adjoints to self-adjoints.

One should note that $\exp : \text{Der}(A) \to \text{Aut}(A)$ defined by

$$\exp(\delta) = 1_{\text{Der}(A)} + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \ldots + \frac{\delta^{n-1}}{(n-1)!}$$

is continuous in the point-norm topology and that we get

$$\exp(\text{ad}(ih)) = \text{Ad}(\exp(ih)).$$

**Definition 2.2.** We say that a derivation $\delta$ is asymptotically inner if there is a norm-continuous bounded family of self-adjoint elements $h_t$ for $t \in [0, \infty)$ such that

$$\lim_{t \to \infty} ||\text{ad}(ih_t)(a) - \delta(a)|| = 0$$

for $a \in A$.

**Lemma 2.3** ([GKP79], 8.6.12/8.6.5). Let $A$ be a unital separable $C^*$-algebra and let $\delta$ be a $*$-derivation of $A$. Then there is an increasing sequence $h_n \in A_+$ that converges to $h$, where $h \in A_+$ such that $\delta = \text{ad}(ih)$ and $||\delta|| = ||h||$. Furthermore

$$||\delta(h_n)|| \to 0, \text{ and } ||\text{ad}(ih_n(a) - \delta(a))|| \to 0$$

for every $a \in A$. 

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Lemma 2.4 ([DE01], 2.12). Let $A$ be a unital separable $C^*$-algebra, then any $*$-derivation $\delta$ is asymptotically inner, in fact one may choose $h_t$ in the definition of asymptotically inner such that $||h_t|| \leq ||\delta||$.

Proof. By Lemma 2.3 there is an increasing sequence $h_n$ of positive elements (self-adjoint) such that $\delta = \text{ad}(ih)$ and $||\delta|| = ||h||$, furthermore

$$\lim_{n \to \infty} ||\text{ad}(ih_n)(a) - \delta(a)|| = 0$$

for all $a \in A$. Since the sequence was increasing then $||h_n|| \leq ||\delta||$.

We can now find a norm-continuous bounded family of those self-adjoints $(h_t)$ for $t \in [0, \infty)$ such that

$$\lim_{t \to \infty} ||\text{ad}(ih_t)(a) - \delta(a)|| = 0$$

so that $i$ correspond with the definition of asymptotically inner. $\square$

Lemma 2.5. The map $\exp : \text{Der}(A) \to \text{Aut}(A)$ is continuous in the point-norm topology.

Proof. Let $\delta_n \to \delta$ be a sequence in $\text{Der}(A)$ such that $\lim_{n \to \infty} ||\delta_n(a) - \delta(a)|| = 0$. Then by Lemma 2.3 $\delta = \text{ad}(ih)$ so we also get the sequence $h_n$ which converges to $h$ and we calculate

$$\lim_{n \to \infty} ||\exp(\delta_n)(a) - \exp(\delta)(a)|| = \lim_{n \to \infty} ||\exp(\text{ad}(ih_n))(a) - \exp(\text{ad}(ih))(a)||$$

$$= \lim_{n \to \infty} ||\text{Ad}(\exp(ih_n))(a) - \text{Ad}(\exp(ih))(a)||$$

$$= 0$$

$\square$

Definition 2.6. An asymptotically inner automorphism $\alpha$ is an automorphism for which there is a continuous path of unitaries $(u_t)$ for $t \in [0, \infty)$ such that

$$\lim_{n \to \infty} ||\text{Ad}(u_t)(a) - \alpha(a)||$$

for all $a \in A$.

Lemma 2.7 ([GKP79], 8.7.8). Given a $C^*$-algebra $A$, let $\text{Aut}(A)$ have the uniform topology. Then any automorphism that can be written as $\exp(\delta) = \alpha$ for some derivation $\delta$, i.e., a derivable automorphism, lies in the connected component of the identity of $\text{Aut}(A)$, i.e. $\text{Aut}_0(A)$. Also any automorphism in the connected component of the identity is a product of derivable automorphisms.

Lemma 2.8 ([DE01] 2.14). If $A$ is a unital separable $C^*$-algebra, then any automorphism in the connected component of the identity is asymptotically inner.
Proof. If $\alpha$ is in the connected component of the identity then by Lemma 2.7 it is the product of derivable automorphisms, i.e. automorphisms of the form $\delta, \exp(\delta) \in \text{Aut}(A)$, so we can assume that $\alpha = \exp(\delta)$. By Lemma 2.4 we have
\[
\lim_{t \to \infty} ||\text{ad}(ih_t)(a) - \delta(a)|| = 0
\]
by exponentiating
\[
0 = \lim_{t \to \infty} ||\exp(\text{ad}(ih_t))(a) - \exp(\delta)(a)|| = \lim_{t \to \infty} ||\text{Ad}(\exp(ih_t))(a) - \alpha(a)|| = 0
\]
and since exponentiating a self-adjoint yields a unitary. We are done since by Lemma 2.7 any derivable automorphism lies in the connected component of the identity.

The last result will be stated but not proved. A proof can be found in [DE01], but it is technical.

Proposition 2.9 ([DE01], Proposition 2.15). Let $A$ be a unital separable $C^*$-algebra. If $(\alpha_t)_{t \in [0, \infty)}$ is a uniformly continuous family in $\text{Aut}(A)$ with $\alpha_0 = \text{id}_A$, then there exists a continuous family $(v_t)_{t \in [0, \infty)}$ of unitaries in $A$ with $v_0 = 1$ such that
\[
\lim_{t \to \infty} ||\alpha_t(a) - \text{Ad}(v_t)(a)|| = 0
\]
for all $a \in A$.

2.2 Absorbing representations

As one suspect “to be absorbing” is a way of forgetting about or ingesting a representation. This is done through unitary equivalence.

Definition 2.10. Let $\pi : A \to \mathcal{L}(E)$ be a representation. The representation $\pi$ is said to be absorbing if $\pi \oplus \sigma \sim \pi$ for all $\sigma : A \to \mathcal{L}(E_2)$. If $A$ is unital and $\sigma$ is a unital representation, $\pi$ is said to be unitaly absorbing.

One might think that to be absorbing is quite strict, however a result from K. Thomsen ensures that if $A$ and $B$ are separable, with $B$ stable, then there is an absorbing representation of $A$. K. Thomsen constructed a setup where a map needed only to fulfill one of 4 equivalent conditions, to deduce that it was unitarily absorbing. In fact

Theorem 2.11 ([KT00], Theorem 2.1). Let $A$ and $B$ be separable $C^*$-algebras, where $A$ is unital and $B$ is stable. Let $\pi : A \to \mathcal{L}(H_B)$, a unital $*$-homomorphism. Then the following conditions are equivalent
1. For any completely positive contraction \( \varphi : A \to B \) there is a sequence \( w_n \in \mathcal{L}(H_B) \) such that
\[
\lim_{n \to \infty} \| \varphi(a) - w_n^* \pi(a) w_n \| = 0 \quad \text{for all } a \in A
\]
\[
\lim_{n \to \infty} \| w_n b \| = 0 \quad \text{for all } b \in B.
\]

2. For any completely positive unital map \( \varphi : A \to \mathcal{L}(H_B) \), there is a sequence of isometries \( v_n \in \mathcal{L}(H_B) \) such that
\[
v_n^* \pi(a) v_n - \varphi(a) \in \mathcal{K}(H_B) \quad \text{for } n \in \mathbb{N} \text{ and } a \in A
\]
\[
\lim_{n \to \infty} \| v_n^* \pi(a) v_n - \varphi(a) \| = 0 \quad \text{for all } a \in A.
\]

3. For any unital \(*\)-homomorphism \( \varphi : A \to \mathcal{L}(H_B) \), there is a sequence of unitaries \( u_n \in \mathcal{L}(B \oplus B, B) \) such that
\[
u_n(\pi(a) \oplus \varphi(a)) u_n^* - \pi(a) \in \mathcal{K}(H_B) \quad \text{for } n \in \mathbb{N} \text{ and } a \in A
\]
\[
\lim_{n \to \infty} \| u_n^* (\pi(a) \oplus \varphi(a)) u_n - \pi(a) \| = 0 \quad \text{for all } a \in A.
\]

4. For any unital \(*\)-homomorphism \( \varphi : A \to \mathcal{L}(H_B) \), there is a sequence of unitaries \( u_n \in \mathcal{L}(B \oplus B, B) \) such that
\[
\lim_{n \to \infty} \| u_n^* (\pi(a) \oplus \varphi(a)) u_n - \pi(a) \| = 0 \quad \text{for all } a \in A.
\]

Note that we here write \( \mathcal{L}(H_B) \) instead of \( M(B) \) which Thomsen wrote in \cite{KT00}. This is merely due to the fact that \( B \) is stable so \( B \cong B \otimes \mathcal{K}(H) \), and in consequence
\[
M(B) \cong M(B \otimes \mathcal{K}(H)) \cong \mathcal{L}(H_B).
\]

K. Thomsen’s definition of unitarily absorbing was to fulfill one of the above requirements, and glancing at Definition 2.10 this is exactly (3).

The next lemma should be well-known.

**Lemma 2.12** \cite{KT00}, Lemma 2.3. *Let \( A \) and \( B \) be separable \( C^*\)-algebras. There is a countable set \( X \) which consists of completely positive contractions from \( A \) to \( B \), such that for any completely positive contraction \( \mu : A \to B \), any finite set \( F \subseteq A \), and any \( \varepsilon > 0 \), there is an element \( l \in X \) such that
\[
\| \mu(f) - l(f) \| \leq \varepsilon \quad \text{for } f \in F.
\]*

This lemma essentially says that the space of all completely positive contractions with the point-norm topology is separable. A proof can be found in \cite{KT00}. 

Lemma 2.13 ([JT89], Lemma 1.3.2). If $B$ is stable, then $H_B \cong B$.

Proof. First off we show that there is a sequence of isometries $v_i \in \mathcal{L}(H_B)$, such that $v_i^* v_j = 0$ when $i \neq j$, and $\sum_{i=1}^{\infty} v_i v_i^* = 1$, in the strict topology. Let $N_i$ be a partition of $\mathbb{N}$ into infinitely many subsets, which are all infinite in size, and let $\varphi_i : N_i \to \mathbb{N}$ be bijections for each $i$. Define $v_i : H_B \to H_B$ by

$$v_i(b_1, b_2, ...) = \begin{cases} b_{\varphi_i(j)} & j \in N_i \\ 0 & \text{else} \end{cases}$$

for $(b_1, b_2, ...) \in H_B$.

Then for $j \in N_j$

$$v_i^* v_j(b_1, b_2, ...) = v_i^*(b_{\varphi_j(j)}) = 0$$

since $j \notin N_i$. Also

$$\sum_{i=1}^{\infty} v_i v_i^*(b_{\varphi_i(j)}) = \sum_{i=1}^{\infty} v_i(b_1, b_2, ...) = v_i(b_1, b_2, ...) = b_{\varphi_i(j)}$$

where the second to last equality came from the fact that $j$ is only in one $N_i$ for some $i$. So $\sum_i v_i^* v_i = 1$.

Since $M(\mathcal{K}(H) \otimes B) \cong \mathcal{L}(H_B)$ we get isometries $u_i \in M(\mathcal{K}(H_B))$ with the same properties. Since $B$ is stable we get isometries $w_i \in M(B)$ again with the same properties. Now finally define $\varphi : H_B \to B$ by

$$\varphi(b_1, b_2, ...) = \sum_{i=1}^{\infty} w_i b_i$$

Which is the isomorphism we were looking for. \qed

Theorem 2.14 ([GK80], Theorem 3(1)). Let $A$ be a unital separable $C^*$-algebra and $B$ be a $\sigma$-unital $C^*$-algebra. Let $\varphi : A \to \mathcal{L}(H_B)$ be completely positive. There is a unital $*$-homomorphism $\pi : A \to \mathcal{L}(H_B)$, and an element $v \in \mathcal{L}(H_B)$ such that $\varphi(a) = v^* \pi(a) v$ for all $a \in A$.

A proof can be found in [GK80].

Theorem 2.15 ([KT00], Theorem 2.4). Let $A, B$ be separable $C^*$-algebras. Assume that $B$ is stable and that $A$ is unital, then there exists a unitarily absorbing $*$-homomorphism $\pi : A \to \mathcal{L}(H_B)$.

Proof. One can deduce from Lemma 2.12 that there is a dense sequence $\{s_n\}_{n \in \mathbb{N}}$ consisting of completely positive contractions from $A$ to $B$. Assume that each $s_n$ is repeated infinitely often in the sequence. By Theorem 2.14...
there are elements $v_n \in \mathcal{L}(H_B)$ and unital $\ast$-homomorphisms

\[ \pi_n : A \to \mathcal{L}(H_B) \]

such that

\[ s_n(\cdot) = v_n^* \pi_n(\cdot) v_n \]

for all $n$. Since $s_n$ is contractive then

\[ 1 \geq ||s_n(1)|| = ||v_n^* \pi_n(1)v_n|| = ||v_n^*v_n|| = ||v_n||^2. \]

Define a unital $\ast$-homomorphism $\pi_\infty : A \to \mathcal{L}(H_B)$, by

\[ \pi_\infty(a)(b_1, b_2, ...) = (\pi_1(a) b_1, \pi_2(a) b_2, \pi_3(a) b_3, ...). \]

Define $L_n \in \mathcal{L}(B, H_B)$ by

\[ L_n b = (0, 0, ..., v_n b, 0, 0, ...) \]

where $v_n b$ occurs at the $n$'th coordinate. Since $s_n$ is repeated infinitely often in $\{s_n\}$, we get for each $n$ a sequence of natural numbers $k_i$, such that $k_1 < k_2 < k_3 < ...$ and such that

\[ s_n(a) = v_n^* \pi(a) v_n = L_{k_i}^* \pi_\infty(a) L_{k_i} \]

for all $a \in A$, $i \in \mathbb{N}$. Also

\[ \lim_{i \to \infty} ||L_{k_i}^*(b_1, b_2, ...)|| = \lim_{i \to \infty} ||v_n^* b_i|| = 0 \]

for $(b_1, b_2, ...) \in H_B$. By Theorem 2.13 there is an isomorphism $S : H_B \to B$.

Set $T_n = SL_n \in \mathcal{L}(H_B)$ and $\pi(\cdot) = s \pi_\infty(\cdot) s^*$. We now wish to prove that $\pi$ satisfies (1) of Theorem 2.11. So let $\varphi : A \to B$ be a completely positive contraction. As $A$ and $B$ are separable it is enough to show that there is some $\epsilon > 0$, and finite subsets $F_1 \subseteq A$ and $F_2 \subseteq A$, and find an element $w \in \mathcal{L}(H_B)$ such that

\[ ||\pi(a) - w^* \pi(a) w|| < \epsilon \text{ for } a \in F_1, \quad ||w^*b|| < \epsilon \text{ for } b \in F_2. \]

Choose $n \in \mathbb{N}$, such that $||\varphi(a) - s_n(a)|| < \epsilon$. Now choose a sequence $k_1 < k_2 < ...$ such that

\[ s_n(a) = v_n^* \pi(a) v_n = L_{k_i}^* \pi_\infty(a) L_{k_i} \]

for all $a \in A$, $i \in \mathbb{N}$. Also

\[ \lim_{i \to \infty} ||L_{k_i}^*(b_1, b_2, ...)|| = 0 \]

for $(b_1, b_2, ...) \in H_B$. Then $T_{k_i}^* \pi(a) T_{k_i} = s_n(a)$ for all $a \in F_1$ and $||T_{k_i}^* b|| < \epsilon$ for $b \in F_2$, for $i$ big enough. Then just set $w = T_{k_i}$ for that big enough $i$. □
If one does not have the property that $A$ is unital, then $\pi$ does not become unital in consequence.

**Remark 2.16.** As with Theorem 2.15, one may do the same again, only unitallity is omitted, and the non-unital elements are unitized, as follows. Let $\pi : A \to \mathcal{L}(E)$ be a representation, and denote by $A_{\text{un}}$ be the unitilization of $A$ by adjoining a unit, and define $\pi_{\text{un}} : A_{\text{un}} \to \mathcal{L}(E)$ by

$$\pi_{\text{un}}(a + \lambda 1_{\text{un}}) = \pi(a) + \lambda 1_E.$$

**Theorem 2.17 ([KT00], Theorem 2.7).** Let $A$ and $B$ be separable $C^*$-algebras, with $B$ stable. Then there is an absorbing $*$-homomorphism $\pi : A \to \mathcal{L}(H_B)$.

**Proof.** By the above remark and Theorem 2.15, the result follows. \qed

**Definition 2.18.** A completely positive contraction $\pi : a \to \mathcal{L}(E)$ is strictly nuclear if there exists $n_\lambda \in \mathbb{Z}$ and generalized sequences $\psi_\lambda : A \to M_{n_\lambda}(\mathbb{C})$ and $\varphi_\lambda : M_{n_\lambda} \to \mathcal{L}(E)$ such that the following diagram commutes in the point-strict topology, when $\lambda \to \infty$

$$\begin{array}{ccc}
A & \xrightarrow{\pi} & \mathcal{L}(E) \\
\downarrow{\psi_\lambda} & & \downarrow{\varphi_\lambda} \\
M_{n_\lambda}(\mathbb{C}) & & \\
\end{array}$$

i.e.

$$\lim_{\lambda \to \infty} ||T^*\pi(a)T - T^*\varphi_\lambda(\psi_\lambda)T|| = 0$$

for all $a \in A$ and $T \in \mathcal{K}(E)$.

Joining Definition 2.10 and Definition 2.18 we get the following definition

**Definition 2.19.** Let $\pi : a \to \mathcal{L}(E_1)$ be a representation. Then $\pi$ is called nuclearly absorbing if $\pi \oplus \sigma \sim \pi$ for all strictly nuclear representations $\sigma : A \to \mathcal{L}(E_2)$. Furthermore, if $A$ is unital and $\sigma$ is unital strictly nuclear, then $\pi$ is called unitarily nuclearly absorbing.

Note that if $\pi : A \to \mathcal{L}(E_1)$ is unital and $\sigma : A \to \mathcal{L}(E_2)$ is non-unital, then $\pi \oplus \sigma \sim \pi$, since if $\pi$ were absorbing then $u_t(1 \oplus \sigma(1))u_t^*$ is a projection which is not 1. So the following is true

$$0 = \lim_{t \to \infty} ||u_t(\pi(1) \oplus \sigma(1))u_t^* - \pi(1)|| = \lim_{t \to \infty} ||u_t(1 \oplus \sigma(1))u_t^* - 1|| = 1$$

which is a contradiction so $\sigma$ must be unital.

It is worthwhile noticing that if a representation $\pi : A \to \mathcal{L}(E)$ is non-unital and nuclearly absorbing, then $\pi_{\text{un}}$ is a unital nuclearly absorbing representation.
Lemma 2.20 ([DE02] 2.17). Let $A$ be a unital separable $C^*$-algebra and let $B$ be a $\sigma$-unital $C^*$-algebra. If $\gamma : A \to \mathcal{L}(H_B)$ is a unital nuclearly absorbing representation then $\gamma = 0 \oplus \gamma : A \to \mathcal{L}(H_B \oplus H)$ is a non-unital nuclearly absorbing representation.

Proof. Assume that $\varphi : A \to \mathcal{L}(E)$ is a strictly nuclear representation and let $\varphi(1) = p$ be a projection. Then $E = pE \oplus p^\perp E$, and $\varphi = p\varphi(\cdot)p \oplus 0_{p^\perp E}$. Then since $\gamma$ is nuclearly absorbing and $p\varphi(\cdot)p$ is strictly nuclear then $\gamma \oplus p\varphi(\cdot)p \sim \gamma$. Since $p^\perp E$ obviously is a countable Hilbert module then by Kasparov’s Stabilization Theorem we get $p^\perp E \oplus H_B \cong H_B$. So by calculation we get

\[
\varphi \oplus \hat{\gamma} = p\varphi(\cdot)p \oplus 0_{p^\perp E} \oplus 0_{H_B} \oplus \gamma \sim p\varphi(\cdot)p \oplus 0_{H_B} \oplus \gamma \sim 0_{H_B} \oplus \gamma = \hat{\gamma}.
\]

So $\hat{\gamma}$ is nuclearly absorbing, since $\varphi$ was nuclear. \hfill \Box

Definition 2.21. Let $A$ be a separable $C^*$-algebra. A faithful scalar representation of infinite multiplicity $\theta : A \to \mathcal{L}(H_B)$ is a $*$-homomorphism which factors as

$A \xrightarrow{\theta'} \mathcal{L}(H) \xrightarrow{\sim} \mathcal{L}(H) \otimes M(B) \hookrightarrow \mathcal{L}(H_B)$

where $\theta' = \infty \cdot \gamma$ is faithful and $\gamma : A \to \mathcal{L}(H)$ is some representation.

Proposition 2.22 ([DE02], Proposition 2.18). Let $A$ be a unital separable $C^*$-algebra. If $\theta : A \to \mathcal{L}(H_B)$ is a unital faithful scalar representation of infinite multiplicity, then $\theta$ is unitarily nuclearly absorbing.

To prove Proposition 2.22 we need an implication from a result from [DE02] and a theorem from Kasparov, which will not be proved. First of we need to define a relation.

Definition 2.23. If $\pi : A \to \mathcal{L}(E_1)$ is a representation and $\varphi : A \to \mathcal{K}(E_2)$ is a completely positive map, then write $\varphi \prec \pi$ if there is a bounded sequence $v_i \in \mathcal{K}(E_2, E_1)$ such that

\[
\lim_{i \to \infty} ||\varphi(a) - v_i^* \pi(a) v_i|| = 0, \quad \lim_{i \to \infty} ||v_i^* \xi|| = 0, \text{ for all } a \in A, \xi \in E_1
\]

Theorem 2.24 ([DE02] 2.13). Let $A$ be a unital separable $C^*$-algebra, and $B$ a $\sigma$-unital $C^*$-algebra. Let $\pi : A \to \mathcal{L}(E_1)$ and $\sigma : A \to \mathcal{L}(E_2)$ be unital representations, then the following are equivalent.

1. $\pi \sim \sigma_\infty \oplus \pi$

2. If $\varphi = v^* \sigma(\cdot)v$ for any $v \in \mathcal{K}(E_2)$, then $\varphi \prec \pi$

3. There is an increasing approximate unit $(e_n) \in \mathcal{K}(E_2)$ with $e_1 = 0$ and quasi-central for $\sigma(A) + \mathcal{K}(E_2)$ such that

\[
(e_{n+k} - e_n)^{\frac{1}{2}} \sigma(\cdot)(e_{n+k} - e_n)^{\frac{1}{2}} \prec \pi
\]

for all $n, k \geq 1$.  

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To be a quasi-central approximate unit in $\mathcal{K}(E_2)$ for $\varphi(A) + \mathcal{K}(E_2)$ is an increasing net of positive operators $\{a_n\} \in \mathcal{K}(E_2)$, which is an approximate unit for $\mathcal{K}(E_2)$ and satisfies

$$||a_n b - b a_n|| \to 0, \text{ for all } b \in \varphi(A) + \mathcal{K}(E_2).$$

The implication we need is 2 implies 1.

**Theorem 2.25 ([GK80], Theorem 4).** Assume that $A$ is separable and let $\theta : A \to \mathcal{L}(H_B)$ be a faithful unital scalar representation of infinite multiplicity and let $\varphi : A \to \text{Mat}_n(B)$ be nuclear. Then $\varphi \prec \theta$.

Now return to the proposition.

**Proof of Proposition 2.22.** As stated in Definition 2.19, we should prove that if $\sigma : A \to \mathcal{L}(E)$ is a strictly nuclear representation then $\sigma \oplus \theta \sim \theta$.

Assume as a start that $E = H_B$. Then define a map $\varphi : A \to \mathcal{K}(H_B)$ by $\varphi = v^* \sigma(\cdot) v$, for $v \in \mathcal{K}(H_B)$ a unitary. Then the set

$$\{ \varphi : A \to \mathcal{K}(H_B) \mid \varphi \prec \theta \}$$

is closed in point-norm. So for some $n$ we may assume that $v \in \text{Mat}_n(B)$ such that $\varphi : A \to \text{Mat}_n(B)$. Since $\varphi$ is nuclear and since $\sigma$ was strictly nuclear, Theorem 2.25 applies to get $\varphi \prec \theta$. By Theorem 2.24 ((2) implies (1)), then $\theta \oplus \sigma \sim \theta$.

What is left to show is that the assumption $E = H_B$ is allowable. If indeed $E \cong H_B$, then let the isomorphism $E \to H_B$ be given by a unitary $u$, then it follows from exchanging $\sigma$ by $u \sigma u^*$ in the previous. Indeed, from unitary equivalence we get $\sigma \oplus \theta \sim u \sigma(\cdot) u^* \oplus \theta$, and by the assumed exchange $u \sigma(\cdot) u^* \oplus \theta \sim \theta$.

If $E \not\cong H_B$ then define $\sigma \oplus \theta : A \to \mathcal{L}(E \oplus H_B)$. By Kasparov’s Stabilization Theorem we already know that $E \oplus H_B \cong H_B$, so let $u' \in \mathcal{L}(H_B)$ be the unitary which implements the isomorphism, then from the instance $E \cong H_B$, we get

$$(\sigma \oplus \theta)_\infty \oplus \theta \sim u'(\sigma \oplus \theta)_\infty u'^* \oplus \theta \sim \theta$$

Furthermore, there is a unitary $\tilde{u} \in \mathcal{L}(H_B)$ such that $\sigma \oplus (\sigma \oplus \theta)_\infty \sim (\sigma \oplus \theta)_\infty$ namely, the one which just includes the first $\sigma$ in the infinite sum of the others. So all in all $(\sigma \oplus \theta)_\infty \oplus \theta \sim \theta$ and $\sigma \oplus (\sigma \oplus \theta)_\infty \sim (\sigma \oplus \theta)_\infty$, so

$$\sigma \oplus \theta \sim \sigma \oplus (\sigma \oplus \theta)_\infty \oplus \theta \sim (\sigma \oplus \theta)_\infty \oplus \theta \sim \theta.$$

$\square$
3 Proper asymptotic unitary equivalence

It is known that there are several equivalences on $KK$-cycles which turn out to yield $KK(A,B)$ depending on which set one permits the equivalences.

**Definition 3.1.** Let $\pi : A \to \mathcal{L}(E)$ and $\sigma : A \to \mathcal{L}(E')$ be representations, then $\pi$ is said to be *asymptotically unitarily equivalent* to $\sigma$, and we write $\pi \sim_{\text{asp}} \sigma$ if there is a norm-continuous path of unitaries $u_t : [0,\infty) \to \mathcal{L}(E,E')$, such that

$$\lim_{t \to \infty} \|\pi(a) - u_t^*\sigma(a)u_t\| = 0, \quad \text{for all } a \in A$$

and

$$\pi(a) - u_t\sigma(a)u_t^* \in \mathcal{K}(E), \quad \text{for all } t \in [0,\infty), \text{ and } a \in A.$$ 

The same is the case if the map $t \mapsto \pi(a) - u_t\sigma(a)u_t^* \in C_0[0,\infty) \otimes \mathcal{K}(E)$ for any $a \in A$.

**Definition 3.2.** Let $\pi,\sigma : A \to \mathcal{L}(E)$ where $A$ is a $C^*$-algebra, then $\pi$ is said to be *properly asymptotically unitarily equivalent* to $\sigma$, and we write $\pi \cong \sigma$, if there is a continuous path of unitaries $u_t \in \mathcal{U}(\mathcal{K}(E) + \mathbb{C}1_E)$ for $t \in [0,\infty)$, such that

$$\lim_{n \to \infty} \|u_t\sigma(a)u_t^* - \pi(a)\| = 0, \quad \text{for all } a \in A$$

and

$$u_t\pi(a)u_t^* - \sigma(a) \in \mathcal{K}(E), \quad \text{for all } t \in [0,\infty), \text{ and } a \in A.$$

**Remark 3.3.** Take $w_\infty : F_\infty \to F \oplus F_\infty$ and an isometry $v : F_\infty \to E$, and let $\pi : A \to \mathcal{L}(E)$ and $\sigma : A \to \mathcal{L}(F)$ be representations. Then if $v\sigma_\infty(a) - \pi(a)v \in \mathcal{K}(F_\infty,E)$ for all $a \in A$ then the unitary $u = (1_F \oplus v)w_\infty v^* + 1_E - vv^* \in \mathcal{L}(E,F \oplus E)$ satisfies

$$\sigma(a) \oplus \pi(a) - u\pi(a)u^* \in \mathcal{K}(F \oplus E), \quad \text{for all } a \in A.$$

**Lemma 3.4 ([DE01], Lemma 2.3).** Let $\pi : A \to \mathcal{L}(E)$ and $\sigma : A \to \mathcal{L}(E')$ be representations. Assume that there is a sequence of isometries $v_i : E'_\infty \to E$ for $i \in \mathbb{N}$ such that

$$v_i\sigma_\infty(a) - \pi(a)v_i \in \mathcal{K}(E'_\infty,E), \quad \|v_i\sigma_\infty - \pi(a)v_i\| \to 0$$

and $v_i^*v_j = 0$ when $i \neq j$. Then $\pi \oplus \sigma \sim_{\text{asp}} \pi$

**Proof.** Extend the sequence of isometries to a family of isometries $v_i : [0,\infty) \to \mathcal{L}(E'_\infty,E)$, for $t \in [0,1]$

by

$$v_{i+t} = (1-t)^{\frac{1}{2}}v_i + t^{\frac{1}{2}}v_{i+1}$$

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Let \( u = (u_t)_{t \in [0, \infty)} \) be a family of continuous unitaries such that

\[
\sigma(a) \oplus \pi(a) - u \pi(a) u^* \in C_0[0, \infty) \oplus \mathcal{K}(E' \oplus E), \quad \text{for all } a \in A.
\]

So \( \sigma \oplus \pi \sim_{\text{asp}} \pi. \)

The next lemma asserts that only with a slight (although infinite) error, that approximate unitary equivalence imply asymptotic unitary equivalence.

**Lemma 3.5 ([DEU1], Lemma 2.4).** Let \( \pi : A \to \mathcal{L}(E) \) and \( \sigma : A \to \mathcal{L}(E') \) be representations. If \( \sigma \oplus \pi \sim \pi \), then \( \sigma \oplus \pi_{\infty} \sim_{\text{asp}} \pi_{\infty} \).

**Proof.** Since \( \sigma \oplus \pi \sim \pi \), there is a sequence of unitaries \( (u_n)_{n \in \mathbb{N}} \in \mathcal{L}(E, E' \oplus E) \) such that

\[
||u_n^*(\sigma(a) \oplus \pi(a))u_n - \pi(a)|| \to 0 \text{ for all } a \in A
\]

Let \( v_n = (u_{n+1}, u_{n+2}, \ldots) \in \mathcal{L}(E_{\infty}, (E' \oplus E)_{\infty}) \), then

\[
||v_n^*(\sigma_{\infty}(a) \oplus \pi_{\infty}(a))v_n - \pi_{\infty}(a)|| = \sup_{k \geq n+1} ||u_k^*(\sigma(a) \oplus \pi(a))u_k - \pi(a)|| \to 0.
\]

Moreover we have that \( v_n^*(\sigma_{\infty}(a) \oplus \pi_{\infty}(a))v_n - \pi_{\infty}(a) \) is equal to

\[
(u_{n+1}^*(\sigma_{\infty}(a) \oplus \pi_{\infty}(a))u_{n+1} - \pi(a), u_{n+2}^*(\sigma_{\infty}(a) \oplus \pi_{\infty}(a))u_{n+2} - \pi(a), \ldots)
\]

which is in \( \prod_{i=1}^{\infty} \mathcal{L}(E) \subseteq \mathcal{L}(E_{\infty}) \). Since each element tend to 0, and since it is a diagonal element, then it sits in \( \mathcal{K}(E_{\infty}) \). So we conclude that \( \sigma_{\infty} \oplus \pi_{\infty} \sim \pi_{\infty} \). When regarding \( E_{\infty} = E_{\infty} \oplus E_{\infty} \), then we may think of the unitaries \( u_n \in \mathcal{L}(E_{\infty} \oplus E_{\infty}, E_{\infty}) \) which assert the above equivalence as isometries \( v_i' : E_{\infty}' \oplus E_{\infty} \to E_{\infty} \oplus E_{\infty} \) such that \( v_i'(v_j')^* = 0 \) when \( i \neq j \). We know that

\[
\sigma_{\infty}(a) \oplus \pi_{\infty}(a) - v_i'(v_j')^* \in \mathcal{K}(E_{\infty}' \oplus E_{\infty})
\]

so multiplying through with \( (v_j')^* \) makes all but the ones where \( i = j \) vanish. Furthermore, since \( v_i'(v_j')^* = 1 \), (because they were isometries) we get

\[
v_i'(\sigma_{\infty} \oplus \pi_{\infty})(a) - \pi_{\infty}(a)v_i' \in \mathcal{K}(E_{\infty}', E), \quad ||v_i'(\sigma_{\infty} \oplus \pi_{\infty})(a) - \pi_{\infty}(a)v_i'|| \to 0
\]

for \( a \in A \).

Define the isometry \( W : E_{\infty}' \to E_{\infty}' \oplus E_{\infty} \) by \( W(e') = \left( \begin{array}{cc} e' & 0 \\ 0 & 0 \end{array} \right) \), then

\[
[\sigma_{\infty} \oplus \pi_{\infty}]W(e') = \left( \begin{array}{cc} \sigma_{\infty} & 0 \\ 0 & \pi_{\infty} \end{array} \right) \left( \begin{array}{cc} e' & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} \sigma_{\infty}e' & 0 \\ 0 & 0 \end{array} \right) = W(e')\sigma_{\infty}
\]

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Now if we scale down each $v'_i$ by $W$, we get a sequence of isometries

$$v_i = v'_iW : E'_\infty \to E_\infty.$$ 

So we get

$$v_i\sigma_\infty(a) - \pi_\infty(a)v_i = v'_iW\sigma_\infty(a) - \pi_\infty(a)v'_iW$$

$$= v'_i(\sigma_\infty \oplus \pi_\infty)(a)W - \pi_\infty(a)v'_iW$$

$$= (v'_i(\sigma_\infty \oplus \pi_\infty)(a) - \pi_\infty(a)v'_i)W$$

$$\in \mathcal{K}(E'_\infty, E)$$

so by Lemma 3.4, $\sigma \oplus \pi_\infty \sim_{asp} \pi_\infty$. 

**Lemma 3.6** ([DE01], Lemma 3.1). Let $\varphi, \psi : A \to \mathcal{L}(H_B)$ be a Cuntz pair, and assume that $\mathcal{K}(H_B)$ is separable. Let $(u_t)_{t \in [1, \infty)}$ be a continuous path of unitaries in $\mathcal{L}(H_B)$, i.e., $u_t : [1, \infty) \to \mathcal{L}(H_B)$ a unitary such that

$$u_t\varphi(a)u_t^* - \psi(a) \in \mathcal{K}(H) \otimes B, \quad \text{for all } t \in [0, \infty)$$

then $[\varphi, \psi] = [\varphi, u_1\varphi u_1^*] \in KK_h(A, B)$. 

**Proof.** So we should define a homotopy $(\Phi_0, \Phi_1) : A \to M(\mathcal{K}(H_B) \otimes C([0, 1]))$ from $(\varphi, \psi)$ to $(\varphi, u_1\varphi u_1^*)$ according to the definition of $KK_h(A, B)$. From Corollary 3.4 in [APT73] then $M(\mathcal{K}(H_B) \otimes C([0, 1])) = C^b_{\text{strict}}([0, 1], \mathcal{L}(E))$, where $C^b_{\text{strict}}$ denotes the bounded strictly continuous functions. This gives a understanding of what the homotopy should do. Since the homotopy $\Phi_0$ is just the identity, this will not be mentioned, however

$$\Phi_1(a)(0) = \psi(a), \quad \Phi_1(a)(s) = u_1\varphi(a)u_1^*$$

is exactly a (continuous) homotopy from $\psi \to u_1\varphi u_1^*$, when $s \to 1$. 

Recalling what it means that two representations are proper asymptotically unitary equivalent we arrive at the crucial part.

**Lemma 3.7** ([DE01], Lemma 3.3). If $\varphi, \psi : A \to \mathcal{L}(H_B)$ are representations and $\varphi \cong \psi$, then $(\varphi, \psi) \in \mathcal{E}_h(A, B)$ and

$$[\varphi, \psi, 1] = 0 \quad \text{in } KK(A, B).$$

**Proof.** By definition of proper asymptotic unitary equivalence there is a continuous path of unitaries $u_t \in \mathcal{K}(H) \otimes B + \mathbb{C}1_{H_B}$ and since $u_t\varphi(a)u_t^* - \psi(a) \in \mathcal{K}(H) \otimes B$, then

$$\varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B, \quad \text{for all } a \in A.$$
By Lemma 3.6 we have that $[\varphi, \psi, 1] = [\varphi, u_1 \varphi u_1^*, 1]$, and since $(\varphi, u_1^*)$ is unitarily equivalent to $(\varphi, u_1^* \varphi u_1^*, 1)$, we get

$$\{\varphi, \psi, 1\} = [\varphi, u_1 \varphi u_1^*, 1] = [\varphi, \varphi, u_1^*].$$

Finally since $u_1^* \in \mathcal{K}(H) \otimes B + \mathbb{C}1_{HB}$ is a unitary, then

$$[\varphi, \varphi, u_1^*] = [\varphi, \psi, 1] = 0$$

so $[\varphi, \psi, 1] = 0$. 

**Lemma 3.8 (DEH), Lemma 3.4.** Let $\varphi, \psi, \gamma : A \to \mathcal{L}(HB)$ be unital representations. If both $\varphi \oplus \gamma \cong \psi \oplus \gamma$ and $\gamma \sim_a \sigma$, then $\varphi \oplus \sigma \cong \psi \oplus \sigma$.

**Proof.** Since $\varphi \oplus \gamma \cong \psi \oplus \gamma$ there is by definition a continuous path of unitaries, of this special “compact + identity” type $u : [0, \infty) \to \mathcal{U}(\mathcal{K}(H \oplus H) \otimes B + \mathbb{C}1_{HB})$ such that

$$u_t(\varphi(a) \oplus \gamma(a))u_t^* - \psi(a) \oplus \gamma(a) \in \mathcal{K}(H \oplus H) \otimes B$$

$$\lim_{t \to \infty} \|u_t(\varphi(a) \oplus \gamma(a))u_t^* - \psi(a) \oplus \gamma(a)\| = 0$$

for all $a \in A$. Likewise since $\gamma \sim_a \sigma$, there is a continuous path of ordinary unitaries $v_t : [0, \infty) \to \mathcal{L}(HB)$ such that

$$v_t \gamma(a)v_t^* - \sigma(a) \in \mathcal{K}(H) \otimes B$$

$$\lim_{t \to \infty} \|v_t \gamma(a)v_t^* - \sigma(a)\| = 0$$

for all $a \in A$. Since $u_t \in \mathcal{U}(\mathcal{K}(H \oplus H) \otimes B + \mathbb{C}1_{HB})$, we may arrange that $v_t = (1 \oplus u_t)u_t(1 \oplus v_t^*) \in \mathcal{U}(\mathcal{K}(H \oplus H) \otimes B + \mathbb{C}1_{HB})$. Then since $\|\sigma\| = \|u_t \sigma v_t^*\|$, we get for $a \in A$

$$\|u_t(\varphi(a) \oplus \sigma(a))u_t^* - \psi(a) \oplus \sigma(a)\|$$

$$= \|u_t(\varphi(a) \oplus v_t \sigma(a) v_t)u_t(1 \oplus v_t^*) - \psi(a) \oplus \sigma(a)\|$$

$$= \|u_t(\varphi(a) \oplus v_t \sigma(a) v_t) - \psi(a) \oplus \sigma(a)\|.$$

Then since $\gamma(a)$ and $v_t \sigma(a) v_t^*$ are close one may replace one by the other, and adding $\gamma(a) \oplus \gamma(a)$ yields

$$\|u_t(\varphi(a) \oplus v_t \sigma(a) v_t)u_t^* - \psi(a) \oplus v_t \sigma(a) v_t^*\|$$

$$= \|2\gamma(a) - 2\gamma(a) + u_t(\varphi(a) \oplus \gamma(a))u_t^* - \psi(a) \oplus \gamma(a)\|$$

$$\leq 2\|v_t \sigma(a) v_t - \gamma(a)\| + \|u_t(\varphi(a) \oplus \gamma(a))u_t^* - \psi(a) \oplus \gamma(a)\| \to 0$$

as $t \to \infty$. Then since the expression tends to 0 and by going to the Calkin algebra, we know that the expression exactly lands in $\mathcal{K}(H \oplus H) \otimes B$ for all $a \in A$. 

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Note that in the proof in [DE01], the authors insist that $\varphi \in C$. However this is remnant of an earlier write, and should be exchanged with $A$ as it has been in the above proof. Also, in the end of the proof in [DE01] a $\psi$ has sneaked its way in, this should be a $\varphi$.

We wish to define the $C^*$-algebra, which determines all elements $\tau$ of $B$ such that $\tau \Phi(A) = \Phi(A)\tau \in \mathcal{K}(H) \otimes B$ of a specific representations image $\Phi : A \to B$. Say such a representation exists and define

$$D_\Phi = \{ \tau \in \mathcal{L}(H_B) \mid \tau \Phi(A) = \Phi(A)\tau \in \mathcal{K}(H) \otimes B \}.$$ 

If $\pi : \mathcal{L}(H_B) \to \mathcal{L}(H_B)/\mathcal{K}(H_B)$ is the quotient map then denote by the following $\Phi_\pi = \pi \circ \Phi : A \to \mathcal{L}(H_B)/\mathcal{K}(H_B)$. For $X \subseteq \mathcal{L}(H_B)/\mathcal{K}(H_B)$ write

$$X^c = \{ \tau \in \mathcal{L}(H_B)/\mathcal{K}(H_B) \mid \tau x - x\tau \in \mathcal{K}(H_B) \text{ for all } x \in X \}$$

to be the commutant of $X$ in $\mathcal{L}(H_B)/\mathcal{K}(H_B)$. It can be shown that

$$0 \to \mathcal{K}(H) \otimes B \to D_\Phi \to \Phi_\pi(A)^c \to 0$$

is a short exact sequence.

**Lemma 3.9 ([DE01], Lemma 3.5).** Let $A$ be a unital separable $C^*$-algebra, and let $B$ be a $\sigma$-unital $C^*$-algebra. Let $\Phi : A \to \mathcal{L}(H_B)$ be a unital representation and let $w_i \in \mathcal{U}(D_k)$. Assume that

$$[\Phi, \Phi, w_1] = [\Phi, \Phi, w_2] \text{ in } KK(A, B).$$

Then there is a unital representation $\gamma : A \to \mathcal{L}(H_B)$ such that

$$(\Phi \oplus \gamma, \Phi \oplus \gamma, w_1 \oplus 1) \sim_{\text{oh}} (\Phi \oplus \gamma, \Phi \oplus \gamma, w_2 \oplus 1).$$

If $\Phi$ is strictly nuclear, then $\gamma$ can be chosen to be strictly nuclear.

**Proof.** Let $\gamma_i : A \to \mathcal{L}(E_i)$ and $v \in \mathcal{L}(E_0, E_1)$ be given such that $(\gamma_0, \gamma_1, v)$ is degenerate, then since adding degenerate cycles do not change the class in $KK(A, B)$, we get

$$(\Phi \oplus \gamma_0, \Phi \oplus \gamma_1, w_1 \oplus v) \sim_{\text{oh}} (\Phi \oplus \gamma_0, \Phi \oplus \gamma_1, w_2 \oplus v)$$

Say that $W_i$ is the operator homotopy from $w_1 \oplus v$ to $w_2 \oplus v$.

Now, let $\gamma_i(1) = p_i \in \mathcal{L}(E_i)$ be a projection. We get, since $(\gamma_0, \gamma_1, v)$ were degenerate,

$$p_0 v - vp_1 = 0, \quad p_0 - p_0 v^* v = 0, \quad p_1 - p_1 v v^* = 0$$

so $v^* v = vv^* = 1$, thus we may assume that $v$ is a unitary. Furthermore, if we substitute $E_i$ by $p_i E_i$, $v$ by $p_1 vp_0$ and $W_i$ by $(1 \oplus p_1)W_i(1 \oplus p_0)$, we may assume that $\gamma_i$ is unital as well.
Now note that the homotopy \((\Phi \oplus \gamma_0, \Phi \oplus \gamma_0, W_t \oplus v^*W_t) \in E(A,B)\) is exactly the operatorial homotopy that makes the following equivalent

\[
(\Phi \oplus \gamma_0, \Phi \oplus \gamma_0, w_1 \oplus 1) \sim_{\text{oh}} (\Phi \oplus \gamma_0, \Phi \oplus \gamma_0, w_2 \oplus 1)
\]

since \((W_t \oplus v^*W_t)(w_1 \oplus v) = w_2 \oplus v^*v = w_2 \oplus 1\) and vice versa.

Now to be done we need that \(E_0 = H_B = E_1\). Let \(\theta : A \to \mathcal{L}(H_B)\) be a unital faithful scalar representation of infinite multiplicity and let \((\theta, \theta, 1)\) be the corresponding degenerate cycle. As before we can add it to an existing operatorial homotopy equivalence to get a new equivalence

\[
(\Phi \oplus \gamma_0 \oplus \theta, \Phi \oplus \gamma_0 \oplus \theta, w_1 \oplus 1 \oplus 1) \sim \text{oh} (\Phi \oplus \gamma_0 \oplus \theta, \Phi \oplus \gamma_0 \oplus \theta, w_2 \oplus 1 \oplus 1)\quad (1)
\]

When proving Kasparovs Stabilization Theorem, which asserts that for a countably generated Hilbert module \(E\), we get that \(E \oplus H_B \cong H_B\), one ends up in a sort of Gram-Schmidt orthonormalization, with a unitary \(z : E \oplus H_B \to H_B\), which asserts the fact. If we take that unitary and construct the following map

\[
\gamma = z(\gamma_0 \oplus \theta)z^* : H_B \to H_B
\]

we get that the equivalence in (1) when conjugating with \(z\) (since \(KK(A,B)\) do not differ with conjugacy of unitaries), is

\[
(\Phi \oplus \gamma, \Phi \oplus \gamma, w_1 \oplus 1) \sim_{\text{oh}} (\Phi \oplus \gamma, \Phi \oplus \gamma, w_2 \oplus 1)
\]

\(\Box\)

**Proposition 3.10** ([DE01], Proposition 3.6). Let \(A\) be a unital separable \(C^*\)-algebra, and \(B\) a \(\sigma\)-unital \(C^*\)-algebra. Let, furthermore \(\varphi, \psi : A \to \mathcal{L}(H_B)\) be a unital Cuntz pair. Then if

\[
[\varphi, \psi, 1] = 0 \text{ in } KK(A,B)
\]

then there is a unital representation \(\gamma : A \to \mathcal{L}(H_B)\) such that

\[
\varphi \oplus \gamma \cong \psi \oplus \gamma.
\]

**Proof.** We need to find a unital representation \(\gamma : A \to \mathcal{L}(H_B)\), the proof will highly depend on the previous lemma.

First off define \(\sigma = \varphi_\infty \oplus \psi_\infty\) and put \(\Phi = \varphi \oplus \sigma = \varphi \oplus \varphi_\infty \oplus \psi_\infty\) and \(\Psi = \psi \oplus \sigma = \psi \oplus \varphi_\infty \oplus \psi_\infty\). Let some unitary \(\tilde{u} \in \mathcal{L}(H_B)\) be given such that \(\tilde{u}\Phi \tilde{u}^* = \Psi\). Here illustrated what that unitary should do
Then since \((\sigma, \sigma, 1)\) is a degenerated cycle, we have \([\Phi, \Psi, 1] = [\varphi, \psi, 1]\), so in total we get

\[
[
\Phi, \Phi, \bar{u}
\] = \[
\Phi, \Psi, 1
\] = [\varphi, \psi, 1] = 0 = [\Phi, \Phi, 1]
\]

By Lemma 3.9, there is a unital representation \(\bar{\gamma} : A \to \mathcal{L}(H_B)\) such that

\[
(\Phi \oplus \bar{\gamma}, \Phi \oplus \bar{\gamma}, \bar{u} \oplus 1) \sim_{\text{oh}} (\Phi \oplus \bar{\gamma}, \Phi \oplus \bar{\gamma}, 1 \oplus 1).
\]

(2)

We may assume that \(\bar{\gamma}_H : A \to \mathcal{L}(H_B)/\mathcal{K}(H_B)\) is injective. If it was not, we could just find some \(\ast\)-homomorphism, say \(\tau : A \to \mathcal{L}(H_B)\) such that \(\pi(\tau)\) was injective and then \(\pi \circ (\tau \oplus \bar{\gamma})\) would be injective.

Define the following \(\Phi = \Phi \oplus \bar{\gamma}, \Psi = \psi \oplus \bar{\gamma}\), and \(u = \bar{u} \oplus 1\), then

\[
u\Psi u^* = (\bar{u} \oplus 1)(\Psi \oplus \bar{\gamma})(\bar{u}^* \oplus 1) = \bar{u}\Psi \bar{u}^* \oplus \bar{\gamma} = \Phi \oplus \bar{\gamma} = \Phi
\]

It follows by \(\text{(2)}\), that \((\Phi, \Phi, u) \sim_{\text{oh}} (\Phi, \Phi, 1)\). Let \(\omega_s \in \mathcal{L}(H_B)\) be the homotopy joining \(u\) to \(1\), i.e. \(\omega_0 = u\) and \(\omega_1 = 1\). We wish to show that \(u\) is homotopic to some unitary, say \(w \in \mathcal{K}(H) \otimes B + C1\), inside 

\[D_\Phi = \{d \in \mathcal{L}(H_B) \mid [d, \Phi(a)] \in \mathcal{K}(H_B)\},\]

with \(\Phi\) defined as above. Then

\[
[\Phi(a), \omega_s] \in \mathcal{K}(H_B)
\]

\[
\Phi(a)(\omega_s \omega_s^* - 1), \Phi(a)(\omega_s^* \omega_s - 1) \in \mathcal{K}(H_B)
\]

for all \(s \in [0, 1]\). Clearly \((\omega_0)_{\pi} = \pi \circ u, \) and also \((\omega_1)_{\pi} = 1 = \Phi(1)\).

So \((\omega_s)_{\pi}\) is a homotopy of unitaries from \(u_{\pi}\) to \(1\) in \(\Phi_{\pi}(a)^c\). Since the map

\[
\mathbb{U}(D_\Phi) \to \mathbb{U}(\Phi_{\pi}(A)^c)
\]

has the homotopy lifting property it is a fibration. Furthermore we had homotopic unitaries \(u_{\pi}\) and \(1\), which by a topological fact gives homotopic elements, namely \(\pi^{-1}(u_{\pi}) = u\) and \(\pi^{-1}(1)\) in the fiber. Since the fiber is \(\mathbb{U}(\mathcal{K}(H_B) + C1)\), we say that \(\pi^{-1}(1) = w\).

Define \(\mathcal{K}_\Phi = \Phi(A) + \mathcal{K}(H) \otimes B\). Let \(\gamma = \sigma \oplus \bar{\gamma}\). Since \(\Phi = \varphi \oplus \gamma\) and \(\Psi = \psi \oplus \gamma\), then to be done we need to find a path of unitaries

\[
u_t \in \mathcal{K}(H) \otimes B + C1\]

such that

\[
\lim_{t \to \infty} \|u_t \Phi(a) u_t^* - \Psi(a)\| = 0
\]

By definition of \(\mathcal{K}_\Phi\), we may find \(x_t \in A\) and \(y_t \in \mathcal{K}(H) \otimes B\) such that we may write \(u_t = \Phi(x_t) + y_t \in \mathcal{K}_\Phi\). We wish to show that \(v_t \in \mathcal{K}_\Phi\) is a path of unitaries such that

\[
\lim_{t \to \infty} \|v_t \Phi(a) v_t^* - \Psi(a)\| = 0.
\]

Since \(u\) and \(w\) are homotopic in \(D_\Phi\), we know that \(\text{Ad}(u)\) and \(\text{Ad}(w)\) are both in \(\text{Aut}(\mathcal{K}_\Phi)\) and moreover they are homotopic. Since the two unitaries are homotopic we know that \(w^*u\) is homotopic to the identity, so \(\text{Ad}(w^*u) \in \text{Aut}_0(\mathcal{K}_\Phi)\). Let \(\alpha_s\) be the uniformly continuous path in \(\text{Aut}(\mathcal{K}_\Phi)\)
which joins the identity to \( \alpha = \text{Ad}(w^* u) \). Now define sub-algebras of \( \mathcal{K}_\varphi \) of the form
\[
\alpha_{s_1}^{j_1} \alpha_{s_2}^{j_2} \ldots \alpha_{s_n}^{j_n} \Phi(A), \quad \text{where } n \geq 1, j_k \in \mathbb{Z} \text{ and } s_k \in [0,1] \cap \mathbb{Q}
\]
and let \( K \) be the \( C^* \)-algebra generated by these. Note that \( K \subseteq \mathcal{K}_\varphi \) and that \( K \) is unital. Furthermore since \( A \) is separable, then \( K \) is separable, and lastly since \( \alpha_t(\mathcal{K}) = \mathcal{K} \), we can conclude that \( \alpha \in \text{Aut}_0(K) \), so it is asymptotically inner. By multiplying through with the unitary \( \phi \), this can be done. We just need that \( \phi \) changes the norm, then changing the norm, then
\[
\Phi(a) \in K \text{ for all } a \in A, \quad \text{then}
\]
\[
\lim_{t \to \infty} ||\text{Ad}(v_t)(k) - \text{Ad}(u)(k)|| = \lim_{t \to \infty} ||v_t kv_t^* - uku^*|| = 0.
\]

Since \( \Phi(a) \in K \) for all \( a \in A \), then
\[
\lim_{t \to \infty} ||v_t \Phi(a) v_t^* - \Psi(a)|| = \lim_{t \to \infty} ||v_t \Phi(a) v_t^* - w \Phi(a) u^*|| = 0.
\]

As \( \Phi \) is unital and since \( \gamma_\varphi \) is injective, we have made sure that \( \Phi_\varphi \) is injective. So \( x_t \) is a continuous path of unitaries and since \( \Phi_\varphi = \Psi_\varphi \) do not change the norm, then
\[
0 = \lim_{t \to \infty} ||x_t a x_t^* - a||.
\]

We finally turn to the actual unitaries we are interested in namely \( u_t = v_t \Phi(x_t)^* = 1 + y_t \Phi(x_t)^* \in \mathbb{C}1 + \mathcal{K}(H) \otimes B \) and calculate
\[
||u_t \Phi(a) u_t^* - \Psi(a)|| = ||v_t \Phi(x_t)^* \Phi(a) \Phi(x_t) v_t^* - \Psi(a)||
\]
\[
= ||v_t \Phi(x_t^* a x_t) v_t^* - \Psi(a) + v_t \Phi(a) v_t^* - v_t \Phi(a) v_t^*||
\]
\[
\leq ||v_t \Phi(a) v_t^* - \Psi(a)|| + ||v_t \Phi(x_t^* a x_t) v_t^* - v_t \Phi(a) v_t^*||
\]
\[
= ||v_t \Phi(a) v_t^* - \Psi(a)|| + ||v_t \Phi(x_t^* a x_t - a) v_t^*|| \to 0
\]

when \( t \to \infty \).

One may wish that we can find a \( \gamma \) that is both unital and strictly nuclear. This can be done, we just need that \( \varphi \) and \( \psi \) are not only unital but also strictly nuclear and \( A \) or \( B \) is nuclear. The proof follows the same strategy. If \( \varphi \) and \( \psi \) are strictly nuclear, so is \( \sigma \), and then one uses that Lemma 3.9 works in the nuclear case to get a strictly nuclear \( \gamma \), then the rest is the same. In fact one need not require either unitallity nor nuclearity to find \( \gamma \). One just replaces \( A \) by its unitilization, and likewise with \( \varphi, \psi \) and \( \gamma \).

**Theorem 3.11** ([DE01], Theorem 3.8). Let \( A \) be a unital separable \( C^* \)-algebra and let \( B \) be a separable \( C^* \)-algebra. Furthermore, let \( \varphi, \psi : A \to \mathcal{L}(H_B) \) be unital representations such that \( \varphi(a) - \psi(a) \in \mathcal{K}(H) \otimes B \) for all \( a \in A \). Then the following are equivalent.
1. The class $[\varphi, \psi, 1] = 0$ in $KK(A, B)$.
2. The class $[\varphi, \psi] = 0$ in $KK_h(A, B)$.
3. There exists a unital representation $\sigma : A \to \mathcal{L}(H_B)$ with
   \[ \varphi \oplus \sigma \cong \psi \oplus \sigma. \]
4. For any unital absorbing representation $\gamma : A \to \mathcal{L}(H_B)$, then
   \[ \varphi \oplus \gamma_{\infty} \cong \psi \oplus \gamma_{\infty}. \]

Proof. (1) implies (2): This relies on Theorem 1.14.
(2) implies (3): This is covered by 3.10.
(3) implies (2): This is covered by 3.7.
(3) implies (4): Since $\varphi \oplus \sigma \cong \psi \oplus \sigma$, we can add $\gamma_{\infty}$ to get
   \[ \varphi \oplus \sigma \oplus \gamma_{\infty} \cong \psi \oplus \sigma \oplus \gamma_{\infty}. \]
Since $\gamma : A \to \mathcal{L}(H_B)$ was assumed absorbing, we get by Lemma 3.5 that $\sigma \oplus \gamma_{\infty} \sim_{asp} \gamma_{\infty}$. By Lemma 3.8, we get that
   \[ \varphi \oplus \gamma_{\infty} \cong \psi \oplus \gamma_{\infty}. \]
(4) implies (3): This uses Theorem 2.15 to justify the fact that a unital absorbing representation do exist. \qed
4 K-Homology

By K-homology, we refer to $KK(A, C)$, when referring to $C^*$-algebras. It is well-known that for a $C^*$-algebra $A$ then

$$KK(A, C) = K^0(A)$$

Voiculescu proved that certain representations will always be approximately unitarily equivalent, this is of course related to the proper kind of asymptotic equivalence via the regular asymptotic kind. However, which representations are we talking about?

**Definition 4.1.** A representation $\varphi : A \to \mathcal{L}(H)$ is said to be admissible if it is faithful, non-degenerate and its image contain no compact operators besides 0, i.e,

$$\varphi(A) \cap \mathcal{K}(H) = \{0\}.$$ 

Observe that since $B = C$ then $H_B = H_C = H$.

It is worthwhile noting that we already encountered such representations namely the faithful scalar representation of infinite multiplicity, as one notes in [DE02], where Proposition 2.22 is phrased in the words of admissibility. In the following we can assume that both $A$, $\varphi$ and $\psi$ are unital. This follows since if $A$ is unital, then admissibility gives unitallity of representations and if $A$ is not unital then $\varphi_{un}$ of $\varphi$ inherits $\varphi$'s admissibility.

**Theorem 4.2 ([DV76], Th. 1.3).** If $A$ is a unital separable $C^*$-algebra. Then if $\sigma : A \to \mathcal{L}(H)$ is an unital admissible representation and $\varphi : A \to \mathcal{L}(H)$ some unital $*$-homomorphism, then

$$\varphi \oplus \sigma \sim \sigma.$$ 

**Theorem 4.3 ([DE01], Theorem 3.11).** If $\varphi, \psi : A \to \mathcal{L}(H)$ are admissible representations, then $\varphi \sim_{asp} \psi$.

**Proof.** By Theorem 4.2 we know that $\psi \sim \psi \oplus \varphi_{\infty} \sim \varphi_{\infty}$. So there is a unitary $u$ such that

$$||u\varphi_{\infty}(a)u^* - \psi(a)|| \to 0 \text{ for all } a \in A$$

$$u\varphi_{\infty}(a)u^* - \psi(a) \in \mathcal{K}(H) \text{ for all } a \in A$$

Define the isometry $s_n(x) = (0, 0, ..., x, 0, ...)$ where $x$ is in the n'th place. Let $v_n = s_n u$, then

$$s_n u\varphi_{\infty}(a)u^* s_n^* - \psi(a) = v_n \varphi_{\infty}(a) v_n^* - \pi(a) \in \mathcal{K}(H).$$

One may calculate that then the following is true (For an example of the same calculation check the proof of Lemma 5.1 p. 33.)

$$v_n \varphi_{\infty}(a) - \psi(a) v_n \in \mathcal{K}(H), \quad \lim_{n \to \infty} ||v_n \varphi_{\infty}(a) - \psi(a) v_n|| = 0.$$
By Lemma 3.4 then $\phi \oplus \psi \sim_{asp} \phi$. Similarly

$$v_n\psi_\infty(a) - \phi(a)v_n \in \mathcal{K}(H), \quad \lim_{n \to \infty} ||v_n\psi_\infty(a) - \phi(a)v_n|| = 0.$$  

again via Lemma 3.4 then $\phi \oplus \psi \sim_{asp} \psi$. So

$$\phi \sim_{asp} \phi \oplus \psi \sim_{asp} \psi.$$  

\[ \square \]

As in [WLP81] we let $U(\pi(A)^c)$ be the unitary group of the commutant of $\pi(A)$, and let $\mathcal{J}(\pi(A)^c)$ be the quotient by the path component of 1. We also let $C^*(\pi(A), u)$ denote the $C^*$-algebra generated by $\pi(A)$ and $u$ a unitary.

**Lemma 4.4** ([WLP81], Lemma 3 (1)). Let $A$ be a separable unital $C^*$-algebra and let $u \in U(\pi(A)^c)$. Suppose that $\tau_0 : C^*(\pi(A), u) \to Q(H)$ is a trivial extension such that $\tau_0(\pi(x)) = \pi(x)$ for $x \in A$. Then $\tau_0(u)$ is homotopic to 1.

We will use this lemma with the trivial extension being the identity. The proof is straightforward but uses some extension theory, see [RH14].

**Proposition 4.5** ([WLP81], Proposition 4). Let $A$ be a separable and unital $C^*$-algebra, then

$$K_1(\pi(A)^c) \cong \mathcal{J}(\pi(A)^c).$$

The following theorem is an improvement of Theorem 3.11 when regarding admissible representations and a separable $A$.

**Theorem 4.6** ([DE01], Theorem 3.12). Let $A$ be a separable $C^*$-algebra and let $\varphi, \psi : A \to \mathcal{L}(H)$ be a Cuntz pair of admissible representations. Then

$$[\varphi, \psi] = 0 \text{ in } KK_h(A, \mathbb{C}) \text{ if and only if } \varphi \cong \psi.$$  

**Proof.** Note that if $\varphi \cong \psi$, then by Lemma 3.7 we get $[\varphi, \psi, 1] = 0$ in $KK(A, \mathbb{C})$, and by the isomorphism in Theorem 1.14 then $[\varphi, \psi] = 0$ in $KK_h(A, \mathbb{C})$.

As before we can assume that both $A$, $\varphi$ and $\psi$ are unital. From Theorem 4.3 we learned that $\varphi \sim_{asp} \psi$, so there is a family of unitaries $(u_t)_{t \in [0, \infty)} \subset \mathcal{L}(H)$ such that

$$u_t\varphi(a)u_t^* - \psi(a) \in C_0([0, \infty)) \otimes \mathcal{K}(H).$$

Assume that $[\varphi, \psi] = 0$ in $KK_h(A, \mathbb{C})$, from Lemma 3.6 we get that $[\varphi, \psi] = [\varphi, u_1\varphi u_1^*]$. Furthermore, since $(\varphi, \varphi, u_1^*) \sim (\varphi, u_1\varphi u_1^*, 1) \sim (\varphi, u_1)$ we get from the isomorphism in Theorem 1.14 that

$$0 = [\varphi, \psi] = [\varphi, u_1\varphi u_1^*] \cong [\varphi, u_1\varphi u_1^*, 1] = [\varphi, \varphi, u_1^*] = [\varphi, \varphi, u_1].$$
in $KK(A, C)$. By Lemma 4.4 then $\pi \circ u_1$ is homotopic to 1 in $\mathcal{U}((\pi \circ \varphi(A))^c) \subset Q(H)$. Finally since

$$KK(A, C) \cong K_1((\pi \circ \varphi(A))^c) \cong \mathbb{I}_1((\pi \circ \varphi(A))^c)$$

then $[\pi \circ u_1] = 0$ since $\pi \circ u_1$ was homotopic to 1.

Then as we did in the case of Proposition 3.10 we observe the fibration

$$\mathcal{U}(D\varphi) \to \mathcal{U}((\pi \circ \varphi(A))^c)$$

with fiber $\mathcal{U}(K(H) + \mathbb{C}1)$, and conclude that $u_1$ is homotopic to 1 in $D\varphi$. Since $(\varphi, \psi)$ was assumed to be a Cuntz pair i.e $\varphi(a) - \psi(a) \in K(H)$, and since we at any point in the homotopy between $u_1$ and 1 have

$$v\varphi(a)v^* - \psi(a) \in K(H)$$

for some $v$ being a step in that homotopy, we can assume that $u_0 = 1$ in

$$u_t\varphi(a)u_t^* - \psi(a) \in C_0([0, \infty)) \otimes K(H). \quad (3)$$

Define by $(\alpha_t)_{t \in [0, \infty)} = \text{Ad}(u_t)$ in the connected component of the identity in $\text{Aut}(K\varphi)$. Since it is a uniformly continuous family there, we get by Proposition 2.9 a continuous family $(v_t)_{t \in [0, \infty)}$ such that

$$\lim_{t \to \infty} \|\alpha_t(x) - \text{Ad}(v_t)(x)\| = \lim_{t \to \infty} \|u_t x u_t^* - v_t x v_t^*\| = 0.$$

So the difference between the unitaries fade, so we can combine with the expression in (1) to get

$$\lim_{t \to \infty} \|v_t\varphi(a)v_t^* - \psi(a)\| = 0$$

Since $\pi \circ \varphi$ is faithful, we can replace $v_t$ with unitaries of the form $\mathcal{K}(H) + \mathbb{C}1$, as we did in Proposition 3.10 to get what we were looking for. \qed
5 Projections in regard to proper asymptotic equivalence

It is assumed that the reader is somewhat accustomed to index theory and Fredholm operators. Some facts will however be of use.

Let \( p, q \in \operatorname{proj}(\mathcal{B}(H)) \) such that \( p - q \in \mathcal{K}(H) \) and \( p = vv^* \) and \( q = ww^* \), for \( v, w \) isometries on \( H \). Then the essential co-dimension \([p : q]\) is defined as

\[
[p : q] = \operatorname{Ind}(v^*w)
\]

So for this to be well-defined \( v^*w \) should be Fredholm, however this follows from the fact \( p - q \in \mathcal{K}(H) \), indeed

\[
\mathcal{K}(H) \ni p - q = vv^* - ww^* \iff w^*vv^*w - 1, v^*ww^*v - 1 \in \mathcal{K}(H).
\]

Which exactly says that \( v^*w \) is a Fredholm operator, so we can calculate the essential co-dimension

\[
[p : q] = \operatorname{Ind}(v^*w) = \dim \ker v^*w - \dim \operatorname{coker} v^*w
\]

Furthermore, in [ASS94] the authors proved that if the index of a pair of projections \([p : q] = 0\) then there is a unitary \( u \) such that \( p = uqu^* \). It will be proved later that this unitary is of the form "identity + compact" under certain restrictions.

Recall that the real rank of a unital \( C^* \)-algebra \( A \) is defined as the smallest integer \( n \) such that

\[
L_n(A) = \left\{ (a_0, a_1, ..., a_n) \in A_{n+1} \sum_{k=0}^{n} Aa_k = A \right\}
\]

is dense in \( A_{n+1} \), we will denote by \( RR(A) \) the real rank. Especially if the real rank is zero then the self adjoint invertible elements are dense in the self-adjoints.

5.1 Connection between projections and Cuntz Pairs

So how does one combine \( KK \)-theory with the essential co-dimension? If we let both \( A = C \) = \( B \), then we get \( KK(C, C) \), and note that for two \(*\)-homomorphisms \( \varphi, \psi : C \to \mathcal{L}(H) \), then \( \varphi(1) \) and \( \psi(1) \) determine \( \varphi \) and \( \psi \), and they are projections. So in the language of essential co-dimension if \( \varphi(a) - \psi(a) \in \mathcal{K}(H) \), then we can define \([\varphi(1) : \psi(1)] = [p : q]\). So to any pair of projections in the above sense, we can associate a Cuntz pair \((\varphi, \psi)\), and in total we will say that

\[
[p : q] \text{ represents the class } [\varphi, \psi] \in KK(C, B).
\]
Lemma 5.1 ([HHL10], Lemma 2.3). Let $A$ be a separable $C^*$-algebra and let \( \psi, \varphi : A \to \mathcal{L}(H_B) \) be representations, with $\psi$ absorbing. Then there exists a sequence of isometries \((v_n) \in \mathcal{L}((H \otimes B)_\infty, H_B)\), such that for each $a \in A$

\[
v_n \varphi(\infty)(a) - \psi(a)v_n \in \mathcal{K}((H \otimes B)_\infty, H_B)\]

\[
\lim_{n \to \infty} \|v_n \varphi(\infty)(a) - \psi(a)v_n\| = 0 \]

\[
v_i^* v_i = 0 \text{ for } i \neq j.\]

Proof. Let \((S_1, S_2, S_3, \ldots)\) be a sequence of isometries such that \(S_i \in \mathcal{L}(H_B)\) for all \(i = 1, 2, 3, \ldots\), and such that \(S_i^* S_j = 0\) if \(i \neq j\) and \(\sum_{i=1}^{\infty} S_i S_i^* = 1\) in the strict topology. This can be done by Lemma 2.13. Note that \(\mathcal{L}(H_B)\) is complete. Define \(\varphi(\infty)(a) = \sum_{i=1}^{\infty} S_i \varphi(a) S_i^*\). Then since $\psi$ is absorbing we know that $\psi \ast \varphi(\infty) \sim \psi$ so there is a sequence of unitaries $u = u_n$ such that $u_n^* \psi(a) u_n - \psi(a) \in \mathcal{K}(H_B)$. Also there is an isometry $R \in \mathcal{L}(H_B)$ such that $R^*(\psi(a) \ast \varphi(\infty))R = \varphi(\infty)$, which just cuts out the bottom right of the sum. So in total we get from these two observations that

\[
R^* u_n^* \psi(a) u_n R - R^* \psi(a) \ast \varphi(\infty) R = R^* u_n^* \psi(a) u_n R - \varphi(\infty) \in \mathcal{K}(H_B)\]

Define the isometry $W = u R$.

Next we define $T : (H \otimes B)_\infty \to H_B$ by $T = (S_1, S_2, \ldots)$. Then

\[
\varphi(\infty)(a) = T \varphi(\infty)(a) T^*\]

Thus we may write

\[
T^* W^* \psi(a) WT - T^* \varphi(\infty)(a) T = T^* W^* \psi(a) WT - \varphi(\infty) \in \mathcal{K}(H_B)\]

Since $\varphi(\infty) = (\varphi(\infty))_{\infty}$ we may split $\mathbb{N}$ into the partition $N_i$, so we get a sequence of isometries \(v_i \in \mathcal{L}((H \otimes B)_\infty, H_B)\), defined from the map $WT = (W S_1, W S_2, \ldots)$. That is a we have a bijection $\nu_i : N_i \to \mathbb{N}$, such that we can define the actual isometries as

\[
v_i = (W S_{\nu_i^{-1}(1)}, W S_{\nu_i^{-1}(2)}, \ldots)\]

Then because of the construction of $v_i$ we get that \(v_i v_i^* = 0\) if \(i \neq j\). Furthermore, then

\[
T^* W^* \psi(a) WT - \varphi(\infty)(a) = v_i^* \psi(a) v_i - \varphi(\infty)(a) \in \mathcal{K}(H_B)\]

and $\lim_{i \to \infty} ||v_i^* \psi(a) v_i - \varphi(\infty)(a)|| = 0$. Finally we calculate

\[
(v_n \varphi(\infty)(a) - \psi(a)v_n)^* (v_n \varphi(\infty)(a) - \psi(a)v_n) = \varphi(\infty)(a^*) \varphi(\infty)(a) - \varphi(\infty)(a^*) v_n^* \psi(a) v_n - v_n^* \psi(a^*) v_n \varphi(\infty)(a) + v_n^* \psi(a^*) v_n = \varphi(\infty)(a^*) (\varphi(\infty)(a) - v_n^* \psi(a) v_n) - v_n^* \psi(a^*) v_n \varphi(\infty)(a) + v_n^* \psi(a^*) v_n.
\]
Adding \( \varphi_\infty(a^*)\varphi_\infty(a) - \varphi_\infty(a^*a) \), yields

\[
\varphi_\infty(a^*)(\varphi_\infty(a) - v_n^*\psi(a)v_n) - \varphi_\infty(a)(\varphi_\infty(a^*) - v_n^*\psi(a^*)v_n) - (\varphi_\infty(a^*a) - v_n^*\psi(a^*a)v_n).
\]

Which finishes the proof.

In the proof of Lemma 5.1 in [HHL10] the author assumes \( W \) to be a unitary, but this will not work, because no such unitary exists, and the lemma would then be wrong, however the lemma is still true when regarding a unitary, since this is all that is needed for conclusion of the lemma.

Next follows a theorem which resembles Theorem 4.3, however applies for absorbing representations

**Theorem 5.2** ([HHL10] Theorem 2.5). Let \( A \) be a separable \( C^* \)-algebra. If \( \varphi, \psi : A \to \mathcal{L}(H_B) \) are absorbing representations, then \( \varphi \sim_{asp} \psi \).

**Proof.** Follows from Lemma 5.1 where you get unitaries, that work in Lemma 3.4 to get

\[
\varphi \sim_{asp} \varphi \oplus \psi \sim_{asp} \psi
\]

Recall the definition of the \( C^* \)-algebra

\[
D_\varphi(A, B) = \{ x \in \mathcal{L}(H_B) \mid x\varphi(a) - \varphi(a)x \in \mathcal{K}(H_B) \}
\]

Note that in the setting where \( A = \mathbb{C} \) and \( \varphi(1) = p \) determines \( \varphi \) then the requirement of \( D_\varphi \) is that \( px - xp \in \mathcal{K}(H_B) \).

**Lemma 5.3** ([HHL10], Lemma 2.6). If \( RR(\mathcal{L}(H_B)) = 0 \), then \( RR(D_\varphi(\mathbb{C}, B)) = 0 \) for any representation \( \varphi : \mathbb{C} \to \mathcal{L}(H_B) \).

In the following we will use the non-trivially graded version of \( KK(A, B) \), where a cycle is given by a Hilbert module \( E \), a graded representation on \( E \), and an odd degree operator on \( E \).

**Lemma 5.4** ([HHL10], Lemma 2.8). Assume that \( B \) is a stable \( C^* \)-algebra and let \( \varphi : A \to \mathcal{L}(H_B) \) be an absorbing representation. Then

\[
K_1(D_\varphi(A, B)) \cong KK(A, B)
\]

via the map where \( v \) is a unitary of \( Mat_n(D_\varphi(A, B)) \).

\[
[v] \mapsto \left[ B^n \oplus B^n, \begin{pmatrix} \varphi^n & 0 \\ 0 & \varphi^n \end{pmatrix}, \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \right]
\]
The proof of the next theorem, namely Theorem 2.11 in [HHL10] is essentially that of Theorem 4.6 however one has another requirement. One demands that $D_\varphi(A, B)$ is $K_1$-injective, which means that the map $\mathcal{U}(D_\varphi(A, B)_{\text{un}})/\mathcal{U}_0(D_\varphi(A, B)_{\text{un}}) \to K_1(D_\varphi(A, B))$ is injective. Here $\mathcal{U}_0(A)$ denotes the connected component of the identity of the $C^*$-algebra in question.

**Definition 5.5.** A purely infinite $C^*$-algebra, is one in which any hereditary sub-algebra is infinite. Moreover, to be hereditary is: If $B \subseteq A$ then for all $a \in A$ and $b \in B$ such that $0 \leq a \leq b$, then $a \in B$.

**Theorem 5.6 ([HHL10], Theorem 2.11).** Let $A$ be a separable $C^*$-algebra and let $\varphi, \psi : A \to \mathcal{L}(H_B)$ be a Cuntz pair of absorbing representations. Suppose that $\pi \circ \varphi$ is injective and $D_\varphi$ is $K_1$-injective. If $[\varphi, \psi] = 0$ in $KK(A, B)$, then $\varphi \sim \psi$.

**Lemma 5.7 ([HHL10], Lemma 2.13).** Let $B$ be a non-unital $\sigma$-unital purely infinite simple $C^*$-algebra. Let $\varphi, \psi : C(X) \to \mathcal{L}(H_B)$ be injective, where $X$ is compact and metrizable. If $\pi \circ \varphi$ and $\pi \circ \psi$ are injective then $\varphi \sim \psi$.

The proof of Lemma 5.7 can be found in [MR01] as Corollary 8.3.2. It is a corollary of Kirchbergs Weyl-Von Neumann Theorem. The Corollary is phrased in terms of Kirchberg algebras, however this is to be a non-unital $\sigma$-unital purely infinite simple nuclear and separable $C^*$-algebra. But the separability and nuclearity can be shifted to $C(X)$, which is separable since $X$ is separable ($X$ metrizable and compact implies separability). So the result adds up.

**Theorem 5.8 ([HHL10], Theorem 2.14).** Let $B$ be a non-unital, $\sigma$-unital purely infinite simple $C^*$-algebra, such that $RR(\mathcal{L}(H_B)) = 0$. Let $\varphi, \psi$ be a Cuntz pair associated to a pair of projections $p, q \in \text{proj}(\mathcal{L}(H_B))$, such that $p, q \notin \mathcal{K}(H_B)$. If $[\varphi : \psi] = 0$ in $K_0(B) = KK(C, B)$ then there is a unitary $u \in 1 + \mathcal{K}(H_B)$, such that

$$upu^* = q.$$

**Proof.** First of $\varphi(\lambda) = 0$ if $p\lambda = 0$, so $\lambda = 0$. So $\varphi$ is injective. Furthermore, since $p \notin \mathcal{K}(H_B)$, then $\varphi(\lambda) \notin \mathcal{K}(H_B)$ for any $0 \neq \lambda \in \mathbb{C}$. So observing that if $\pi \circ \varphi(\lambda) = 0$, then $\varphi(\lambda) \in \ker \, \pi$. From the canonical short exact sequence, we know that $\ker \, \pi = \text{im} \, i = \mathcal{K}(H_B) \subset \mathcal{L}(H_B)$, where $i$ is the inclusion. Since $\varphi(\lambda) \notin \mathcal{K}(H_B)$ then $\lambda = 0$, so $\pi \circ \varphi$ is injective. Let $\psi^\infty(\lambda) = \sum_i S_i \psi(a)S_i^*$ be defined as in the proof of Lemma 5.1 and suppose that $\psi^\infty(\lambda) = 0$, then by multiplying on either side with $S_i$ and $S_i^*$ respectively we get

$$S_i^* \psi^\infty(\lambda) S_i = \psi(\lambda) = 0.$$

So as before this implies that $q\lambda = 0$, so $\lambda = 0$. With the same arguments as with $\varphi$, we get that $\pi \circ \psi$ is injective. So by using Lemma 5.7 we get
\( \varphi \sim \psi^\infty \), when \( X = \{x\} \). So we have a sequence of unitaries \( w_n \in \mathcal{L}(H_B) \) such that
\[
 w_n \varphi(a) w_n^* - \psi^\infty(a) \in \mathcal{K}(H_B).
\]
As in the proof of Lemma 5.1, we defined some isometry \( W \). Let that isometry be \( w = (w_n) \). Since a unitary is especially an isometry we can follow the same procedure as in Lemma 5.1 to find another sequence of isometries fulfilling
\[
 v_n \varphi^\infty(a) - \psi(a) v_n \in \mathcal{K}(H \otimes B) \text{, } \mathcal{H}_B
\]
\[
 \lim_{n \to \infty} ||v_n \varphi^\infty(a) - \psi(a) v_n|| = 0
\]
\[
 v_j^* v_i = 0 \text{ for } i \neq j.
\]
By Lemma 3.4 we have
\[
 \varphi \sim_{\text{asp}} \varphi \oplus \psi \sim_{\text{asp}} \psi.
\]
By definition, there is a continuous family of unitaries \( (u_t)_{t \in [0, \infty)} \in \mathcal{L}(H_B) \) such that
\[
 u_t \varphi(a) u_t^* - \psi(a) \in C_0([0, \infty) \otimes \mathcal{K}(H_B)).
\]
Since \( RR(\mathcal{L}(H_B)) = 0 \), then \( RR(D\varphi(C, B)) = 0 \), so by Lemma 2.2 in [HL96] \( D\varphi(C, B) \) satisfies \( K_1 \) injectivity. So following the same procedure as in Theorem 2.11 from [HHL10], then \( \varphi \cong \psi \).

Finally for some \( t \), then \( ||u_t pu_t^* - q|| < 1 \), where \( u_t = u \) is of the form “identity + compact”. Since \( p - q \in \mathcal{K}(H_B) \), then
\[
 z = pq + (1 - p)(1 - q) = 1 + (2pq - q - p) \in 1 + \mathcal{K}(H_B)
\]
is invertible. Furthermore,
\[
 pz = p(1 + (2pq - q - p)) = p + 2pq - p = pq
\]
and
\[
 zq = (1 + (2pq - q - p)q = q + 2pq - q - pq = pq
\]
so \( pz = zq \). Now observe the polar decomposition of \( z \), i.e. there is a unitary (since \( z \) is invertible) \( v \), such that \( z = v|z| \). Then note that \( v = \frac{z}{|z|} \in 1 + \mathcal{K}(H_B) \), and
\[
 vpz^{-1} = q \frac{z^{-1}}{|z|} = \frac{qzz^{-1}}{|z|} = q.
\]

Then \( w = vu \in 1 + \mathcal{K}(H_B) \) is also a unitary such that
\[
 wpz = q
\]
.\]
5.2 Lifting projections in the corona algebra to the multiplier algebra

For this section we see an application of proper asymptotic unitary equivalence of two projections. We will give necessary conditions for lifting a projection of the corona algebra to a projection in the multiplier algebra.

Assume that $B$ is a stable $C^*$-algebra, such that $RR(M(B)) = 0$ for the remainder of this section if not otherwise instructed. Let $X = [0, 1]$ and let $I = C(X) \otimes B$, i.e the $C^*$-algebra of norm-continuous functions from $X$ to $B$. Then by the result of [APT73]

$$M(I) = M(B \otimes C([0, 1])) = C^b_{\text{strict}}([0, 1], M(B)).$$

Define by $\mathcal{Q}(I) = M(I)/I$ the corona algebra, and as before $\pi : M(I) \to \mathcal{Q}(I)$ be the quotient map. Now if $f$ in the corona algebra is a projection, we can not be sure that the element, $f$, it lifts to is a projection a priori. However, if an $f \in C^b_{\text{strict}}([0, 1], M(B))$ represents a projection $\pi f$ in the corona algebra, then fine, assuming that it is a projection. If it is not a projection then we can find a sequence $f_i$ of functions on subintervals, which can be chosen such that $f_i$ are projections. Finally it is shown that if the essential codimension of two successive $f_i$’s is 0 then $f$ lifts to a projection in $M(I)$.

However, we need a recipe for choosing such functions which are strictly continuous, and then extend those to a function which is (strictly) continuous on all of $X$: Consider a finite partition of $X$, by points $0 < x_1 < x_2 < \ldots < x_n < 1$ which divides $X$ into $n + 1$ closed subintervals $X_0, X_1, X_2, \ldots, X_n$ i.e. $X_i = [x_i, x_{i+1}]$. Then $f_i \in C^b_{\text{strict}}(X_i, M(B))$, such that $f_i(x_i) - f_{i-1}(x_i) \in B$ for $i = 1, 2, 3, \ldots n$. Furthermore as mentioned, $f_i$ can be chosen such that it is a projection in $B$ for all $i$ - this will be shown in a lemma later on. Define a function $m_i : X \to B$ by

$$m_i(x) = \begin{cases} \frac{x-x_{i+1}}{x_i-x_{i+1}} (f_i(x_i) - f_{i-1}(x_i)), & \text{if } x_{i-1} \leq x \leq x_i \\ \frac{x-x_{i+1}}{x_i-x_{i+1}} (f_i(x_i) - f_{i-1}(x_i)), & \text{if } x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots n$, and let $m_0 = m_{n+1} = 0$. Subsequently we define

$$\tilde{f}_i(x) = f_i(x) - m_i(x)/2 + m_{i+1}(x)/2, \text{ for } x \in X_i.$$  

Then let $\tilde{f}(x) = \bigcup_{i=0}^n \bigcup_{x \in X_i} \tilde{f}_i(x)$, then $\tilde{f} \in C^b_{\text{strict}}([0, 1], M(B))$.

Then any element in the corona algebra is of the form

$$\tilde{f} + I = \{f \in C^b_{\text{strict}}([0, 1], M(B)) \mid f - \tilde{f} \in C(X) \otimes B \}$$

where $\tilde{f}$ was defined as above, but via the recipe above, we can define the following.
**Lemma 5.9** ([HHL10], Lemma 3.1). The co-set \( \tilde{f} + I \subset \mathcal{Q}(I) \), represented by \((f_0, f_1, \ldots, f_n)\) is
\[
\tilde{f} + I = \left\{ f \in C^0_{\operatorname{strict}}([0, 1], M(B)) \mid f - \tilde{f} \in C(X_i) \otimes B \right\}
\]

**Proof.** That it is a coset, is obvious. So we need to show that \( \tilde{f} \) is well-defined, and that \( f - \tilde{f} \) is actually norm-continuous on \( X \). So assume that \( x_j \in X_{i-1} \) and \( x_i \in X_i \), such that \( x_i = x_j \), this happens only when \( j = i \), so we check what happens on the intersection of \( X_i \) and \( X_{i-1} \), which is \( x_i \). Observe
\[
\tilde{f}_{i-1}(x_i) = f_{i-1}(x_i) - m_{i-1}(x_i)/2 + m_i(x_i)/2.
\]
Then \( m_{i-1}(x_i) \), when \( x_i \in X_{i-1} \), has \( x_i - x_i = 0 \) in the numerator, so vanishes. Now observe
\[
\tilde{f}_i(x_i) = f_i(x) - m_i(x_i)/2 + m_{i+1}(x_i)/2.
\]
As before when \( x_i \in X_i \) then \( m_{i+1}(x_i) \) has \( x_i - x_i = 0 \) in the numerator, so vanishes. So to show that \( \tilde{f}_i(x_i) = \tilde{f}_{i-1}(x_i) \), we show that
\[
2f_{i-1}(x_i) + m_i(x_i)) = 2f_i(x) - m_i(x_i),
\]
or rather that \( f_{i-1}(x_i) + m_i(x_i)) = f_i(x_i) \). So since \( m_i(x_i) = f_i(x_i) - f_{i-1}(x_i) \), the result follows, so \( \tilde{f}_i \) is well-defined, and we can expand it to get
\[
\tilde{f} = \bigcup_{i=0}^n \tilde{f}_i.
\]
Note that the condition \( f - \tilde{f} \in C(X_i) \otimes B \) for all \( i \) imply that \( f - \tilde{f} \in C(X) \otimes B \), since
\[
f_i(x_i) - \tilde{f}_i(x_i) = f_{i-1}(x_i) - \tilde{f}_{i-1}(x_i)
\].

The relation between elements of \( \mathcal{Q}(I) \), when we represent them as described, in Lemma 5.9 is defined as follows: The tuple \((f_0, f_1, \ldots, f_n)\) represent the same element as \((g_0, g_1, \ldots, g_n)\) if and only if \( f_i - g_i \in C(X_i) \otimes B \) for \( i = 1, 2, \ldots, n \).

The next theorem actually show that we can find these intermediate points \( x_i \)'s, such that the \( f_i \)'s are projections.

**Theorem 5.10** ([HHL10], Theorem 3.2). A projection \( f \in \mathcal{Q}(I) \) can be represented by \((f_0, f_1, \ldots, f_n)\) as described where \( f_i(x) \) is a projection in \( B \) for every \( i \) and \( x \in X_i \).

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Proof. Assume that $f \in M(I)$ such that $\pi(f) = \mathbb{1}$, and assume without loss of generality that $f$ is a self-adjoint and $0 \leq f \leq 1$.

Let $t_0 \in X$. Then by the proof of Theorem 3.4 of [BP91] there is a self-adjoint element $T \in M(B)$ such that $T - f(t_0) \in B$, and so that $(1/2 - \epsilon, 1/2 + \epsilon) \not\subset \sigma(T)$ for some $\epsilon > 0$. Also $\pi(T - f(t_0)) = 0$ so we get

$$\pi(f(t) + T - f(t_0)) = \pi(f(t)) + \pi(T - f(t_0)) = \pi(f(t)).$$

Furthermore, since $T$ had a gap around $1/2$ then the same counts for $f(t) + T - f(t_0)$.

Define the norm-continuous function $r : C(X) \to B$ by $r(f(t)) = f(t) - f(t)^2$. Pick a point $z \in (0, 1/2)$ and since $f(t)$ was self-adjoint we know that we can pick the $z$ such that

$$z \notin \sigma(f(t_0) - f(t_0)^2) = \sigma(r(f(t_0))) \subset \mathbb{R}.$$ 

By the continuous functional calculus then $\sigma(r(f(t_0))) = r(\sigma(f(t_0)))$ and since the spectrum is closed, we realize that for $\delta > 0$, then

$$r^{-1}([z - \delta, z + \delta]) \not\subset \sigma(f(s))$$

when $\|s - t_0\| < \eta$ for some $\eta > 0$. Now let $f_{t_0}(s) = 1_{(z + \delta, 1]}(f(s))$ for $s \in (t_0 - \eta, t_0 + \eta)$, then this is a continuous function, such that $f_{t_0}(s)$ is a projection, and such that $f_{t_0} - f \in C(t_0 - \eta, t_0 + \eta) \otimes B$.

By repeating the procedure we produce an open cover of $X$. Then since $X$ is compact we can pass to a finite sub-cover such that we produce $n + 1$ points $t_0, t_1, ..., t_n$ and $n + 1$ functions $f_{t_0}, f_{t_1}, ..., f_{t_n}$ and a finite open covering $\{O_1\}$ which covers $X$, such that $t_i \in O_i$ and $O_i \cap O_{i-1} \neq \emptyset$ for $i = 1, 2, ..., n$. Then $f_i$ is a projection on $O_i$. Define $f_i(x_i) = f_{t_i}(x_i) = 1_{(z + \delta, 1]}(f(x_i))$ for $x_i \in (t_i - \eta, t_i + \eta)$. Then we can deduce that since $f_i - f = f_{t_i} - f \in B$ for all $i$, then for $x_i \in O_i \cap O_{i-1}$ we have

$$f_i(x_i) - f_{i-1}(x_i) = f_i(x_i) - f(x_i) + f(x_i) - f_{i-1}(x_i) \in B.$$ 

Define $X_i$ as we have already done in the start of the section by these points. Then since $f_i$ is also defined on $X_i$, then $(f_0, f_1, ..., f_n)$ is what represents $f$. \hfill \Box

Theorem 5.11 (Zhangs dichotomy). If $B$ is non-unital $\sigma$-unital purely infinite simple $C^*$-algebra, then $B$ is stable.

If we represent a projection as described in Theorem 5.10, the requirement for $f_i(x_i)$ is that it is a projection in $M(B)$ and $f_i(x_i) - f_{i-1}(x_i) \in B$. So we can observe the essential co-dimension $[f_i(x_i) : f_{i-1}(x_i)] \in KK(C, B)$. In the next theorem it is shown that if the essential co-dimension is 0 for all $i$, then we can lift a projection in $Q(I)$ to one in $M(I)$. We will also drop $B$'s stability in the next theorem since it comes from the above theorem.
Theorem 5.12 ([HHL10], Theorem 3.3). Let $B$ be a non-unital $\sigma$-unital purely infinite simple $C^*$-algebra, such that $RR(M(B)) = 0$ and $K_1(B) = 0$. Represent a projection $\mathbb{I} \in \mathcal{O}(I)$ by $(f_0, f_1, ..., f_n)$, where $f_i(x)$ is a projection in $C(X) \otimes M(B)$, for $x \in X_i$ and for each $i$. If $k_i = [f_i(x_i) : f_{i-1}(x_i)] = 0$ for all $i$, then we can lift the projection $\mathbb{I}$, to a projection in $M(I)$.

Proof. By Theorem 5.11 $B$ is stable. Assume for induction that $f_{j-1}(x_j) = f_j(x_j)$, for $j = 1, 2, ..., i - 1$. Then we wish to show that it is true for $i$.

As $f_i(x_i)$ was a projection for all $i$, we let them be denoted by $f_i(x_i) = p_i$, and $f_{i-1}(x_i) = p_{i-1}$. Since we have assumed that $[p_i : p_{i-1}] = 0$, we get from Theorem 5.8 that there is a unitary $u$ of the form "identity + compact" such that

$$||p_i - up_{i-1}u^*|| < \frac{1}{4}.$$ 

Then since $RR(M(B)) = 0$, then $RR(B) = 0$ and one may deduce from Theorem 5 of [HL93] that then there is a unitary $v \in 1_C + B$ with finite spectrum, such that for $0 < \epsilon < \frac{1}{4}$, then

$$||u - v|| < \epsilon < \frac{1}{4}.$$ 

It follows that

$$||p_i - vp_{i-1}v^*|| = ||p_i - up_{i-1}u^* + up_{i-1}u^* - vp_{i-1}v^*||$$

$$< ||p_i - up_{i-1}u^*|| + ||up_{i-1}u^* - vp_{i-1}v^*||$$

$$< \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$ 

Thus we get a unitary $w \in C_1 + B$ such that $wp_iw^* = vp_{i-1}v^*$. Since $w$ is of the correct form we can define $g_i = wf_iw^*$ and then $f_i - g_i \in C(X_i) \otimes B$.

Since $v$ has finite spectrum then $v = e^{ih}$ where $h \in B$ is some self-adjoint. Now, for $t \in [0, 1]$ there is a homotopy of unitaries $t \mapsto e^{ith}$ that maps $v$ to $1$ and for every $t \in [0, 1]$, the element is a unitary of the form "identity + compact". Now we can define

$$g_{i-1}(t) = e^{\frac{t-x_i}{x_i-x_{i-1}}h}f_{i-1}(t)e^{\frac{t-x_i}{x_i-x_{i-1}}h}$$

for $t \in [x_{i-1}, x_i]$. Then

$$g_{i-1}(x_i) = e^{\frac{x_i-x_{i-1}}{x_i-x_{i-1}}h}f_{i-1}(x_i)e^{\frac{x_i-x_{i-1}}{x_i-x_{i-1}}h}$$

$$= vf_{i-1}(x_i)v^* = wf_i(x_i)w^*$$

$$= g_i(x_i).$$
Furthermore $g_{i-1} - f_{i-1} \in C(X_i) \otimes B$, and
\[ g_{i-1}(x_{i-1}) = e^{i \frac{x_{i-1} - x_i}{h}} f_{i-1}(x_{i-1}) e^{-i \frac{x_{i-1} - x_i}{h}} = f_{i-1}(x_{i-1}). \]

If we also let $g_{i+1} = w f_{i+1} w^*$, then conclusively
\[ [g_{i+1}(x_{i+1}) : g_i(x_{i+1})] = [w f_{i+1}(x_{i+1}) w^* : w f_i(x_{i+1}) w^*] = [f_{i+1}(x_{i+1}) : f_i(x_{i+1})] = 0 \]

by assumption. So we have just shown that the family $(f_0, f_1, f_2, ..., f_n)$ and $(f_0, f_1, ..., f_{i-2}, g_{i-1}, g_i, g_{i+1}, f_{i+2}, ..., f_n)$ define the same projection $f$, and do not change the elements $k_i$ and it now solves the problem at $i$ i.e. also $f_i(x_i) = f_{i-1}(x_i)$. So from now on we will use the latter as $(f_0, f_1, f_2, ..., f_n)$. Now do the procedure for the remaining $j \in [i+1, n]$, it now follows that $(f_0, f_1, f_2, ..., f_n) \in \mathcal{L}(H_B)$ lifts $f$. 

\[ \square \]
6 A topology on $KK(A, B)$

This section will be concerned with constructing a pseudometric $d_K$ on $KK(A, B)$ and then proving that $KK(A, B)$ with $A$ and $B$ separable, is complete and separable. One may ask what this has to do with the essential theme of proper asymptotic unitary equivalence. However as it turns out, the pseudometric will measure how far $\phi \oplus \psi' \oplus \gamma$ is from being proper asymptotic unitary equivalent to $\phi' \oplus \psi \oplus \gamma$, when $(\phi, \psi)$ and $(\phi', \psi')$ are Cuntz pairs, and $\gamma : A \to \mathcal{L}(F)$ is an absorbing representation. Note that from Theorem 3.11, we get that $(\phi \oplus \psi', \phi' \oplus \psi) = 0$ in $KK_h(A, B)$.

6.1 The metric structure

Recall Theorem 3.11, this leads to the following definition

**Definition 6.1.** Let $(a_i)_{i=1}^{\infty}$ be a dense sequence in the unit ball of $A$, where $A$ is a separable $C^*$-algebra. Let $\phi, \psi : A \to \mathcal{L}(E)$ be $*$-homomorphisms and define

$$d_0(\phi, \psi) = \sum_{i=1}^{\infty} \frac{1}{2^i} ||\phi(a_i) - \psi(a_i)||.$$

Then define the following for $(\phi, \psi) \in \mathcal{E}_h(A, B)$

$$d_{\gamma}(\phi, \psi) = \inf\{d_0(\phi \oplus \gamma, u_n(\psi \oplus \gamma)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F)\}$$

where $\gamma : A \to \mathcal{L}(F)$ is an absorbing $*$-homomorphism.

**Lemma 6.2 ([MD03]).** The map $d$ is a pseudometric on $\mathcal{E}_h(A, B)$.

**Proof.** Since we are only dealing with a pseudometric, then the requirement that $d_{\gamma}(\phi, \psi) = 0 \Rightarrow \phi = \psi$ is not necessarily true. However if $\phi = \psi$, then

$$d_{\gamma}(\phi, \psi) = d_{\gamma}(\phi, \phi) = \inf\{d_0(\phi \oplus \gamma, u_n(\phi \oplus \gamma)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F)\} = 0.$$ 

Also

$$d_{\gamma}(\phi, \psi) = \inf\{d_0(\phi \oplus \gamma, u_n(\psi \oplus \gamma)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F)\} = d_{\gamma}(\psi, \phi).$$

Finally if $d_{\gamma}(\phi, \eta) = \inf\{d_0(\phi \oplus \gamma, u_n(\eta \oplus \gamma)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F)\}$ then if we observe $d_0(\phi \oplus \gamma, u_n(\eta \oplus \gamma)u_n^*)$, we see
\[ \delta_0(\varphi \oplus \gamma, u_n(\eta \oplus \gamma)u_n^*) = \sum_{i=1}^{\infty} \frac{1}{2^i} ||\varphi(a_i) \oplus \gamma(a_i) - u_n(\eta(a_i) \oplus \gamma(a_i))u_n^*|| \]
\[ = \sum_{i=1}^{\infty} \frac{1}{2^i} ||\varphi(a_i) \oplus \gamma(a_i) - u_n(\psi(a_i) \oplus \gamma(a_i))u_n^*|| + ||\psi(a_i) \oplus \gamma(a_i) - u_n(\eta(a_i) \oplus \gamma(a_i))u_n^*|| \]
\[ \leq \sum_{i=1}^{\infty} \frac{1}{2^i} ||\varphi(a_i) \oplus \gamma(a_i) - u_n(\psi(a_i) \oplus \gamma(a_i))u_n^*|| + \sum_{i=1}^{\infty} ||\psi(a_i) \oplus \gamma(a_i) - u_n(\eta(a_i) \oplus \gamma(a_i))u_n^*|| \]
\[ = \delta_0(\varphi \oplus \gamma, u_n(\psi \oplus \gamma)u_n^*) + \delta_0(\psi \oplus \gamma, u_n(\eta \oplus \gamma)u_n^*) \]
and conclude that \( \delta_\gamma(\varphi, \eta) \leq \delta_\gamma(\varphi, \psi) + \delta_\gamma(\psi, \eta) \).

It is also true that if \( \varphi_n(a) \to \varphi(a) \), for all \( a \in A \), then
\[ \delta_\gamma(\varphi_n, \psi) \to \delta_\gamma(\varphi, \psi). \]

One may have the question, whether this depends on the choice of absorbing representations. The surprising and relevant thing is that it only does up to approximate unitary equivalence, of those representations. Note that the following lemma is proven in all generality.

**Lemma 6.3** ([MD05], Lemma 2.2). If \( \gamma_i : A \to \mathcal{L}(F_i) \) for \( i = 1, 2 \), are \( * \)-homomorphisms. If \( \gamma_1 \sim \gamma_2 \), then \( \delta_{\gamma_1}(\varphi, \psi) = \delta_{\gamma_2}(\varphi, \psi). \)

**Proof.** Let \( w_n \in \mathcal{L}(F_1, F_2) \) be a unitary such that
\[ \lim_{n \to \infty} ||w_n\gamma_1(a)w_n^* - \gamma_2(a)|| = 0, \]
then clearly \( \delta_{w_n\gamma_1 w_n^*}(\varphi, \psi) \to \delta_{\gamma_2}(\varphi, \psi) \), so what is left to prove is that \( \delta_{w_n\gamma_1 w_n^*}(\varphi, \psi) = \delta_{\gamma_1}(\varphi, \psi) \). First off note that
\[ (1 \oplus w_n)(1 + \mathcal{K}(E \oplus F_1))(1 \oplus w_n^*) = 1 + \mathcal{K}(E \oplus w_n F_1 w_n^*) = 1 + \mathcal{K}(E \oplus F_2). \]
So we can calculate
\[ \delta_{w_n\gamma_1 w_n^*}(\varphi, \psi) = \inf \{ \delta_0(\varphi \oplus w_n \gamma_1 w_n^*, u_n(\psi \oplus w_n \gamma_1 w_n^*)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F_1) \} \]
\[ = \inf \{ \delta_0(\varphi \oplus \gamma_1, u_n(\psi \oplus \gamma_1)w_n^*) \mid (1 \oplus w)u_n(1 \oplus w^*) \oplus \mathcal{K}(E \oplus F_1) \} \]
\[ = \delta_{\gamma_1}(\varphi, \psi). \]

\[ \Box \]

It should be noted that if the \( \gamma_i \)'s are absorbing then they are indeed approximately unitarily equivalent. In fact if \( \gamma_i \) is absorbing then \( \gamma_1 \oplus \varphi \sim \gamma_1 \)
and \( \gamma_2 \oplus \varphi \sim \gamma_2 \) for all \( \varphi \) so especially for \( \gamma_i \), so \( \gamma_1 \sim \gamma_1 \oplus \gamma_2 \sim \gamma_2 \). So for the future we define
\[
\delta(\varphi, \psi) = \delta_n(\varphi, \psi).
\]

**Lemma 6.4** ([MD05], Lemma 2.3(a)). Let \( w \in \mathcal{L}(E, F) \) be a unitary, then
\[
\delta(w \varphi w^*, w \psi w^*) = \delta(\varphi, \psi).
\]

**Proof.** As in Lemma 6.3, then
\[
(w \oplus 1)(1 + \mathcal{K}(E \oplus F))(w^* \oplus 1) = 1 + \mathcal{K}(wEw^* \oplus F) = 1 + \mathcal{K}(F \oplus F).
\]
So as before we can calculate
\[
\delta(w_n \varphi w_n^*, w_n \psi w_n^*) = \inf \{ \delta_0(w_n \varphi w_n^* \oplus \gamma, u_n(w_n \psi w_n^* \oplus \gamma)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F) \}
= \inf \{ \delta_0(\varphi \oplus \gamma, u_n(\psi \oplus \gamma)u_n^*) \mid (w \oplus 1)u_n(w^* \oplus 1) \in 1 + \mathcal{K}(F \oplus F) \}
= \delta(\varphi, \psi).
\]

**Lemma 6.5** ([MD05], Lemma 2.3(b)). Let \( \eta : A \to \mathcal{L}(F) \) be a \(*\)-homomorphism, then
\[
\delta(\varphi, \psi) = \delta(\varphi \oplus \eta, \psi \oplus \eta) = \delta(\eta \oplus \varphi, \eta \oplus \psi)
\]

**Proof.** The last of the equalities is easy.

Note that if \( \gamma \oplus \varphi \sim \gamma \), then \( \eta \oplus \gamma \oplus \varphi \sim \eta \oplus \gamma \). So \( \delta_\gamma = \delta_{\eta \oplus \gamma} \). So
\[
\delta(\varphi \oplus \eta, \psi \oplus \eta) = \inf \{ \delta_0(\varphi \oplus \eta \oplus \gamma, u_n(\psi \oplus \eta \oplus \gamma)u_n^*) \mid u_n \in 1 + \mathcal{K}(E \oplus F \oplus F) \}
= \delta(\varphi, \psi).
\]

As one would suspect, then proper asymptotic unitary equivalent pairs of \(*\)-homomorphisms, yields equality of \( \delta \), in fact we have the following lemma

**Lemma 6.6** ([MD05], Lemma 2.4). Let \( \varphi, \psi, \varphi', \psi' : A \to \mathcal{L}(E) \) be \(*\)-homomorphisms, such that \( \varphi \cong \varphi' \) and \( \psi \cong \psi' \), then
\[
\delta(\varphi, \psi) = \delta(\varphi', \psi').
\]

**Proof.** If we observe \( \delta(\varphi, \psi) = \delta(u_n \varphi u_n^*, \psi) \), where \( u_n \in 1 + \mathcal{K}(E) \) is a unitary and let \( v_n \in 1 + \mathcal{K}(E) \) be another unitary, then
\[
\delta_0(u_n \varphi u_n^*, v_n \psi v_n) = \sum_{i=1}^{\infty} \frac{1}{2^{i}} ||u_n \varphi(a_i)u_n^* - v_n \psi(a_i)v_n^*||
= \sum_{i=1}^{\infty} \frac{1}{2^{i}} ||\varphi'(a_i) - v_n \psi(a_i)v_n^*||
= \delta_0(\varphi', v_n \psi v_n^*)
\]
which is an element of \( \{ \delta_0(\varphi', v_nv_n^*) \mid v_n \in 1 + \mathcal{K}(E) \} \). By symmetry we get \( \delta(\varphi, \psi) = \delta(\varphi', \psi') \).

We are now ready to define our pseudometric, not on random \(*\)-homomorphisms, but on Cuntz pairs i.e. a pseudometric on \( E_h(A, B) \).

**Definition 6.7.** Let \((\varphi, \psi) \in E_h(A, B)\) and \((\varphi', \psi') \in E_h(A, B)\) and define the map

\[
d(((\varphi, \psi), (\varphi', \psi'))) = \delta(\varphi \oplus \psi', \psi \oplus \varphi')
\]

To actually prove that this is an invariant pseudometric on the desired space, we first need a rather intuitive result.

**Lemma 6.8 (MD05, Lemma 2.6).** Let \((\varphi, \psi), (\varphi', \psi') \in E_h(A, B)\), then if \([\varphi, \psi] = [\varphi', \psi']\) in \(KK_h(A, B)\), then

\[
d(((\varphi, \psi), (\varphi', \psi'))) = 0.
\]

**Proof.** Note that

\[
0 = [\varphi, \psi] - [\varphi', \psi'] = [\varphi, \psi] + [\psi', \varphi']
\]

and

\[
[\varphi, \psi] + [\psi', \varphi'] = [\varphi \oplus \psi', \psi \oplus \varphi'] = 0.
\]

By Theorem 3.11 we conclude that \( \delta(\varphi \oplus \psi', \psi \oplus \varphi') = 0 \), so

\[
d(((\varphi \oplus \psi), (\varphi' \oplus \psi))) = 0.
\]

**Proposition 6.9 (MD05).** The map \( d \) is a pseudometric on \( E_h(A, B) \).

**Proof.** So we need to show the regular requirements of a metric, except that distance 0 imply same argument.

We already know that if \((\varphi, \psi) \in E_h(A, B)\), then from Lemma 6.8 then

\[
d(((\varphi, \psi), (\varphi', \psi'))) = 0.
\]

To prove the symmetry of \( d \), we note that the symmetry comes from the symmetry of \( \delta \), in fact

\[
d(((\varphi, \psi), (\varphi', \psi'))) = \delta(\varphi \oplus \psi', \psi \oplus \varphi') = \delta(\psi \oplus \varphi', \varphi \oplus \psi')
\]

Now letting \( w \) be the unitary which flips each sum, then it follows that by Lemma 6.4 then \( \delta(w\psi \oplus \varphi'w, w\varphi \oplus \psi'w) = \delta(\psi \oplus \varphi', \varphi \oplus \psi') \), so we get

\[
\delta(w\psi \oplus \varphi'w, w\varphi \oplus \psi'w) = \delta(\varphi' \oplus \psi, \psi' \oplus \varphi) = d(((\varphi', \psi'), (\varphi, \psi))).
\]

From the triangle inequality we remember that for any triple of representations \( \varphi, \varphi', \varphi'' : A \to \mathcal{L}(E) \), then

\[
\delta(\varphi, \varphi'') \leq \delta(\varphi, \varphi') + \delta(\varphi', \varphi''). \tag{4}
\]
As before the result follows from \( \delta \)'s properties. We wish to show that

\[
\delta(\varphi \oplus \psi'', \psi \oplus \varphi'') \leq \delta(\varphi' \oplus \psi'', \psi' \oplus \varphi'') + \delta(\varphi \oplus \psi', \psi \oplus \varphi').
\]

From Lemma 6.5 we see that \( \delta(\varphi \oplus \psi'', \psi \oplus \varphi'') = \delta(\psi \oplus \varphi' \oplus \psi'', \psi \oplus \varphi' \oplus \varphi'') \) and \( \delta(\varphi \oplus \psi', \psi \oplus \varphi') = \delta(\varphi \oplus \psi' \oplus \psi'', \psi \oplus \varphi' \oplus \varphi'') \). So all in all we get from \( \mathbb{H} \) and from Lemma 6.5

\[
\delta(\varphi \oplus \psi'', \psi \oplus \varphi'') = \delta(\varphi \oplus \psi'' \oplus \psi', \psi \oplus \varphi'' \oplus \psi') = \delta(\varphi \oplus \psi' \oplus \psi'', \psi \oplus \varphi' \oplus \varphi'') + \delta(\psi \oplus \varphi' \oplus \psi'', \psi \oplus \varphi' \oplus \varphi'')
\]

which proves the assertion. \( \square \)

**Lemma 6.10** ([MD05], Proposition 3.7). \( d \) descends to an invariant pseudometric \( d_K \) on \( KK_h(A, B) \).

**Proof.** First off by the symmetry of \( d \), it is enough to show that if \( (\varphi, \psi), (\varphi', \psi'), (\varphi'', \psi'') \in \mathbb{E}_h(A, B) \), and \( [\varphi', \psi'] = [\varphi'', \psi''] \) in \( KK_h(A, B) \), then

\[
d((\varphi, \psi), (\varphi', \psi')) \leq d((\varphi, \psi), (\varphi'', \psi'')).
\]

By Lemma 6.8, we get \( d((\varphi', \psi'), (\varphi'', \psi'')) = 0 \), and since \( d \) is a pseudometric, we can apply the triangle inequality to see that

\[
d((\varphi, \psi), (\varphi'', \psi'')) \leq d((\varphi, \psi), (\varphi', \psi')) + d((\varphi', \psi'), (\varphi'', \psi''))
\]

So \( d \) descends to a pseudometric on \( KK_h(A, B) \).

To see that it is invariant to adding Cuntz pairs, we show that

\[
d((\varphi, \psi) \oplus (\alpha, \beta), (\varphi', \psi') \oplus (\alpha, \beta)) = d((\varphi, \psi), (\varphi', \psi')).
\]

Let \( d_k([\varphi, \psi], [\varphi', \psi']) = d((\varphi, \psi), (\varphi', \psi')) \). Then it is enough to show that \( d_k([\varphi, \psi] - [\varphi', \psi'], 0) = d((\varphi, \psi), (\varphi', \psi')) \). One should see as in Lemma 6.8 that \( [\varphi, \psi] - [\varphi', \psi'] = [\varphi, \psi] + [\psi', \varphi'] \) so

\[
d_k([\varphi, \psi] - [\varphi', \psi'], 0) = d_k([\varphi, \psi] + [\psi', \varphi'], 0)
\]

\[
= d_k([\varphi \oplus \psi', \psi \oplus \varphi'], 0)
\]

\[
= d_k((\varphi \oplus \psi', \psi \oplus \varphi'), (0, 0))
\]

\[
= \delta(\varphi \oplus \psi', \psi \oplus \varphi')
\]

\[
= d((\varphi, \psi), (\varphi', \psi')).
\]

\[ \square \]
We already know from the isomorphism $KK(A,B) \cong KK_h(A,B)$ that any element $(\varphi, \psi, 1) \in KK(A,B)$, has a representative $(\varphi, \psi) \in KK_h(A,B)$. But by Lemma 5.4 we can do better, we can in fact represent any element as $(\alpha, \gamma) \in E_h(A,B)$, where $\gamma$ is a fixed absorbing $*$-homomorphism such that it commutes with $\alpha$ up to compacts.

We are now ready to prove that this pseudometric makes $KK(A,B)$ into a really nice type of spaces, namely a separable and complete space, which has extensive uses. In fact if we take the closure of 0, and mod it out, then we get a Polish space. This is required since we can not be sure that distance 0 imply that the arguments are the same.

**Theorem 6.11.** Let $A$ and $B$ be separable $C^*$-algebras. The topology defined by the pseudometric $d$, makes $KK(A,B)$ separable and complete.

**Proof.** As just mentioned it is the case that any element of $KK(A,B)$ can be represented by $(\alpha, \gamma) \in E_h(A,B)$, where $\gamma$ is a fixed absorbing $*$-representation. So since $\alpha(a) - \gamma(a) \in K(H_B)$ for all $a \in A$, then

$$\alpha(A) \subseteq \gamma(A) + K(H_B).$$

Since the algebra $\gamma(A) + K(H_B)$ is separable, we can conclude that $KK(A,B)$ is second countable so $KK(A,B)$ is separable.

Represent some elements of $KK(A,B)$, as $(\alpha_n, \gamma) \in E_h(A,B)$ being a Cauchy sequence, where as before, $\gamma : A \to L(H_B)$ is a fixed absorbing $*$-homomorphism. So by definition of Cauchy sequences then

$$d((\alpha_n, \gamma), (\alpha_m, \gamma)) = \delta(\alpha_n \oplus \gamma, \gamma \oplus \alpha_m) \to 0, \text{ when } m, n \to \infty.$$  

Also since $\delta(\alpha_m \oplus \gamma, \gamma \oplus \alpha_m) = 0$ by Lemma 6.2 We get by the triangle inequality that

$$\delta(\alpha_n \oplus \gamma, \alpha_m \oplus \gamma) \leq \delta(\alpha_n \oplus \gamma, \gamma \oplus \alpha_m) + \delta(\gamma \oplus \alpha_n, \alpha_m \oplus \gamma)$$

$$= \delta(\gamma \oplus \alpha_n, \alpha_m \oplus \gamma) \to 0$$

so $\delta(\alpha_n \oplus \gamma, \alpha_m \oplus \gamma) \to 0$, when $m, n \to \infty$. Furthermore since

$$[\alpha_n, \gamma] + [\gamma, \gamma] = [\alpha_n \oplus \gamma, \gamma \oplus \gamma] \text{ in } KK(A,B),$$

then since $\gamma$ is absorbing

$$[\alpha_n, \gamma] = [\alpha_n \oplus \gamma, \gamma \oplus \gamma] - [\gamma, \gamma]$$

$$= [\alpha_n \oplus \gamma, \gamma \oplus \gamma] + [\gamma, \gamma]$$

$$= [\alpha_n \oplus \gamma, \gamma \oplus \gamma] + [\gamma, \gamma]$$

$$= [\alpha_n \oplus \gamma, \gamma \oplus \gamma].$$

So we may assume that $\delta(\alpha_n, \alpha_m) \to 0$, when $m, n \to \infty$.

We may need to pass to a subsequence of $\alpha_n$, and use that to show that the sequence converges, because then the sequence itself converges. So
if necessary pass to a subsequence, and then find a sequence of unitaries $u_n \in 1 + \mathcal{K}(H_B)$, such that

$$
\delta_0(\alpha_n, u_{n+1} \alpha_{n+1} u_{n+1}^*) = \sum_{i=1}^{\infty} \frac{1}{2^i} ||\alpha_n(a_i) - u_{n+1} \alpha_{n+1}(a_i) u_{n+1}^*|| < \frac{1}{2^n}.
$$

We can do this since $\delta(\alpha_n, \alpha_m) \to 0$, when $m, n \to \infty$. Now define

$$
\alpha'_n(a) = (u_2 u_3 \ldots u_n) \alpha_n(a) (u_2 u_3 \ldots u_n)^*.
$$

Then since

$$
\delta_0(\alpha'_n, \alpha'_{n+1}) = \sum_{i=1}^{\infty} \frac{1}{2^i} ||\alpha'_n(a_i) - \alpha'_{n+1}(a_i)||
$$

$$
= \sum_{i=1}^{\infty} \frac{1}{2^i} ||(u_2 u_3 \ldots u_n) \alpha_n(a_i) (u_2 u_3 \ldots u_n)^* - (u_2 u_3 \ldots u_{n+1}) \alpha_n(a_i) (u_2 u_3 \ldots u_{n+1})^*||
$$

$$
= \sum_{i=1}^{\infty} \frac{1}{2^i} ||\alpha_n(a_i) - u_{n+1} \alpha_{n+1}(a_i) u_{n+1}^*|| < \frac{1}{2^n}
$$

it follows that $\alpha'_n$ is a Cauchy sequence in $\operatorname{Hom}(A, \mathcal{L}(H_B))$. Since $\operatorname{Hom}(A, \mathcal{L}(H_B))$ is complete $\alpha'_n \to \alpha$ for $n \to \infty$. Furthermore, $\alpha$ is a $*$-homomorphism such that $\alpha(a) - \gamma(a) \in \mathcal{K}(H_B)$, since we had that the following was true $\alpha'_n(a) - \gamma(a) \in \mathcal{K}(H_B)$. It now follows that the sequence $[\alpha'_n, \gamma] = [\alpha_n, \gamma]$ converges to $[\alpha, \gamma]$ in $KK(A, B)$. So $KK(A, B)$ is complete.

It is then easy to define an almost identical statement as Theorem 3.11, only omitting the asymptotic part of the equivalence, so we get a “proper approximate unitary equivalence”. The proof follows by definition of $d$ and the separability of $A$.

**Theorem 6.12** ([MD05], Theorem 3.3/Corollary 3.7). *Let $A$ and $B$ be separable $C^*$-algebras, and let $(\varphi, \psi) \in E_h(A, B)$. Then the following are equivalent.*

- $[\varphi, \psi] \in 0$ in $KK_h(A, B)$.
- $[\varphi, \psi, 1] \in 0$ in $KK(A, B)$.
- There is a absorbing $*$-homomorphism $\gamma : A \to \mathcal{L}(H_B)$ such that

$$
\varphi \oplus \gamma \sim \psi \oplus \gamma
$$

where the unitary is of the form “identity + compact”.

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References


