Property Gamma and inner amenability

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This is a corrected version of the original project. A small amount of errors have been corrected, none of which was very serious, mostly bad spelling, bad grammar and bad formulation. The original project can be found through the author.
Abstract

This project contains an investigation of the relationship between inner amenability of a group and property $\Gamma$ of its associated group von Neumann algebra. We prove that a group with infinite conjugacy classes is inner amenable if its associated group von Neumann algebra has property $\Gamma$. Some theory concerning ways of constructing groups and tensor products of von Neumann algebras is covered, including infinite tensor products of von Neumann algebras. Lastly we give an example of an inner amenable group with infinite conjugacy classes whose associated group von Neumann algebra does not have property $\Gamma$; thus showing that the converse implication to the result proved in the first part is not true.

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1 Introduction

The group von Neumann algebra associated to a group has been the subject of much interest throughout the years. Much energy has been spent on describing group theoretical properties of a group $G$ by operator algebraic properties of its associated group von Neumann algebra $L^G$ and vice versa. In this project we investigate the relationship between a particular group theoretical property called inner amenability, and a particular operator algebraic property called property $\Gamma$. For many years it has been known that property $\Gamma$ of the group von Neumann algebra implied inner amenability of the group, but it was only recent proved that the converse is not true. The goal of this project is to give a proof of the first implication and a counterexample showing that the other one is false.

The group von Neumann algebra associated to a group is always a $\text{II}_1$ factor if the group has infinite conjugacy classes, but in many cases, these factors are isomorphic. It is a result by A. Connes [Con] that all the injective $\text{II}_1$ factor are in fact isomorphic, and this unique (up to isomorphism) $\text{II}_1$ factor is called the Hyperfinite $\text{II}_1$ factor. A corollary of Connes result was that all countable amenable groups with infinite conjugacy classes have isomorphic group von Neumann algebras. So indeed a large class of groups have isomorphic group von Neumann algebras.

In general it is hard to determine whether two $\text{II}_1$ factors are isomorphic, and in 1943 property $\Gamma$ was introduced by F. J. Murray and J. von Neumann [MvN] to give the first example of two non-isomorphic $\text{II}_1$ factors. Murray and von Neumann were interested in methods to determine when factors were isomorphic, and they introduced a property for $\text{II}_1$ factors called approximately finite. They showed that all approximately finite $\text{II}_1$ factors were isomorphic, but it was hard to find examples of $\text{II}_1$ factors that were not approximately finite. This unique approximately finite factor satisfied property $\Gamma$ and Murray and von Neumann found that the group von Neumann algebra of the free group on two generators did not, thus proving the existence of two non-isomorphic factors of type $\text{II}_1$.

E. G. Effros proposed in his article [Eff] of 1975, that property $\Gamma$ of the group von Neumann algebra might be characterized by a group theoretical property he introduced as inner amenability. He was able to prove that if the group von Neumann algebra of a discrete group with infinite conjugacy classes had property $\Gamma$ then the group was in fact inner amenable, and he expected that the converse was also true. As it turned out the converse implication was not true, but the problem was unsolved for a long time, and it is only recently a counterexample has been produced. This counterexample was produced by S. Vaes, and makes up the main part of this project.

Besides this introduction, the project starts with some preliminaries and notation. The reader is assumed to be familiar with the basic theory of von Neumann algebras, some fundamental functional analysis and maybe to be
acquainted with the algebraic tensor product. There are a few results needed during the proof of Theorem 6.13 that is not mentioned in the preliminaries, but relegated to the appendix.

Section 3 starts by introducing the reader to the concepts amenability, inner amenability and property \( \Gamma \), and during this, discusses the relationship between amenability and inner amenability. Afterwards the main result of the section is proved, namely that property \( \Gamma \) of \( L^G \) implies inner amenability of \( G \), when \( G \) is a discrete group with infinite conjugacy classes. The proof follows [Eff] but is separated in two parts presented in Theorem 3.6 and Theorem 3.7. The motivation for this separation is that Theorem 3.6 is used not only to prove Theorem 3.7, but also in the proof of Theorem 6.13.

Sections 4 contains methods of constructing groups, namely the semidirect product of groups, the amalgamated free product of groups, and a special case of the inductive limit of groups. Section 5 is an introduction to the tensor products of von Neumann algebras. In particular the infinite tensor product of von Neumann algebras is of interest, and defining this is a bit more tricky than the finite case.

The really interesting part of the project lies in Section 6, and it is here the counterexample to Effros conjecture is produced, as in [Vae]. This section starts with a small introduction to property (T) for groups, since property (T) of \( SL(3,\mathbb{Z}) \) is used to show that the particular group in question does not have property \( \Gamma \). After this the group is constructed, i.e. the group we will show to be inner amenable but with a group von Neumann algebra that does not have property \( \Gamma \). The main theorem of the section, which states this particular group is a counterexample to Effros conjecture, is presented slightly different than in [Vae]. The proof is the same, but is divided in smaller pieces.

2 Preliminaries and notation

As usual \( \mathbb{N}, \mathbb{Z}, \mathbb{R} \) and \( \mathbb{C} \) will denote the natural numbers, the integers, the real numbers and the complex numbers respectively, but besides these \( \mathbb{N}_0 \) denotes the non-negative integers. Let \( \mathfrak{X} \) be a normed space. We denote by \( \mathfrak{X}^* \) the Banach dual of \( \mathfrak{X} \) and we will denote the duality between \( \mathfrak{X} \) and \( \mathfrak{X}^* \) by \( \langle \cdot, \cdot \rangle \), i.e. \( \langle f, v \rangle = f(v) \) when \( f \in \mathfrak{X}^* \) and \( v \in \mathfrak{X} \). The locally convex topology on \( \mathfrak{X} \) induced by \( \mathfrak{X}^* \) will be referred to as the weak topology and the locally convex topology on \( \mathfrak{X}^* \) induced by \( \mathfrak{X} \) will be referred to as the weak* topology. It is a well-known consequence of the Hahn-Banach Theorem that the weak closure and the norm closures of any convex subset of \( \mathfrak{X}^* \) coincide. If \( X \) is any topological space, then \( C(X;\mathbb{C}) \) will be used to denote the continuous functions on \( X \) with values in \( \mathbb{C} \). The letters \( G, H, K \) and \( N \) will be used
for groups, and $g, h$ and $k$ for elements in these groups.\footnote{Sometimes we will also use $k$ as an index of natural numbers, but this should not cause any confusion.} The letter $e$ will be used for the neutral element of a group, and sometimes this will be implicit.

The letter $\mathcal{H}$ will be used to denote Hilbert spaces whose inner product we denote by $(\cdot | \cdot)$, and $B(\mathcal{H})$ will denote the set of bounded operators from the Hilbert space $\mathcal{H}$ to itself. There are several interesting topologies on $B(\mathcal{H})$ and those we will use in this project are the weak operator topology, the ultraweak operator topology, the strong operator topology and the ultrastrong operator topology. They are defined as follows:

(wo) The \textit{weak operator topology} is the weak topology induced by the linear functionals on $B(\mathcal{H})$ of the form $x \mapsto \sum_{n=1}^{k} (x \xi_n | \eta_n)$ with $k \in \mathbb{N}$ and $\xi_n, \eta_n \in \mathcal{H}$ ($n = 1, \ldots, k$);

(uwo) the \textit{ultraweak operator topology} is the weak topology induced by the linear functionals on $B(\mathcal{H})$ of the form $x \mapsto \sum_{n=1}^{\infty} (x \xi_n | \eta_n)$ with $\xi_n, \eta_n \in \mathcal{H}$ ($n \in \mathbb{N}$) chosen so that $\sum_{n=1}^{\infty} \|\xi_n\|^2 + \|\eta_n\|^2 < \infty$.

(so) the \textit{strong operator topology} is the weak topology induced by the seminorms on $B(\mathcal{H})$ of the form $x \mapsto \sum_{n=1}^{k} \|x \xi_n\|$ with $k \in \mathbb{N}$ and $\xi_n \in \mathcal{H}$ ($n = 1, \ldots, k$);

(uso) the \textit{ultrastrong operator topology} is the weak topology induced by the seminorms on $B(\mathcal{H})$ of the form $x \mapsto \sum_{n=1}^{\infty} \|x \xi_n\|$ with $\xi_n \in \mathcal{H}$ ($n \in \mathbb{N}$) chosen such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

A selfadjoint subalgebra of $B(\mathcal{H})$ containing the identity is called a von Neumann algebra if it is closed in the weak operator topology. The letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ and $\mathcal{N}$ will be used to denote von Neumann algebras (or in a few cases just selfadjoint algebras of operators) and $C'$ will be used to denote the commutant of a subset $C$ of $B(\mathcal{H})$. We recall some useful facts about the four mentioned topologies on a von Neumann algebra inherited from $B(\mathcal{H})$:

(i) the weak operator (resp. ultraweak operator) and the strong operator (resp. ultrastrong operator) continuous linear functionals are the same, and are all of the form described in (wo) (resp. (uwo)) above.

(ii) the weak operator (resp. ultraweak operator) and the strong operator (resp. ultrastrong operator) closures of a convex set agree.

(iii) the weak operator (resp. strong operator) and ultraweak operator (resp. ultrastrong operator) topologies agree on bounded sets, i.e. any bounded net converges weakly (resp. strongly) to a point $x$ if and only if it converges ultraweakly (resp. ultrastrongly) to $x$. 

\[ \sum_{n=1}^{k} (x \xi_n | \eta_n) \]

\[ \sum_{n=1}^{\infty} (x \xi_n | \eta_n) \]

\[ \sum_{n=1}^{k} \|x \xi_n\| \]

\[ \sum_{n=1}^{\infty} \|x \xi_n\| \]
(iv) the closure of a selfadjoint subalgebra of $B(H)$ containing the unit is
the same in all the four locally convex topologies mentioned above.

The letters $\xi, \eta$ will be used for elements in a Hilbert space and $x, y$ will be
used for operators on a Hilbert space. The identity operator on $H$ will be
denoted $1_H$ or $1$ if we want to emphasize on what Hilbert space it is the
identity.

A trace on a von Neumann algebra $M$ is a positive linear functional on
$M$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in M$, and a normalized trace is a
trace of norm 1. If $\tau(x) > 0$ for all non-zero positive elements $x \in M$ we say
that $\tau$ is faithful and if $\tau$ is ultraweakly continuous we say that $\tau$ is normal.
If $\tau$ is a trace we can define a semi-norm on $M$ by $x \mapsto (\tau(x^*x))^{1/2}$ which we
will denote $\|x\|_\tau$, and in the case where $\tau$ is faithful this becomes an actual
norm. In the case where $\tau$ has norm 1 we have the following well-known
inequalities

$$|\tau(x)| \leq \|x\|_\tau, \quad \|xy\|_\tau \leq \|x\| \|y\|_\tau \quad \text{and} \quad \|xy\|_\tau \leq \|y\| \|x\|_\tau.$$  

A $*$-homomorphism $\varphi: N \to M$ between von Neumann algebras is required
to map the identity operator to the identity operator, and we say that $\varphi$ is
ultraweakly continuous if $\varphi$ is continuous when both $N$ and $M$ are equipped
with the ultraweak topologies.

Suppose that $G$ is any old group with neutral element $e$. As usual $\ell^2(G)$
denotes the Hilbert space of square-summable complex-valued functions on
$G$ and $\ell^\infty(G)$ denotes the Banach space of bounded complex-valued functions
on $G$. We will think of $\ell^\infty(G)$ as a von Neumann algebra acting as
multiplication operators on $\ell^2(G)$ in the natural way. By the standard or-
thonormal basis of $\ell^2(G)$ we refer to the set $\{\delta_g : g \in G\}$ where $\delta_g$
denotes the indicator function on $g$. Besides $\ell^2(G)$ and $\ell^\infty(G)$ we will also consider
$\ell^1(G)$ which is the Banach $*$-algebra of summable complex-valued functions
on $G$.

### 2.1 A brief introduction to the group von Neumann algebra

Let $G$ be a group still, with neutral element $e$, and let us for simplicity denote
$\ell^2(G)$ by $H$. This section contains (as the title suggests) a brief introduction
to the group von Neumann algebra. The construction is rather elementary,
and almost all arguments are left out. More on the group von Neumann
algebra (including more details) may be found in [KR2, Section 6.7].

For $\xi, \eta \in H$, we define the convolution of $\xi$ and $\eta$, as in $\ell^1(G)$, to be the
map $\xi * \eta : G \to \mathbb{C}$ given by

$$(\xi * \eta)(g) = \sum_{h \in G} \xi(gh^{-1})\eta(h)$$
for all $g \in G$. It is easy to check that the sum is absolutely convergent so that the convolution of $\xi$ and $\eta$ is well-defined. The convolution of $\xi$ and $\eta$ is always a bounded map, but it is not always true that $\xi * \eta \in \mathcal{H}$ whenever $\xi, \eta \in \mathcal{H}$. If $\xi \in \ell^2(G)$ and $g \in G$, then we see that

$$(\delta_g * \xi)(h) = \xi(g^{-1}h) \quad \text{and} \quad (\xi * \delta_g)(h) = \xi(hg^{-1})$$

for all $h \in G$, which shows that both $\delta_g * \xi$ and $\xi * \delta_g$ are elements in $\ell^2(G)$ of the same norm as $\xi$. Thus

$$\lambda_g : \xi \mapsto \delta_g * \xi \quad \text{and} \quad \rho_g : \xi \mapsto \xi * \delta_g^{-1}$$

defines linear isometries on $\mathcal{H}$ which turns out to be unitaries. In fact the maps $\lambda : g \mapsto \lambda_g$ and $\rho : g \mapsto \rho_g$ are unitary representations which we refer to at the left regular representation and the right regular representation respectively. These two representations are closely related in the sense that the von Neumann algebras they generate are each others commutants, so in particular $\lambda_g \rho_g = \rho_g \lambda_g$ for all $g \in G$. Another unitary representation of interest, is the adjoint representation of $G$ which we throughout this project will denote $\pi$, i.e.

$$\pi : g \mapsto \lambda_g \rho_g$$

This representation will be used a lot, and we often denote $\pi(g)$ by $\pi_g$ as with $\lambda_g$ and $\rho_g$. It is straight forward to check that the map

$$L_\xi : \eta \mapsto \xi * \eta, \quad \mathcal{H} \rightarrow \ell^\infty(G)$$

defines a linear operator $\mathcal{H} \rightarrow \ell^\infty(G)$ if $\xi \in \mathcal{H}$, and for certain $\xi$, such as $\xi = \delta_g$, the range of $L_\xi$ is contained in $\mathcal{H}$. By [KR1, Theorem 6.7.2.] the set

$$LG = \{L_\xi : \xi \in \mathcal{H}, L_\xi \in B(\mathcal{H})\}$$

is a von Neumann algebra which we call the group von Neumann algebra associated to $G$. By the same theorem we know that the span of $\{\lambda_g : g \in G\}$ is weak operator dense in $LG$, and that

$$L_\xi + L_\eta = L_{\xi + \eta}, \quad aL_\xi = L_{a\xi}, \quad L_\xi L_\eta = L_{\xi \eta}, \quad L^*_\xi = L_{\xi^*} \quad \text{and} \quad L_{\delta_g} = \lambda_g$$

if $\xi, \eta \in \mathcal{H}$ is such that $L_\xi, L_\eta \in LG$. Here the element $\xi^* \in \mathcal{H}$ is defined by $\xi^*(g) := \xi(g^{-1})$ as in $\ell^1(G)$. A handy thing to know is that for $x \in LG$ and $\xi \in \ell^2(G)$ we have

$$x = L_\xi \iff x \delta_e = \xi,$$

which also implies that $L_\xi = L_\eta \iff \xi = \eta$. The linear functional on $LG$ given by $x \mapsto (x \delta_e | \delta_e) = x(e)$ is a faithful normal trace of norm 1, which we will refer to as the standard trace on $LG$. If $H \subseteq G$ is a subgroup of $G$,
then we can identify $LH$ with the von Neumann subalgebra of $LG$ consisting
of all those $L_\xi \in LG$ for which $\xi(g) = 0$ when $g \notin H$ in a natural trace
preserving way. More explicitly, the inclusion of $LH$ into $LG$ is given by the
map $L_\xi \mapsto L_{\xi'}$ where $\xi' : G \to \mathbb{C}$ is the map given by $\xi'(g) = \xi(g)$ when
$g \in H$ and $\xi'(g) = 0$ elsewhere. It is straightforward to check that this map
is an injective trace preserving $*$-homomorphism. In fact it is ultraweakly
continuous and the image is a von Neumann algebra which is generated by
the set $\{\delta_h : h \in H\}$.

Before ending this brief introduction we will consider when elements of
$LG$ commute with each other. An element $L_\xi \in LG \cap (LH)'$ is an element
in $LG$ that commutes with $LH$ when $LH$ is considered a subalgebra of $LG.$
Since $\{\lambda_h : h \in H\}$ generates $LH$ this happens if and only if $\lambda_h L_\xi = L_\xi \lambda_h$
for all $h \in H.$ In other words $\xi \in LG \cap (LH)'$ if and only if $\delta_h \ast \xi \ast \delta_{h^{-1}} = \xi$
for all $h \in H,$ but since $(\delta_h \ast \xi \ast \delta_{h^{-1}})(g) = \xi(h^{-1}gh)$ for all $g \in G$ and
$h \in H,$ this is equivalent to saying that $\xi$ is constant on sets of the form
$\{gh^{-1}h : h \in H\}$ with $g \in G.$ In particular when $H = G$ we see that
the center of $LG$ is the set of elements $L_\xi \in LG$ such that $\xi$ is constant on
conjugacy classes. Thus $LG$ is a factor if and only if $G$ is i.c.c., where i.c.c.
means that every non-trivial element in $G$ has infinite conjugacy class. In
fact $LG$ is a factor of type $\text{II}_1$ when $G$ is i.c.c..

3 Property Gamma implies inner amenability

As one might guess from the title, this section serves the purpose of proving
that a group whose group von Neumann algebra has property $\Gamma$ is inner
amenable. This is the result Effros proved in [Eff], and it is proved in The-
orem 3.6 and Theorem 3.7. The latter contains the main part of the proof
and the former is an application of Theorem 3.7 to the case we are interested
in.

3.1 Amenability, inner amenability and property Gamma

Before proving the main result of this section, it is relevant to know what it
means for a group to be inner amenable and for a factor of type $\text{II}_1$ to have
property $\Gamma.$ Besides this we also define amenability and discuss some of the
relations between amenability and inner amenability.

Definition 3.1. A mean on a discrete group $G$ is a state $m$ on $\ell^\infty(G),$ i.e.
an element $m \in \ell^\infty(G)^*$ with $m \geq 0$ and $m(1) = 1.$ If the mean $m$ satisfies
$$
\langle m, \lambda_g f \rangle = \langle m, f \rangle \quad \text{and} \quad \langle m, \rho_g f \rangle = \langle m, f \rangle
$$
for all $g \in G$ and $f \in \ell^\infty(G),$ then it is called \textit{left invariant} and \textit{right
invariant} respectively. A mean $m$ that satisfies
$$
\langle m, \pi_g f \rangle = \langle m, f \rangle
$$
for all \( g \in G \) and \( f \in \ell^\infty(G) \) is called inner invariant, and it is called non-trivial, if \( \langle m, \delta_e \rangle \neq 1 \).

**Remark 3.2.** It is not hard to show that having a left invariant mean is equivalent to having a right invariant mean. In fact both conditions are equivalent to having a mean that is left and right invariant, and with a little abuse of notation we can point to concrete constructions. If \( G \) is a discrete group and \( m \) is a left (resp. right) invariant mean on \( G \), then

\[
x \mapsto \langle m, g \mapsto x(g^{-1}) \rangle
\]

will be a right (resp. left) invariant mean on \( G \). Thus having a left invariant mean is equivalent to having a right invariant mean. If \( G \) has both a left invariant mean and a right invariant mean, denoted by \( m_l \) and \( m_r \) respectively, then

\[
x \mapsto \langle m_r, g \mapsto \langle m_l, \rho_g x \rangle \rangle
\]

defines a mean that is both left and right invariant. To check that these two constructions work is a small (and fun, some would say) task. In particular this shows that if a group \( G \) has a left or right invariant mean then it has a non-trivial inner invariant mean\(^2\), namely a mean that is both left and right invariant.

**Definition 3.3.** A discrete group \( G \) is said to be amenable if it has a left invariant mean, and inner amenable if it has a non-trivial inner invariant mean.

By Remark 3.2 we know that amenability implies inner amenability\(^3\), but the converse it not true. In fact if \( G \) is inner amenable, then \( G \times H \) is inner amenable for any group \( H \). To realize this, let \( m \) be a non-trivial inner invariant mean on \( G \), and let \( e \) denote the neutral element in \( H \). Then the map (again with some abuse of notation)

\[
f \mapsto \langle m, g \mapsto f(g, e) \rangle, \quad \ell^\infty(G \times H) \to \mathbb{C}
\]

will be a non-trivial inner invariant mean on \( G \times H \). Now if \( G \) is amenable and \( H \) is not, then \( G \times H \) is inner amenable but not amenable. An example of this could be \( \Pi \times F_2 \), where \( \Pi \) is the group of permutations of \( \mathbb{N} \) that leaves all but finitely many elements fixed and \( F_2 \) is the free group on two generators.

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\(^2\)This is provided that \( G \) is not the trivial group, for if this is the case then all means are trivial.

\(^3\)Again provided that the group is non-trivial.
Remark 3.4. At first it might seem a little strange to require that $\langle m, \delta_e \rangle \neq 1$, but there is a perfectly good reason for this requirement. The good reason is that without it, every group $G$ would be inner amenable, since

$$m_e : f \mapsto f(e), \quad \ell^\infty(G) \longrightarrow \mathbb{C}$$

always defines an inner invariant mean, and the requirement $\langle m, \delta_e \rangle \neq 1$ is equivalent to saying that $m \neq m_e$. Because of this requirement we can always assume that the mean $m$ satisfies $\langle m, \delta_e \rangle = 0$, for if $m'$ is any non-trivial inner invariant mean on $G$, then

$$m = \frac{m' - \langle m', \delta_e \rangle m_e}{1 - \langle m', \delta_e \rangle}$$

defines an inner invariant mean on $G$ with $\langle m, \delta_e \rangle = 0$.

Before moving on to proving the main theorem we need to introduce Property $\Gamma$. As mentioned in the introduction this was introduced in [MvN] by Murray and von Neumann. Recall from Section 2 that we denote by $\| \cdot \|_\tau$ the pre-Hilbert norm on a von Neumann algebra $\mathcal{M}$ with normalized trace $\tau$ defined by

$$\| x \|_\tau = \tau(x^*x)^{1/2}.$$

Definition 3.5. Let $\mathcal{M}$ be a factor of type $\text{II}_1$ with a normalized trace $\tau$, then $\mathcal{M}$ is said to have property $\Gamma$ if for each $x_1, \ldots, x_n \in \mathcal{M}$ and $\varepsilon > 0$ there exists a unitary $u \in \mathcal{M}$ with the properties that $\tau(u) = 0$ and

$$\|ux_j - x_ju\|_\tau < \varepsilon \quad \text{for all } j = 1, \ldots, n.$$

This definition deserves a comment for it is not a priori clear that it does not depend on the choice of trace. It does not, and what saves the day is that $\mathcal{M}$ is required to be a $\text{II}_1$ factor. It is a non-trivial result that there always is a trace on a $\text{II}_1$ factor and that this trace is unique up to normalization, and because of this there is actually no choice involved. A reference for this could be [KR2, Proposition 8.5.3].

Uniqueness of the normalized trace on a $\text{II}_1$ factor also has the effect that property $\Gamma$ only depends on the algebraic structure of the von Neumann algebra. For if $\mathcal{N}$ is another $\text{II}_1$ factor and $\varphi$ is a $*$-isomorphism of $\mathcal{N}$ onto $\mathcal{M}$, then $\tau \circ \varphi$ will be the unique normalized trace on $\mathcal{N}$ and property $\Gamma$ of $\mathcal{M}$ will naturally imply property $\Gamma$ for $\mathcal{N}$.

3.2 Proving the theorem

We start by proving the following theorem, which gives a sufficient condition for a group to be inner amenable.

**Theorem 3.6.** Suppose that $G$ is a discrete group with neutral element $e$. If there exists a net $(\xi_\alpha)_{\alpha \in A}$ of unit vectors in $\ell^2(G)$ such that

$$\lim_{\alpha \in A} \xi_\alpha(e) = 0 \quad \text{and} \quad \lim_{\alpha \in A} \| \pi_g \xi_\alpha - \xi_\alpha \|_2 = 0$$

for all $g \in G$, then $G$ is inner amenable.
Proof. Suppose that \((\xi_\alpha)_{\alpha \in A}\) is a net in \(l^2(G)\) satisfying the requirements of the theorem. If for each \(\alpha \in A\) we define \(\eta_\alpha : G \to \mathbb{C}\) by

\[
\eta_\alpha(h) = |\xi_\alpha(h)|^2 \quad (h \in G)
\]

then \(\eta_\alpha \in l^1(G)\) since \(\xi_\alpha \in l^2(G)\) and it is readily seen that \(\eta_\alpha \geq 0\), \(\|\eta_\alpha\|_1 = 1\) and \(\lim_{\alpha \in A} \eta_\alpha(e) = 0\) by our assumptions on \(\xi_\alpha\). Suppose that \(g \in G\), then for each \(h \in G\) we get, using the triangle inequality, that

\[
|\pi_g \eta_\alpha(h) - \eta_\alpha(h)| = |\|\pi_g \xi_\alpha(h)\|^2 - |\xi_\alpha(h)|^2|
\]

\[
= |\|\pi_g \xi_\alpha(h)\| - |\xi_\alpha(h)||(\|\pi_g \xi_\alpha(h)\| + |\xi_\alpha(h)|)|
\]

\[
\leq |\pi_g \xi_\alpha(h) - \xi_\alpha(h)|(\|\pi_g \xi_\alpha(h)\| + |\xi_\alpha(h)|),
\]

so using the Cauchy-Schwarz Inequality we see that

\[
\|\pi_g \eta_\alpha - \eta_\alpha\|_1 \leq \sum_{h \in G} |\pi_g \xi_\alpha(h) - \xi_\alpha(h)|(\|\pi_g \xi_\alpha(h)\| + |\xi_\alpha(h)|)
\]

\[
\leq \|\pi_g \xi_\alpha - \xi_\alpha\|_2 \|\pi_g \xi'_\alpha + \xi'_\alpha\|_2,
\]

where \(\xi'_\alpha \in l^2(G)\) is defined by \(\xi'_\alpha(g) = |\xi_\alpha(g)|\) for all \(g \in G\). Clearly \(\|\pi_g \xi'_\alpha\|_2 = \|\xi'_\alpha\| = \|\xi_\alpha\| = 1\), so using the triangle inequality again we obtain

\[
\|\pi_g \eta_\alpha - \eta_\alpha\|_1 \leq \|\pi_g \xi_\alpha - \xi_\alpha\|_2 \|\pi_g \xi'_\alpha + \xi'_\alpha\|_2
\]

\[
\leq 2\|\pi_g \xi_\alpha - \xi_\alpha\|_2.
\]

Thus \(\lim_{\alpha \in A} \|\pi_g \eta_\alpha - \eta_\alpha\|_1 = 0\) since \(\lim_{\alpha \in A} \|\pi_g \xi_\alpha - \xi_\alpha\|_2 = 0\) by assumption. Now let us summarize what we have proved so far. If we consider \((\eta_\alpha)_{\alpha \in A}\) as a net in the unit ball of \(l^\infty(G)^*\), then \((\eta_\alpha)_{\alpha \in A}\) is a net of positive linear functionals of norm 1 satisfying

\[
\lim_{\alpha \in A} \langle \eta_\alpha, \delta_g \rangle = 0 \quad \text{and} \quad \lim_{\alpha \in A} \|\pi_g \eta_\alpha - \eta_\alpha\|_1 = 0
\]

for all \(g \in G\). By Alaoglu’s Theorem [Dou, Theorem 1.23] the unit ball of \(l^\infty(G)^*\) is weak*-compact and thus the net has a weak*-convergent subnet \((\eta_{\alpha'})_{\alpha' \in A'}\). If \(m\) denotes the limit of this subnet in \(l^\infty(G)^*\), then \(m\) is our candidate for a non-trivial inner invariant mean on \(G\). Let is first check that \(m\) is a state on \(l^\infty(G)\). Suppose that \(f \in l^\infty(G)\) with \(f \geq 0\). Then

\[
\langle m, f \rangle = \lim_{\alpha' \in A'} \langle \eta_{\alpha'}, f \rangle \geq 0,
\]

since \(\eta_{\alpha'} \geq 0\) for all \(\alpha' \in A'\). This shows that \(m\) is positive, and in the same manner we see that

\[
\langle m, 1 \rangle = \lim_{\alpha' \in A'} \langle \eta_{\alpha'}, 1 \rangle = 1,
\]

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since \( \langle \eta_{\alpha'}, 1 \rangle = 1 \) for all \( \alpha' \in A' \) because \( \eta_{\alpha'} \) is a state on \( \ell^\infty(G) \). So \( m \) is indeed a state on \( \ell^\infty(G) \). Let us check for inner invariance. For all \( g \in G \) and \( f \in \ell^\infty(G) \) we see by a change of variable that

\[
\langle \eta_{\alpha'}, \pi_g f \rangle = \sum_{h \in G} (\pi_g f)(h)\eta_{\alpha'}(h) = \sum_{h' \in G} f(h')(\pi_g^{-1}\eta_{\alpha'})(h') = \langle \pi_g^{-1}\eta_{\alpha'}, f \rangle
\]

for all \( \alpha' \in A' \), so in particular

\[
\langle \eta_{\alpha'}, \pi_g f - f \rangle = \langle \pi_g^{-1}\eta_{\alpha'} - \eta_{\alpha'}, f \rangle \leq \|\pi_g^{-1}\eta_{\alpha'} - \eta_{\alpha'}\|_1 \|f\|_\infty.
\]

Since \( \lim_{\alpha \in A} \pi_{g^{-1}}\eta_{\alpha} - \eta_{\alpha} = 0 \), this must also hold for the subnet \( (\eta_{\alpha'})_{\alpha' \in A'} \), i.e. \( \lim_{\alpha' \in A'} \|\pi_{g^{-1}}\eta_{\alpha'} - \eta_{\alpha'}\| = 0 \), so from the display above we conclude that

\[
\langle m, \pi_g f - f \rangle = \lim_{\alpha' \in A'} \langle \eta_{\alpha'}, \pi_g f - f \rangle = 0,
\]

which exactly shows that \( m \) is inner invariant. Now we only need to show that \( m \) is a non-trivial mean, but this follows from the calculation

\[
\langle m, \delta_e \rangle = \lim_{\alpha' \in A'} \langle \eta_{\alpha'}, \delta_e \rangle = \lim_{\alpha \in A} \langle m, \delta_e \rangle = 0.
\]

Thus we have shown that \( G \) is inner amenable since it has the non-trivial inner invariant mean \( m \).

The converse implication of Theorem 3.6 actually also holds, i.e. if \( G \) is inner amenable, then there exist net satisfying the conditions in the theorem. Alas the proof of this takes up too much space and we do not really need it, so it is omitted. The proof can be found in [Eff].

**Theorem 3.7.** If \( G \) is a discrete ICC group so that \( LG \) has property \( \Gamma \), then \( G \) is inner amenable.

**Proof.** Let \( \tau \) denote the standard trace on the \( \Pi_1 \) factor \( LG \). Let \( \mathcal{F} \) denote the set of finite subsets of \( G \) considered as a directed set with inclusion, and for each \( \alpha \in \mathcal{F} \) choose a unitary \( u_\alpha \in LG \) such that \( \tau(u_\alpha) = 0 \) and

\[
\|u_\alpha \lambda_g - \lambda_g u_\alpha\|_\tau < \frac{1}{|\alpha|} \quad \text{for} \quad g \in \alpha.
\]

This is possible since \( LG \) has property \( \Gamma \) by assumption. Now if we choose \( \xi_\alpha \in \ell^2(G) \) such that \( L_{\xi_\alpha} = u_\alpha \) for all \( \alpha \in \mathcal{F} \), then the net \( (\xi_\alpha)_{\alpha \in \mathcal{F}} \) is our candidate for a net satisfying the condition of Theorem 3.6. Since \( L_{\xi_\alpha^*\xi_\alpha} = u_\alpha^*u_\alpha = \lambda_e \), implies that \( \xi_\alpha^* \xi_\alpha = \delta_e \), we get and so we get

\[
\|\xi_\alpha\|^2 = \sum_{g \in G} |\xi_\alpha(g)|^2 = \sum_{g \in G} \xi_\alpha^*(g^{-1})\xi_\alpha(g) = \xi_\alpha^* \xi_\alpha(e) = 1
\]

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for all $\alpha \in F$, which shows that $(\xi_{\alpha})_{\alpha \in F}$ is a sequence of unit vectors in $\ell^2(G)$. We also see that $\xi_{\alpha}(e) = \tau(u_{\alpha}) = 0$ for all $\alpha \in F$ so indeed the net satisfies the first requirements. If we note that
\[
\|ux\|_\tau = \|xu\|_\tau = \|x\|_\tau = \|x\delta e\|_2
\]
whenever $u, x \in LG$ with $u$ unitary, then it follows that
\[
\|u_\alpha \lambda_g - \lambda_g u_\alpha\|_\tau = \|u_\alpha - \lambda_g u_\alpha \lambda_g^{-1}\|_\tau = \|(u_\alpha - \lambda_g u_\alpha \lambda_g^{-1})\delta e\|_2
\]
for all $g \in \alpha$ and $\alpha \in F$. Since $\lambda_g x \lambda_g^{-1} \delta e = \pi_g x \delta e$ for all $x \in LG$ and $g \in G$, we get
\[
\|u_\alpha \lambda_g u_\alpha^* - \lambda_g\|_\tau = \|u_\alpha \delta e - \pi_g u_\alpha \delta e\|_2 = \|\xi_{\alpha} - \pi_g \xi_{\alpha}\|_2
\]
for all $g \in \alpha$ and $\alpha \in F$. From this it follows that
\[
\|\pi_g \xi_{\alpha} - \xi_{\alpha}\| < \frac{1}{|\alpha|} \quad \text{for all } g \in \alpha
\]
whenever $\alpha \in F$, and so we conclude that $\lim_{\alpha \in A} \|\pi_g \xi_{\alpha} - \xi_{\alpha}\|_2 = 0$, which proves that the net $(\xi_{\alpha})_{\alpha \in F}$ satisfies the requirement of Theorem 3.6. Hence $G$ is inner amenable.

4 Ways of constructing groups

In Section 6 we will need to construct an inner amenable group whose group von Neumann algebra does not have property $\Gamma$. There are a lot of non-pleasant groups, but often those that you can think off are those that behave nicely in most ways. In this section three methods of constructing groups from other groups is presented: the semidirect product of groups; the amalgamated free product of groups; and the inductive limit of group (in a special case).

4.1 The semidirect product

The semidirect product of groups is a generalization of the direct product of groups. If $G$ and $H$ are both groups, then an action of $G$ on $H$ by automorphisms is a homomorphisms $\theta : G \to \text{Aut}(H)$. In this situation $\theta(g)$ is often denoted by $\theta_g$, and sometimes when there is no doubt about which action is considered, $\theta$ is entirely suppressed, and $\theta_g(h)$ is written $g \cdot h$.

If $\theta$ is such an action of $G$ on $H$, then the semidirect product of $G$ and $H$ with respect to $\theta$, consists of the usual Cartesian product $H \times G$ together with the composition defined by the rule
\[
(h, g)(h', g') = (h \theta_g(h'), gg')
\]
for all \( h, h' \in H \) and \( g, g' \in G \). The semidirect product of \( G \) and \( H \) with respect to \( \theta \) is denoted \( H \rtimes \theta G \). Often when there is no doubt about which actions the semidirect product is made with respect to, it is denoted \( H \rtimes G \) instead of \( H \rtimes \theta G \). The semidirect product of groups is indeed a generalization of the direct product, since the ordinary direct product of groups is just the case where \( \theta \) is the trivial homomorphism, i.e. \( \theta(g) = \text{id}_H \) for all \( g \in G \).

The semidirect product has some nice properties. First of all, if \( e_G \) and \( e_H \) denote the neutral elements of \( G \) and \( H \) respectively, then we can identify \( G \) and \( H \) with the subgroups \( \{ e_H \} \times G \) and \( H \times \{ e_G \} \) respectively via the maps

\[
g \mapsto (e_H, g) \quad \text{and} \quad h \mapsto (h, e_G).
\]

Moreover, this identification has the property that \( H \) sits as a normal subgroup of \( H \rtimes \theta G \) in such a way that the action of \( G \) on \( H \) via \( \theta \) is just conjugation inside \( H \rtimes \theta G \) in the sense that

\[
\theta_g(h) = ghg^{-1}
\]

for all \( g \in G \) and \( h \in H \). It is also worth to note that as in the direct product every element in \( H \rtimes \theta G \) can be written uniquely as a product of an element of \( H \) and an element of \( G \). More precisely, every element has the form \( hg \) with \( h \in H \) and \( g \in G \), since \( hg = (h, g) \) when the multiplication is carried out in \( H \rtimes \theta G \).

### 4.2 Amalgamated free products

Let us start by recalling what the free product of groups is. If \( G \) and \( H \) are groups with neutral elements \( e_G \) and \( e_H \) respectively, then the \textit{free product} of \( G \) and \( H \), denoted \( G \ast H \), is the set of reduced words in the alphabet \((G \setminus \{ e_G \}) \cup (H \setminus \{ e_H \})\),\(^4\) composition given as juxtaposition followed by reduction, and neutral element given by the empty word. A word \( k_1k_2 \cdots k_n \) is called reduced, if for each \( i = 1, 2, \ldots, n - 1 \), it is not the case that \( k_i \) and \( k_{i+1} \) both belong to \( G \) or both belongs to \( H \). The juxtaposition of two reduced words \( k_1k_2 \cdots k_n \) and \( k'_1k'_2 \cdots k'_m \) is the word \( k_1 \cdots k_nk'_1 \cdots k'_m \), and the reduction of such a word is a recursive process that goes as follows:

\[\triangleright\text{ if } k_1 \cdots k_nk'_1 \cdots k'_m \text{ is reduced, then the process stops and output is the word } k_1 \cdots k_nk'_1 \cdots k'_m;\]

\[\triangleright\text{ if the word } k_1 \cdots k_nk'_1 \cdots k'_m \text{ is not reduced, i.e. if either } k_n \text{ and } k'_1 \text{ both belongs to } G \text{ or both belongs to } H, \text{ and if } k_n \neq (k'_1)^{-1} \text{ then we let } k := k_nk'_1 \text{ (where the composition is carried out) and the output is the new reduced word } k_1 \cdots k'_{n-1}kk'_2 \cdots k'_m;\]

\[\text{ and}\]

\(^4\text{It is important that the sets } G \setminus \{ e_G \} \text{ and } H \setminus \{ e_H \} \text{ are considered as disjoint sets.}\]
if \( k_1 \cdots k_n k'_1 \cdots k'_m \) is not reduced and \( k'_1 = k_n^{-1} \), then we consider the word \( k_1 \cdots k_{n-1} k'_2 \cdots k'_m \) instead, and apply the same procedure as we did to the word \( k_1 \cdots k_n k'_1 \cdots k'_m \).

Eventually we will end up with either a reduced word or the empty word, and this new word is then the composition of the two.

We are now ready to define the amalgamated free product. If \( G \) and \( H \) has a common subgroup \( K \), and \( i \) and \( j \) denote the inclusion of \( K \) into \( G \) and \( H \) respectively, then we define the amalgamated free product of \( G \) and \( H \) with the common subgroup \( K \) to be the group \( G * H \) with the relations \( i(k) = j(k) \) for all \( k \in K \). In other words it is the group \((G * H)/\bar{K}\), where \( \bar{K} \) denotes the smallest normal subgroup of \( G * H \) containing \( \{i(k)j(k)^{-1} : k \in K\} \). We denote this amalgamated free product by \( G *_K H \).

Above we required \( K \) to be a common subgroup of \( G \) and \( H \), but we allow taking the amalgamated free product of \( G \) and \( H \) when they just contain an isomorphic image of \( K \). In this situation the definition is the same as above with the twist that the maps \( i \) and \( j \) need not be inclusions, but only injective homomorphisms. We will think of the free product with amalgamation as reduced words (like in the free product) but where we are allowed to carry out the multiplication between two elements if one of them is in \( K \).

The free product of groups is a very non-commutative thing, in the sense that even if both groups are abelian, then the center of the free product will be trivial. In the case of amalgamated free products there are a small possibility of having a non-trivial center, in fact it is not hard to see that the center is given by \( \mathcal{Z}(G) \cap K \cap \mathcal{Z}(H) \), where \( \mathcal{Z}(G) \) and \( \mathcal{Z}(H) \) denotes the centers of \( G \) and \( H \) respectively. Clearly all elements not in \( K \) must necessarily have infinite conjugacy classes, and so we only need to check this requirement for elements in \( K \), to check if the group is i.c.c.. Another property we will use about the amalgamated free product is the following. If we are in the situation above, and \( K' \subseteq K \) is a normal subgroup of \( G \) and a normal subgroup of \( H \) (or the isomorphic image is), then \( K' \) will be a normal subgroup of the amalgamated free product \( G *_K H \).

### 4.3 Inductive limits of groups

Inductive systems is a categorical concept that is defined in the categorical language via objects, morphisms and universal properties. In this project there will not be much use of categories and we will put more emphasis on the actual construction of the inductive limit than the universal property. In fact we will entirely omit the universal property and take the group constructed as a definition of the inductive limit. Besides this we will also restrict to a special case of inductive systems, namely the linear ones.

Suppose that \( G_1, G_2, \ldots \) is a sequence of groups together with group
homomorphisms $i_n : G_n \to G_{n+1}$. Then we have the situation

$$G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} \cdots \xrightarrow{i_k} G_{k+1} \xrightarrow{i_{k+1}} \cdots,$$

which is called an inductive system of groups. Let $n \in \mathbb{N}$ and let $i_n^k$ denote the map $i_{k-1} \circ i_{k-2} \circ \cdots \circ i_n : G_n \to G_k$ when $n < k$ and $i_n^n$ the identity on $G_n$. Clearly

$$i_k^l \circ i_n^k = i_n^l \quad (4.1)$$

for all $n, k, l \in \mathbb{N}$ with $n \leq k \leq l$. Let us define a relation $\sim$ on the disjoint union $\bigsqcup_{n=1}^{\infty} G_n$ of the $G_n$’s as follows. For any $g_n \in G_n$ and $g_m \in G_m$ ($n, m \in \mathbb{N}$) we say that

$$g_n \sim g_m \iff i^k_n(g_n) = i^k_m(g_m) \quad \text{for some } k \geq n, m.$$

Because of the identity (4.1) this turns out to be an equivalence relation. If $G$ denotes the set of equivalence classes of $\sim$, then there is a natural choice of composition making $G$ into a group. This composition is given by the following rule. If $g_n \in G_n$ and $g_m \in G_m$ ($n, m \in \mathbb{N}$) are any two representatives (of some equivalence classes), then

$$[g_n] \cdot [g_m] = \left[ i^k_n(g_n) \cdot i^k_m(g_m) \right]$$

for some $k \geq n, m$. It is not hard to check that this is a well-defined composition and we call $G$ the inductive limit of this inductive system of groups. For all $n \in \mathbb{N}$ there is a natural map $j_n : G_n \to G$ given by $j_n(g) = [g]$. It is straightforward to check that all the maps $j_n$ ($n \in \mathbb{N}$) are homomorphisms. Moreover, if all the maps $i_n$ ($n \in \mathbb{N}$) are injective, then all the maps $j_n$ ($n \in \mathbb{N}$) are injective. In this case we identify $G_n$ with its image under $j_n$.

## 5 Tensor products

This section serves as a short review on the finite and the infinite tensor product of von Neumann algebras, and naturally we will also discuss the finite and infinite tensor product of Hilbert spaces in the process.

### 5.1 Finite tensor products

We start by recalling how the setup is in the case of finite tensor products of Hilbert spaces and finite tensor products of von Neumann algebras.

**Definition 5.1.** For Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, with $(\cdot \mid \cdot)_i$ denoting the inner product of $\mathcal{H}_i$, we can consider the algebraic tensor product

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$$
as a pre-Hilbert space with the inner product defined on basic tensors by

\[ (\xi_1 \otimes \cdots \otimes \xi_n | \eta_1 \otimes \cdots \otimes \eta_n) = (\xi_1 | \eta_1)_1 \cdots (\xi_n | \eta_n)_n. \]

We define the tensor product of the Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_n \) to be the completion of this pre-Hilbert space with respect to this inner product. We denote this tensor product by \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \) or sometimes \( \bigotimes_{k=1}^n \mathcal{H}_k \).

As always the basic tensors are easier to work with, and it helps their span constitute an everywhere dense subset of the tensor product \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \). It is straightforward to check that if all the Hilbert spaces \( \mathcal{H}_i \) (\( i = 1, \ldots, n \)) are equipped with an orthonormal basis \( \{ e^i_j : j \in I_i \} \), then there is a canonical choice of orthonormal basis for the tensor product, namely the set

\[ \{ e^1_{j_1} \otimes \cdots \otimes e^n_{j_n} : j_i \in I_i \ (i = 1, \ldots, n) \}. \]

It follows directly from this that

\[ \dim(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) = \dim(\mathcal{H}_1) \cdots \dim(\mathcal{H}_n). \]

Just for now we denote the elements in this orthonormal basis \( f^1_j \otimes \cdots \otimes f^n_j \) when \( j \in I := I_1 \times \cdots \times I_n \) in the sense that \( f^k_j = e^k_{j_k} \) if \( j = (j_1, \ldots, j_n) \).

Suppose that we are given \( x_i \in B(\mathcal{H}_i) \ (i = 1, \ldots, n) \), then we can define a linear operator \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \to \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \), by specifying how it acts on the orthonormal basis as follows:

\[ f^1_j \otimes \cdots \otimes f^n_j \mapsto (x_1 f^1_j) \otimes \cdots \otimes (x_n f^n_j) \]

for all \( j \in I \). This is clearly a linear operator, and we will denote it by \( x_1 \otimes \cdots \otimes x_n \). We shall see that this is a bounded operator on the tensor product. It is easy to see that this tensor product of the \( x_i \)'s is linear in each variable (i.e. in each \( x_i \)). If we are also given \( y_i \in B(\mathcal{H}_i) \ (i = 1, \ldots, n) \), then by construction

\[ (x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_n) = x_1 y_1 \otimes \cdots \otimes x_n y_n, \]

so to show that the operator \( x_1 \otimes \cdots \otimes x_n \) is bounded, it suffices to show that the operator \( \hat{x}_i := z_1 \otimes \cdots \otimes z_n \) where \( z_k = 1_{\mathcal{H}_k} \) when \( k \neq i \) and \( z_i = x_i \) is bounded. We can without loss of generality assume that \( i = 1 \). First note that all elements in the algebraic tensor product can be written of the form

\[ \sum_{l=1}^k \xi_l \otimes e^2_{j_1} \otimes \cdots \otimes e^n_{j_n}, \]

where \( (j_1^1, \ldots, j_n^1) \in I_2 \times \cdots \times I_n \) are different for all \( l = 1, \ldots, k \) and \( \xi_l \in \mathcal{H}_1 \).

This can be done by gathering terms in the expansion in the orthonormal
basis. Now with this condition the elements $x_1 \xi \otimes e_{j_1}^2 \otimes \cdots \otimes e_{j_n}^n$ are all orthogonal, and thus

$$\left\| \hat{x}_1 \sum_{i=1}^k \xi_i \otimes e_{j_1}^2 \otimes \cdots \otimes e_{j_n}^n \right\| = \left\| x_1 \xi_1 \otimes e_{j_1}^2 \otimes \cdots \otimes e_{j_n}^n \right\|^2 = \sum_{i=1}^k \| x_1 \xi_i \otimes e_{j_1}^2 \otimes \cdots \otimes e_{j_n}^n \|^2.$$

This shows that $\hat{x}_1$ is bounded by $\| x_1 \|$ on the algebraic tensor product, and therefore also on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. From this we see that $x_1 \otimes \cdots \otimes x_n$ is bounded by $\| x_1 \| \cdots \| x_n \|$ by writing

$$x_1 \otimes \cdots \otimes x_n = \hat{x}_1 \cdots \hat{x}_n.$$

This operator is called the tensor product of the $x_i$'s, and we will show that its norm is actually $\| x_1 \| \cdots \| x_n \|$. For each $i = 1, \ldots, n$ choose a sequence $(\xi_m^i)_{m \in \mathbb{N}}$ of unit vectors in $\mathcal{H}_i$ so that $\| x_i \xi_m^i \| \rightarrow \| x_i \|$ as $m \rightarrow \infty$. Since

$$\left\| (x_1 \otimes \cdots \otimes x_n) \xi_m^1 \otimes \cdots \otimes \xi_m^n \right\| = \left\| x_1 \xi_m^1 \right\| \cdots \left\| x_n \xi_m^n \right\|,$$

and $\xi_m^1 \otimes \cdots \otimes \xi_m^n$ is a unit vector for all $m \in \mathbb{N}$, the fact that

$$\left\| (x_1 \otimes \cdots \otimes x_n) \xi_m^1 \otimes \cdots \otimes \xi_m^n \right\| \rightarrow \left\| x_1 \right\| \cdots \left\| x_n \right\|$$

as $m \rightarrow \infty$, proves $\| x_1 \otimes \cdots \otimes x_n \| \geq \| x_1 \| \cdots \| x_n \|$. From this we conclude that $\| x_1 \otimes \cdots \otimes x_n \| = \| x_1 \| \cdots \| x_n \|$, since we already know the other inequality to be true. Tensor products also behave nicely with respect to adjoints, in fact

$$(x_1 \otimes \cdots \otimes x_n)^* = x_1^* \otimes \cdots \otimes x_n^*$$

which is straightforward to check. Now we are ready to define the tensor product of von Neumann algebras.

**Definition 5.2.** Suppose that $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are von Neumann algebras acting on the Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$. Then their tensor product is the von Neumann subalgebra of $B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ generated by the set

$$\{ x_1 \otimes \cdots \otimes x_n : x_i \in \mathcal{M}_i \ (i = 1, \ldots, n) \}.$$ 

We denote this von Neumann algebra by $\mathcal{M}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{M}_n$.

Let $\mathcal{M}_1 \odot \cdots \odot \mathcal{M}_n$ denote the span of the generating set above. Clearly this is a selfadjoint subalgebra, and the tensor product of the von Neumann algebras is just the weak operator closure of this algebra.
There is a natural way of identifying \( \mathcal{M}_i \) \((i = 1, \ldots, n)\) with a von Neumann subalgebra of \( \mathcal{M}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{M}_n \), namely by the map \( x \mapsto \hat{x} \). \( \mathcal{M}_i \rightarrow \mathcal{M}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{M}_n \) (with the \( \hat{x} \) notation from page 18). This is clearly a injective \(*\)-homomorphism, and it is not hard to realize that it is a homeomorphism with respect to the weak operator topology on \( \mathcal{M}_i \) and \( \mathcal{M}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{M}_n \).

The following result relates the direct sum of groups to the tensor product of their associated group von Neumann algebras.

**Proposition 5.3.** If \( n \in \mathbb{N} \) and \( G_1, \ldots, G_n \) are groups, then the von Neumann algebras \( L(G_1 \oplus \cdots \oplus G_n) \) and \( L(G_1 \overline{\otimes} \cdots \overline{\otimes} G_n) \) are unitarily equivalent. More precisely there is a unitary

\[
U : \ell^2(G_1 \oplus \cdots \oplus G_n) \rightarrow \ell^2(G_1) \otimes \cdots \otimes \ell^2(G_n)
\]

with \( U(L(G_1 \oplus \cdots \oplus G_n)U^*) = L\overline{G_1} \overline{\otimes} \cdots \overline{\otimes} L\overline{G_n} \) satisfying \( U\lambda_{g_1, \ldots, g_n}U^* = \lambda_{g_1} \otimes \cdots \otimes \lambda_{g_n} \) when \( g_i \in G_i \) \((i = 1, \ldots, n)\).

**Proof.** We start by constructing the unitary \( U \). We already know that

\[
\{ \delta_{g_1, \ldots, g_n} : g_i \in G_i(i = 1, \ldots, n) \}
\]

is an orthonormal basis for \( \ell^2(G_1 \oplus \cdots \oplus G_n) \) and that

\[
\{ \delta_{g_1} \otimes \cdots \otimes \delta_{g_n} : g_i \in G_i(i = 1, \ldots, n) \}
\]

is an orthonormal basis for \( \ell^2(G_1) \otimes \cdots \otimes \ell^2(G_n) \). We may therefore define the unitary operator \( U \) by \( U\delta_{g_1, \ldots, g_n} = \delta_{g_1} \otimes \cdots \otimes \delta_{g_n} \) for \( g_i \in G_i \) \((i = 1, \ldots, n)\) and extend by linearity. Suppose that \( g_i, h_i \in G_i \) \((i = 1, \ldots, n)\), then

\[
U\lambda_{g_1, \ldots, g_n}U^*\delta_{h_1} \otimes \cdots \otimes \delta_{h_n} = U\lambda_{g_1, \ldots, g_n}\delta_{h_1, \ldots, h_n}
\]

\[
= U\delta_{g_1, \ldots, g_n}h_{1, \ldots, n}
\]

\[
= \delta_{g_1, h_1} \otimes \cdots \otimes \delta_{g_n, h_n}
\]

\[
= (\lambda_{g_1} \otimes \cdots \otimes \lambda_{g_n})\delta_{h_1} \otimes \cdots \otimes \delta_{h_n}
\]

which proves that \( U\lambda_{g_1, \ldots, g_n}U^* = \lambda_{g_1} \otimes \cdots \otimes \lambda_{g_n} \) since \( h_i \in G_i \) was arbitrary \((i = 1, \ldots, n)\). Now it follows that \( U(L(G_1 \oplus \cdots \oplus G_n)U^*) = L\overline{G_1} \overline{\otimes} \cdots \overline{\otimes} L\overline{G_n} \) since the map

\[
x \mapsto UxU^* , \quad L(G_1 \oplus \cdots \oplus G_n) \rightarrow L\overline{G_1} \overline{\otimes} \cdots \overline{\otimes} L\overline{G_n}
\]

is a weak operator continuous \(*\)-homomorphism. \( \square \)

**Remark 5.4.** One thing worth noting is that with the identification of Proposition 5.3 the natural inclusion of \( L\overline{G_k} \) into \( L(G_1 \oplus \cdots \oplus G_n) \) corresponds to the natural inclusion of \( L\overline{G_k} \) into \( L\overline{G_1} \overline{\otimes} \cdots \overline{\otimes} L\overline{G_n} \) for all \( n \in \mathbb{N} \).
We will need the following lemma later. It holds in more general settings than stated, but this version fits the application of it later. For one thing we do not even use the assumption that the \(G_i\)'s are groups.

**Proposition 5.5.** If \(n \in \mathbb{N}\) and \(G_1, \ldots, G_n\) are all finite groups, then the von Neumann algebras \(\ell^\infty(G_1 \oplus \cdots \oplus G_n)\) and \(\ell^\infty(G_1) \overline{\otimes} \cdots \overline{\otimes} \ell^\infty(G_n)\) acting as multiplication operators on \(\ell^2(G_1 \oplus \cdots \oplus G_n)\) and \(\ell^2(G_1) \otimes \cdots \otimes \ell^2(G_n)\) respectively, are unitarily equivalent. More precisely, the unitary \(U\) from Proposition 5.3 satisfies

\[
U \ell^\infty(G_1 \oplus \cdots \oplus G_n) U^* = \ell^\infty(G_1) \overline{\otimes} \cdots \overline{\otimes} \ell^\infty(G_n)
\]

in such a way that \(U \delta_{g_1 + \cdots + g_n} U^* = \delta_{g_1} \otimes \cdots \otimes \delta_{g_n}\), when \(g_i \in G_i\) \((i = 1, \ldots, n)\).

**Proof.** The spaces \(\ell^\infty(G_1 \oplus \cdots \oplus G_n)\) and \(\ell^\infty(G_1) \overline{\otimes} \cdots \overline{\otimes} \ell^\infty(G_n)\) has the same finite linear dimension. Furthermore, the set

\[
\{ \delta_{g_1 + \cdots + g_n} : g_i \in G_i(i = 1, \ldots, n) \}
\]

is a basis for \(\ell^\infty(G_1 \oplus \cdots \oplus G_n)\) and the set

\[
\{ \delta_{g_1} \otimes \cdots \otimes \delta_{g_n} : g_i \in G_i(i = 1, \ldots, n) \}
\]

is a basis for \(\ell^\infty(G_1) \overline{\otimes} \cdots \overline{\otimes} \ell^\infty(G_n)\). Therefore it suffices to show that \(U \delta_{g_1 + \cdots + g_n} U^* = \delta_{g_1} \otimes \cdots \otimes \delta_{g_n}\), when \(g_i \in G_i(i = 1, \ldots, n)\), which is a straightforward computation. \(\square\)

So this was the story about finite tensor products. We have avoided to mention properties of the tensor product like associativity and commutativity, but these properties naturally holds up to unitary equivalence. For more details or further properties the reader can consult [KR1, Section 2.6]. We now move on to infinite tensor products. In the infinite case one has to be a lot more careful than in the finite case, since there is going to be all sorts of restrictions due to convergence related issues.

### 5.2 Countable tensor products

To define the infinite tensor product of Hilbert spaces, we need to work with "based" Hilbert spaces in the sense that all the Hilbert spaces will need to have some distinguished unit vector. Because of this choice, the infinite tensor products of both Hilbert spaces and von Neumann algebras will be less canonical than in the finite case, since the construction will depend on the choice of distinguished vector.

Let \(\mathcal{H}_1, \mathcal{H}_2, \ldots\) be a sequence of Hilbert spaces with distinguished unit vectors \(\xi_1, \xi_2, \ldots\) (i.e. \(\xi_i \in \mathcal{H}_i\)) and let

\[
\mathcal{H}^n = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \quad (n \in \mathbb{N}).
\]
We shall see that for \( n \in \mathbb{N} \), there is a natural way of identifying \( H^n \) with a closed subspace of \( H^{n+1} \), in such a way that the inner product on \( H^n \) is the same as the one inherited from \( H^{n+1} \). Define \( v_n : H_1 \otimes \cdots \otimes H_n \to H^{n+1} \) by

\[
\eta_1 \otimes \cdots \otimes \eta_n \mapsto \eta_1 \otimes \cdots \otimes \eta_n \otimes \xi_{n+1}
\]
on basic tensors and extend by linearity. Since \( \xi_{n+1} \) was a unit vector, the map \( v_n \) is an isometry, and we can therefore extend it to an isometry from \( H^n \) to \( H^{n+1} \), which of course preserves the inner product. With these maps we get an inductive system

\[
H^1 \hookrightarrow H^2 \hookrightarrow \cdots \hookrightarrow \cdots
\]
where the arrows are the maps \( v_n \) (\( n \in \mathbb{N} \)). From this inductive system we can take the inductive limit Hilbert space, which is constructed as follows: first we see that we have an inductive system of groups with injective group homomorphisms. We can take the inductive limit of these groups as in Section 4.3, and denote it by \( H_0 \). Like in the case of groups we let \( v^k_n \) denote the map \( v_n \circ \cdots \circ v_{k-1} : H^n \to H^k \) when \( k > n \) and \( v^n_n = 1_{H_n} \) when \( k = n \). Let us describe a natural pre-Hilbert structure on \( H_0 \). We already know how the additive structure works, so let us start with the scalar multiplication. For \( \eta \in H^n \) and \( \mu \in \mathbb{C} \) we let \( \mu[\eta] = [\mu \eta] \). This makes a well-defined scalar multiplication, since all the maps \( v^k_n \) are linear. Now the inner product of two elements \([\eta]\) and \([\eta']\) with \( \eta \in H^n \) and \( \eta' \in H^m \) for some \( n,m \in \mathbb{N} \) is defined by

\[
([\eta] \mid [\eta']) = (v^k_n(\eta) \mid v^k_m(\eta')) \quad \text{for some} \ k \geq n,m.
\]
This makes a well-defined inner product since all the maps \( v^k_n \) are isometries, and like in the case of groups we will identify \( H^n \) with its image in \( H_0 \) under the map \( \eta \mapsto [\eta] \). This identification is clearly isometric, and will often be implicit. Now if we let \( \hat{H} \) denote the completion of \( H_0 \) with respect to this new inner product, then we arrive at the following definition.

**Definition 5.6.** In the situation above \( \hat{H} \) is called the tensor product of the Hilbert spaces \( \hat{H}_1, \hat{H}_2, \ldots \) with distinguished vectors \( \xi_1, \xi_2, \ldots \). This we will write

\[
\hat{H} = \bigotimes_{n=1}^{\infty} (\hat{H}_n, \xi_n) \quad \text{or sometimes} \quad (\hat{H}, \xi) = \bigotimes_{n=1}^{\infty} (\hat{H}_n, \xi_n)
\]
to emphasize the vector \( \xi = \xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \cdots \).

Now the notation \( \xi = \xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \cdots \) needs some explanation. If \( \eta_1 \otimes \cdots \otimes \eta_n \) is some basic tensor in \( H^n \) for some \( n \in \mathbb{N} \), then the equivalence class in \( H_0 \) containing this basic tensor will be

\[
\{ \eta_1 \otimes \cdots \otimes \eta_n \otimes \xi_{n+1} \otimes \cdots \otimes \xi_{n+k} : k \in \mathbb{N} \}.
\]
Naturally we will think of these equivalence classes as “infinite basic tensors” of the form
\[ \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \cdots, \]
where \( \eta_n \neq \xi_n \) for only finitely many \( n \in \mathbb{N} \). With this notation, the element \( \xi = \xi_1 \otimes \xi_2 \otimes \xi_3 \otimes \cdots \) is just \([\xi_1]\) or even \([\xi_1 \otimes \cdots \otimes \xi_n]\) for some \( n \in \mathbb{N} \). Clearly the span of the basic tensors of this form with \( \eta_k = \xi_k \) for all \( k > n \) for some fixed \( n \in \mathbb{N} \) span a dense subset of \( \mathcal{H}_n \) as a subset of \( \mathcal{H}_o \). Since \( \mathcal{H}_o = \bigcup_{n=1}^{\infty} \mathcal{H}_n \), it follows that the span of the basic tensors described above, i.e.
\[ \bigcup_{n=1}^{\infty} \mathcal{H}_1 \circ \cdots \circ \mathcal{H}_n, \]
form an everywhere dense subset of \( \mathcal{H} \). When working with the infinite tensor product of Hilbert spaces, it will often be more convenient to work with these elements, and then use some kind of continuity argument.

Suppose that, in addition to the Hilbert spaces, we are given a sequence of von Neumann algebras \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots \), such that \( \mathcal{M}_i \) acts on \( \mathcal{H}_i \) (\( i \in \mathbb{N} \)). If we let
\[ \mathcal{M}^n = \mathcal{M}_1 \overline{\otimes} \cdots \overline{\otimes} \mathcal{M}_n \]
for all \( n \in \mathbb{N} \) then we can view \( \mathcal{M}^n \) as a von Neumann subalgebra of \( B(\mathcal{H}) \), by letting \( \mathcal{M}^n \) act on \( \mathcal{H} \) as follows. If \( x \in \mathcal{M}^n \), we write \( \mathcal{H} = \mathcal{H}_n \otimes \mathcal{H}' \) with
\[ \mathcal{H}' = \bigotimes_{k=n+1}^{\infty} (\mathcal{H}_k, \xi_k) \]
and let \( x \) act on \( \mathcal{H} \) as \( x \otimes 1_{\mathcal{H}'} \). With these identifications we get an inclusion \( \mathcal{M}^n \subseteq \mathcal{M}^{n+1} \) for all \( n \in \mathbb{N} \), and so we have an increasing sequence of von Neumann algebras
\[ \mathcal{M}^1 \subseteq \mathcal{M}^2 \subseteq \cdots \subseteq \mathcal{M}^n \subseteq \cdots \subseteq B(\mathcal{H}) \]
If we consider their union \( \bigcup_{n=1}^{\infty} \mathcal{M}^n \) in \( B(\mathcal{H}) \), then this is a selfadjoint subalgebra of \( B(\mathcal{H}) \). Let \( \mathcal{M} \) denote the weak closure of this subalgebra inside \( B(\mathcal{H}) \).

**Definition 5.7.** In the setup from above, where \( \mathcal{M}_1, \mathcal{M}_2, \ldots \) are von Neumann algebras acting on the Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2, \ldots \) with distinguished vectors \( \xi_1, \xi_2, \ldots \), we call \( \mathcal{M} \) the tensor product of the von Neumann algebras \( \mathcal{M}_1, \mathcal{M}_2, \ldots \) and denote it by \( \bigotimes_{n=1}^{\infty} \mathcal{M}_n \).

---

\(^5\)Remember the identification of \( \mathcal{H}_n \) as a subspace of \( \mathcal{H}_o \).
Since $\mathcal{M}_1 \odot \cdots \odot \mathcal{M}_n$ is weak operator dense in $\mathcal{M}^n$ for each $n \in \mathbb{N}$, the subset
\[
\mathcal{M}_o = \bigcup_{n=1}^{\infty} \mathcal{M}_1 \odot \cdots \odot \mathcal{M}_n
\]
is a weak operator dense selfadjoint subalgebra. Like in the case of Hilbert spaces this subset $\mathcal{M}_o$ is much more convenient to work with, so we will often prove something about $\mathcal{M}_o$ and then use some continuity argument. The identifications $\mathcal{M}_i \subseteq \mathcal{M}_n \subseteq \mathcal{M}$ (with $n \geq i$) will be used without mentioning, and elements of $\mathcal{M}_o$ will be denoted as elements of $\mathcal{M}_i$ or $\mathcal{M}_n$ for some $i \in \mathbb{N}$ or $n \in \mathbb{N}$.

As in the finite case we have a theorem connecting the group von Neumann algebra of a countable sum of groups to the group von Neumann algebras of the individual groups, namely by the tensor product.

**Proposition 5.8.** If $G_1, G_2, \ldots$ is a sequence of groups, then the von Neumann algebras $L(\bigoplus_{n=1}^{\infty} G_n)$ and $\bigotimes_{n=1}^{\infty} LG_n$ are unitarily equivalent. More precisely there is a unitary $U : \ell^2(\bigoplus_{n=1}^{\infty} G_n) \to \bigotimes_{n=1}^{\infty} (\ell^2(G_n), \delta_e)$ with $UL(\bigoplus_{n=1}^{\infty} G_n)U^* = \bigotimes_{n=1}^{\infty} LG_n$ and $U\lambda_{g_1+\cdots+g_k}U^* = \lambda_{g_1} \otimes \cdots \otimes \lambda_{g_k}$ when $k \in \mathbb{N}$ and $g_i \in G_i$ ($i = 1, \ldots, k$).

**Proof.** The proof of this proposition is essentially the same as in the finite case of Proposition 5.3 with the orthogonal basis
\[
\{\delta_{g_1+\cdots+g_k} : k \in \mathbb{N}, g_i \in G_i (i = 1, \ldots, k)\}
\]
for the Hilbert space $\ell^2(\bigoplus_{n=1}^{\infty} G_n)$ and the orthonormal basis
\[
\{\delta_{g_1} \otimes \cdots \otimes \delta_{g_k} : k \in \mathbb{N}, g_i \in G_i (i = 1, \ldots, k)\}
\]
for the Hilbert space $\bigotimes_{n=1}^{\infty} (\ell^2(G_n), \delta_e)$, and the unitary $U$ defined on the orthonormal basis by $U\delta_{g_1+\cdots+g_k} = \delta_{g_1} \otimes \cdots \otimes \delta_{g_k}$ for all $k \in \mathbb{N}$ and $g_i \in G_i$ ($i = 1, \ldots, k$).

**Remark 5.9.** Note that, as in the finite case, when identifying $L(\bigoplus_{n=1}^{\infty} G_n)$ and $\bigotimes_{n=1}^{\infty} LG_n$, the natural inclusion of $LG_k$ into $L(\bigoplus_{n=1}^{\infty} G_n)$ corresponds to the natural inclusion of $LG_k$ into $\bigotimes_{n=1}^{\infty} LG_n$. 

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6 Inner amenability without property Gamma

This section is the main part of the project. In Section 3 we proved that property \( \Gamma \) of the group von Neumann algebra implied inner amenability of the group if the group is i.c.c. In this section we prove that the converse is not true. This requires a bit more work, and the group we use as a counterexample is not a pretty one.

In proving that the constructed group is actually a counterexample, we will use that a certain group has the property called property \((T)\). The group in question is \( \text{SL}(3, \mathbb{Z}) \), and the proof of this is not present in this project. None the less we will start by defining property \((T)\) for groups and state the fact that \( \text{SL}(3, \mathbb{Z}) \) has property \((T)\).

6.1 Property \((T)\)

This section is a short introduction to property \((T)\) for groups. Much can be said about property \((T)\), but we will make do with a few definitions, a small lemma and the statement about \( \text{SL}(3, \mathbb{Z}) \) having property \((T)\).

**Definition 6.1.** Let \( G \) be a discrete group and \( \varphi : G \to B(\mathcal{H}) \) a unitary representation of \( G \). A vector \( \xi \in \mathcal{H} \) is said to be \( \varphi \)-invariant if

\[
\varphi(g)\xi = \xi
\]

for all \( g \in G \). A sequence \( \xi_n (n \in \mathbb{N}) \) in \( \mathcal{H} \) is said to be a sequence of almost \( \varphi \)-invariant vectors if

\[
\|\varphi(g)\xi_n - \xi_n\| \to 0 \quad \text{as} \quad n \to \infty
\]

for all \( g \in G \). The representation \( \varphi \) is said to have almost invariant vectors if there exists a sequence of almost \( \varphi \)-invariant unit vectors.

With this definition clear we have a context in which we can define property \((T)\) for a group \( G \).

**Definition 6.2.** A group \( G \) is said to have Kazhdan's property \((T)\), or just property \((T)\), if every unitary representation which has almost invariant vectors actually has non-trivial invariant vectors.

The set of \( \varphi \)-invariant vectors of a unitary representation \( \varphi : G \to B(\mathcal{H}) \) of a group \( G \) on a Hilbert space \( \mathcal{H} \) form a closed subspace of \( \mathcal{H} \) no matter if the group has property \((T)\) or not. The following lemma shows that a sequence of almost \( \varphi \)-invariant vectors must necessarily get close to this subspace.

**Proposition 6.3.** If \( \varphi : G \to B(\mathcal{H}) \) is a unitary representation of a group \( G \) with property \((T)\), and \( (\xi_n)_{n \geq 1} \) is a sequence of almost \( \varphi \)-invariant vectors, then

\[
\|\xi_n - P\xi_n\| \to 0 \quad \text{as} \quad n \to \infty,
\]
where $P$ denotes the projection onto the closed subspace of $\varphi$-invariant vectors.

Proof. Let $\mathcal{M}$ denote the set of $\varphi$-invariant vectors, and let $\xi'_n := \xi_n - P\xi_n$. It is easy to check that

$$\varphi(G)\mathcal{M} \subset \mathcal{M} \quad \text{and} \quad \varphi(G)\mathcal{M}^\perp \subset \mathcal{M}^\perp,$$

in fact both inclusions are equalities. Because $\varphi(G)\mathcal{M}^\perp \subset \mathcal{M}^\perp$ we can consider the unitary representation $\tilde{\varphi} : G \to B(\mathcal{M}^\perp)$ defined by $\tilde{\varphi}(g) := \varphi(g)|_{\mathcal{M}^\perp}$. Assume towards a contradiction that $\xi'_n$ does not converge to 0. This means that there exists some $\varepsilon > 0$ and a subsequence $(\xi'_n_k)_{k \geq 1}$ such that $\|\xi'_n_k\| > \varepsilon$ for all $k \geq 0$. Now if $\xi''_n_k := \frac{\xi'_n_k}{\|\xi'_n_k\|}$, then

$$\|\tilde{\varphi}(g)\xi''_n_k - \xi''_n_k\| \leq \frac{1}{\varepsilon}\|\varphi(g)\xi'_n_k - \xi'_n_k\| = \frac{1}{\varepsilon}\|\varphi(g)\xi_n_k - \xi_n_k\|,$$

which shows that

$$\|\tilde{\varphi}(g)\xi''_n_k - \xi''_n_k\| \to 0 \quad \text{as} \quad n \to \infty$$

for all $g \in G$. Thus the representation $\tilde{\varphi}$ has almost invariant vectors and since $G$ is assumed to have property (T), this means that $\tilde{\varphi}$ must have a non-trivial $\tilde{\varphi}$-invariant vector $\eta$. Clearly $\eta$ is then also a $\varphi$-invariant vector, but this the fact that $\mathcal{M}^\perp$ was chosen so that it does not contain any non-trivial $\varphi$-invariant vectors. Hence we must have $\xi'_n \to 0$ when $n \to \infty$. \qed

**Theorem 6.4.** The group $\text{SL}(3, \mathbb{Z})$ has property (T).

Here $\text{SL}(3, \mathbb{Z})$ denotes the group of $3 \times 3$ matrices with entries in $\mathbb{Z}$ and determinant one. This is a non-trivial result and as mentioned we will not prove this theorem. A proof can be found in [BO, Theorem 12.1.14].

### 6.2 Constructing the group

This part is devoted to constructing the group we will show is inner amenable with a von Neumann algebra that does not have property $\Gamma$. The construction use the methods of Section 4 on $\text{SL}(3, \mathbb{Z})$ and some finite cyclic groups. From this point forth

$$p_0, p_1, p_2, \ldots$$

will be a fixed sequence of distinct primes. The group will be constructed using this sequence, and naturally it will depend on this choice, but this is not important for our purpose. Now let

$$H_n := (\mathbb{Z}/p_n\mathbb{Z})^3 \quad \text{and} \quad K := \bigoplus_{n=0}^\infty H_n,$$

Hence we must have $\xi'_n \to 0$ when $n \to \infty$. \qed
and denote SL(3, Z) by Λ. There is a natural action of Λ on \( H_n \) \( (n \in \mathbb{N}_0) \), which is given by the usual matrix product when representatives are chosen in \( \mathbb{Z}^3 \), \( A \cdot [v] = [Av] \) when \( A \in \text{SL}(3, \mathbb{Z}) \) and \( v \in \mathbb{Z}^3 \). This action induces an action on \( K \), by letting \( \Lambda \) act diagonally, that is

\[
A \cdot (g_0 + g_1 + \cdots + g_k) = A \cdot g_0 + A \cdot g_2 + \cdots + A \cdot g_k
\]

when \( A \in \Lambda \), \( k \in \mathbb{N}_0 \) and \( g_n \in H_n \) \( (n = 1, \ldots, k) \). Since \( \Lambda \) acts on each \( H_n \) by automorphisms it also acts on \( K \) by automorphisms, and we may form the semidirect product \( K \rtimes \Lambda \) with respect to this action. Let \( G_0 \) denote this semidirect product, and for each \( N \in \mathbb{N}_0 \) define

\[
K_N := \bigoplus_{n=N}^{\infty} H_n,
\]

which of course sits as a subgroup of \( K \). From \( G_0 \), \( K \) and \( K_N \) we will construct an inductive system of groups

\[
G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_N \hookrightarrow G_{N+1} \hookrightarrow \cdots
\]

and then take the inductive limit. We will construct this system inductively. Suppose therefore that we are in the situation

\[
G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_N
\]

for some \( N \in \mathbb{N}_0 \), and want to construct the group \( G_{N+1} \) and the inclusion \( G_N \hookrightarrow G_{N+1} \). By the composition of all the maps above we get an inclusion \( G_0 \hookrightarrow G_N \), so since \( K_N \subseteq K \subseteq G_0 \) we get an inclusion of \( K_N \) as a subgroup of \( G_N \). We also have a natural inclusion of \( K_N \) as a subgroup of \( K_N \times \mathbb{Z} \) by the map \( g \mapsto (g, 0) \), and thus we may form the amalgamated free product

\[
G_{N+1} := G_N \ast_{K_N} (K_N \times \mathbb{Z})
\]

Now the map \( G_N \hookrightarrow G_{N+1} \) is just the natural inclusion of \( G_N \) as a subgroup of \( G_{N+1} \). In this way we have constructed an inductive system and we let \( G \) denote the inductive limit of this system. From this point and forth \( G \) will denote this group.

Before moving on we show that the group \( G \) is countable with infinite conjugacy classes, but it is not finitely generated. To show that the group has infinite conjugacy classes we use that this is true for \( \text{SL}(3, \mathbb{Z}) \), which is easy to show.

**Lemma 6.5.** The group \( G \) is countable with infinite conjugacy classes, but not finitely generated. Moreover, for each \( g \in G \setminus K \) the set \( \{hgh^{-1} : h \in \Lambda\} \) is infinite.
Proof. Clearly $K$ and $\Lambda$ are countable. The free product of two countable groups is again countable, since there are only countably many words of each length, and it follows by induction that $G_N$ is countable for all $N \geq 0$. Remembering how the inductive limit of the groups $G_N$ is constructed, and that a countable union of countable sets is again countable, we see that $G$ must indeed be countable.

To show that $G$ has infinite conjugacy classes we start by showing the last statement. Suppose first that $g \in G \setminus G_0$. This means that we can find some $N \in \mathbb{N}$ such that $g \in G_{N+1} \setminus G_N$. Since $\Lambda$ and $K_N$ intersect trivially as subgroups of $G_N$ and

$$G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z}),$$

the elements $hgh^{-1} (h \in \Lambda)$ are all different, so the set $\{hgh^{-1} : h \in \Lambda\}$ must be infinite. Suppose instead that $g \in G_0 \setminus K$. Since $g \in G_0$ we can write $g = kg'$ for unique $k \in K$ and $g' \in \Lambda$. We know that $g'$ must be different from the neutral element since $g \notin K$, and for $h \in \Lambda$ we see that

$$hgh^{-1} = hkh^{-1} = (hh^{-1})(hg'h^{-1}) = (h,k)(hg'h^{-1}). \quad (6.1)$$

Since $\text{SL}(3, \mathbb{Z})$ has infinite conjugacy classes, the set $\{hg'h^{-1} : h \in \Lambda\}$ must be infinite, but since $h,k \in K$ and all elements in $G_0$ can be written uniquely as a product of an element from $\Lambda$ and one from $K$ (6.1) shows that the set $\{hg'h^{-1} : h \in \Lambda\}$ must also be infinite. This partly proves that $G$ has infinite conjugacy classes, since we then know that elements in $G \setminus K$ have infinite conjugacy classes. We only need to show that elements in $K$ also has infinite conjugacy classes. Suppose that $g \in K$. By choosing $N \in \mathbb{N}$ large enough, we can make sure that $g \in G \setminus G_N$. We know that $g \in G_N$ since $K \subset G_N$, so if $g$ is not the neutral element, $g$ must have an infinite conjugacy class in

$$G_{N+1} = G_N *_{K_N} (K_N \times \mathbb{Z})$$

since $g \notin K_N$ and thus also in $G$. This shows that $G$ has infinite conjugacy classes.

Let us now argue that $G$ is not finitely generated. Suppose towards a contradiction that $G$ actually is finitely generated, then we can choose generators $g_1, \ldots, g_k \in G$ for some $k \in \mathbb{N}$. Now since

$$G = \bigcup_{n=0}^{\infty} G_n,$$

we can choose $N$ large enough so that all these generators are in $G_N$. But since $G_N$ is a group in it self it must contain the group generated by these generators, i.e. $G \subseteq G_N$. This though is clearly a contradiction, and conclude that $G$ is not finitely generated. $\square$
6.3 Some preparations

In the proof of the main theorem we need a few technical results, and they may at this point seem a little out of place. They are placed here to get a better flow in the proof, and the reader may just skip them for now, and then return when they are needed.

Lemma 6.6. Let \( n \in \mathbb{N}_0 \). The orbits under the diagonal action \( \Lambda \curvearrowright H_1 \oplus \cdots \oplus H_n \) are exactly the sets \( \mathcal{U}_i \times \cdots \times \mathcal{U}_n \) where, for each \( i = 1, \ldots, n \), the set \( \mathcal{U}_i \) is either \( H_i \setminus \{0\} \) or \( \{0\} \).

Proof. Suppose that \((g_1, \ldots, g_n) \in H_1 \oplus \cdots \oplus H_n\). What the lemma says, is that for each \( i \in \{1, \ldots, n\} \) such that \( g_i \neq 0 \) we may change the \( i \)th coordinate in \((g_1, \ldots, g_n)\) freely by the action of \( \Lambda \) without changing the rest of the coordinate, with the exception of changing it to 0. To show this suppose that \( i \in \{1, \ldots, n\} \) with \( g_i \neq 0 \), and that \( g_i' \in H_i \) is just some element with \( g_i' \neq 0 \). We want to find an element \( h \in \Lambda \) so that

\[
\begin{align*}
\lambda \cdot g_j &= \begin{cases} g_j & \text{if } j \neq i \\ g_i' & \text{if } j = i. \end{cases}
\end{align*}
\]

Choose \( a, b, c \in \mathbb{Z} \) so that \( g_i = [a, b, c] \) and \( a', b', c' \in \mathbb{Z} \) so that \( g_i' = [a', b', c'] \). What we do is that we change one of the values \( a, b, c \) at a time, so there are various cases to consider. We will not go through them all, since they are very much alike. By assumption either \( a, b \) or \( c \) is relatively prime with \( p_i \), and we will consider the case where \( a \) is relatively prime with \( p_i \) and we want to change \( b \) to \( b' \), that is to find some \( h \in \Lambda \) such that \( h \cdot [a, b, c] = [a, b', c] \) and \( h \cdot g_j = g_j \) when \( j \neq i \).

We know that all the primes \( p_1, \ldots, p_n \) are different, so we know that

\[
m := p_0 \cdots p_{i-1}p_{i+1} \cdots p_n \equiv 0 \pmod{p_i}.
\]

We also have \( a \not\equiv 0 \pmod{p_i} \) so we can choose some \( d \in \mathbb{Z} \) with \( dm \equiv 1 \pmod{p_i} \) since \( \mathbb{Z}/\mathbb{Z}p_i \) is cyclic. Note here that

\[
dm \equiv 0 \pmod{p_j}
\]

when \( j \neq i \), since \( p_j \) is then a divisor of \( dm \). Consider the element \( h \) in \( \Lambda \) given by the following matrix

\[
h = \begin{bmatrix} 1 & 0 & 0 \\ (b' - b)dm & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

This satisfies \( h \cdot [a, b, c] = [a, (b+(b'-b)dm), c] = [a, b', c] \), and it is straightforward to check that \( h \cdot g_j = g_j \) when \( j \neq i \) since \( dm \equiv 0 \pmod{p_j} \). Now as mentioned the rest of the cases are analogue, and in this way the statement follows, just by changing one coordinate at a time. \( \square \)
Lemma 6.7. $H_n$ is a normal subgroup of $G_{n+1}$ for all $n \in \mathbb{N}_0$.

Proof. Let $n \in \mathbb{N}_0$. First we show that $H_n$ is a normal subgroup of $G_0$. All elements in $G_0$ can be written as a product of an element of $\Lambda$ and one of $K$, so it suffices to show that $gH_ng^{-1} = H_n$ whenever $g \in \Lambda$ or $g \in K$. If $g \in K$ the case is clear since $K$ is commutative. So suppose that $g \in \Lambda$. Since $ghg^{-1} = g \cdot h$ when $h \in K$ and the action of $\Lambda$ on $K$ is defined diagonally it follows that $gH_ng^{-1} = H_n$.

We know that if $H$ and $H'$ are groups with a common subgroup $S$, then if $N$ is a subgroup of $S$ such that $N$ is a normal subgroup of both $H$ and $H'$, then $N$ is a normal subgroup of $H \ast_S H'$. So since $H_n$ is normal in $K_N$ for all $N \leq n + 1$ and since $H_n$ is normal in $G_0$, it follows by induction that $H_n$ is normal in $G_N$ for all $N \leq n + 1$ since

$$G_{n+1} = G_N \ast_{K_N} (K_N \times \mathbb{Z}).$$

In particular $H_n$ is a normal subgroup of $G_{n+1}$. \hfill \Box

Remark 6.8. Let $n \in \mathbb{N}_0$. For two elements $g$ and $h$ in $H_n$ we can choose unique representatives in $(v_1, v_2, v_3)$ and $(w_1, w_2, w_3)$ in $\{1, \ldots, p_n\}^3$ respectively, and we will use the notation $(g \mid h)$ to denote the integer $v_1w_1 + v_2w_2 + v_3w_3$, i.e., the standard inner product of the representatives in $\mathbb{R}^3$. \footnote{There is no particular reason to choose representatives in $\{1, \ldots, p_n\}$ for our purpose, but to make $(g \mid h)$ well-defined we need some choice of representatives.}

Now for each $g \in H_n$ define $\psi(g) : H_n \to \mathbb{C}$ by

$$\psi(g)(h) = \exp(2\pi ip_n^{-1}(h \mid g))$$

when $h \in H_n$, then it is easy to check that $\psi(g)$ is in fact a group homomorphism, and that the map $\psi : H_n \to H_n$ is in fact an isomorphism. This obviously gives us a $\ast$-isomorphism $C(H_n; \mathbb{C}) \to C(H_n; \mathbb{C})$, namely

$$f \mapsto f \circ \psi.$$

We know from Proposition A.5 that the Gelfand transform $\Gamma_n : \ell^1(H_n) \to C(H_n; \mathbb{C})$ is a $\ast$-isomorphism, since $H_n$ is finite, and we will consider it as a map $\ell^1(H_n) \to C(H_n; \mathbb{C})$, using the isomorphism from $C(H_n; \mathbb{C}) \to C(H_n; \mathbb{C})$. In this situation the Gelfand transform is given by

$$\Gamma_n(f)(h) = \sum_{g \in H_n} f(g) \exp(2\pi ip_n^{-1}(g \mid h))$$

for all $f \in \ell^1(H_n)$ and $h \in H_n$.

Lemma 6.9. For each $n \in \mathbb{N}$ and each $h \in H_n$, the sum

$$\sum_{h' \in H_n} \exp \left( 2\pi ip_n^{-1}(h' \mid h) \right)$$

(6.2)

equals zero if $h \neq e$ and $p_n^3$ if $h = e$. 

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Proof. Let \( h \in H_n \) and choose representatives \( a, b, c \in \mathbb{Z} \) such that \( h = [a, b, c] \). The sum can be rewritten as

\[
\sum_{h' \in H_n} \exp \left( 2 \pi ip_n^{-1} (h' \mid h) \right) = \sum_{a' = 1}^{p_n} \sum_{b' = 1}^{p_n} \sum_{c' = 1}^{p_n} \exp \left( 2 \pi ip_n^{-1} (a'a + bb' + cc') \right)
\]

and if we let \( \zeta \) denote \( \exp(2 \pi ip_n^{-1}) \), then

\[
\sum_{h' \in H_n} \exp \left( 2 \pi ip_n^{-1} (h' \mid h) \right) = \left( \sum_{a' = 1}^{p_n} \zeta^{aa'} \right) \left( \sum_{b' = 1}^{p_n} \zeta^{bb'} \right) \left( \sum_{c' = 1}^{p_n} \zeta^{cc'} \right).
\]

Now in the case where \( h = e \) we have that \( \zeta^a = \zeta^b = \zeta^c = 1 \), why the three sums above all equal \( p_n \), and (6.2) equals \( p_n^3 \). If we are in the case where \( h \neq e \) then it suffices to show that either

\[
\sum_{a' = 1}^{p_n} \zeta^{aa'}, \quad \sum_{b' = 1}^{p_n} \zeta^{bb'} \quad \text{or} \quad \sum_{c' = 1}^{p_n} \zeta^{cc'}
\]

is zero. Since \( h \neq e \) either \( a, b \) or \( c \) must be relatively prime with \( p_n \). We can assume, without loss of generality, that \( a \) is relatively prime with \( p_n \). That \( a \) and \( p_n \) are relatively prime implies that the numbers \( \zeta^a, \zeta^{2a}, \ldots, \zeta^{p_na} \) are exactly the \( p_n \)'th roots of unity, or in other words that

\[
\prod_{a' = 1}^{p_n} (X - \zeta^{a'a}) = X^{p_n} - 1.
\]

Now if we expand the left hand side, we get a polynomial of the form

\[
X^{p_n} - \left( \sum_{a' = 1}^{p_n} \zeta^{a'a} \right) X^{p_n-1} + \ldots
\]

so since this agrees with \( X^{p_n} - 1 \) we must have that \( \sum_{a' = 1}^{p_n} \zeta^{a'a} = 0 \). Thus (6.2) is zero in the case where \( h \neq e \).

Lemma 6.10. Let \( H \) be a discrete group and let \( \tau \) be a normalized trace on \( \ell^\infty(H) \). Then for all \( g \in H \) and every \( f \in \ell^\infty(H) \) with \( \|f\|_\infty \leq 1 \) it holds that

\[
\|f - \tau(f)1\|_\tau \leq 4(1 - \tau(\delta_g))^{1/2}
\]

Proof. Recall that by \( \| \cdot \|_\tau \) we mean the seminorm defined by \( \|f\|_\tau = \tau(f^* f) \) for all \( f \in \ell^\infty(H) \). So suppose that \( g \in H, f \in \ell^\infty(H) \) with \( \|f\|_\infty \leq 1 \) and let \( \chi = 1 - \delta_g \). Then

\[
\begin{align*}
f - \tau(f)1 &= f\delta_g + f\chi - \tau(f\delta_g + f\chi)\delta_g - \tau(f)\chi \\
&= (f\delta_g - \tau(f\delta_g)\delta_g) - \tau(f\chi)\delta_g + (f\chi - \tau(f)\chi)
\end{align*}
\]

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and by the triangle inequality we get
\[ \| f - \tau(f)1 \|_\tau \leq \| f\delta_g - \tau(f\delta_g)\delta_g\|_\tau + \| \tau(f\chi)\delta_g\|_\tau + \| f\chi - \tau(f)\chi\|_\tau \quad (6.3) \]

Let us consider each of the three terms separately. We start with the term \( \| f\delta_g - \tau(f\delta_g)\delta_g\|_\tau \). Observing that \( f\delta_g = f(g)\delta_g \) we get
\[ \| f\delta_g - \tau(f\delta_g)\delta_g\|_\tau = \| f(g)(1 - \tau(\delta_g))\delta_g\|_\tau = |\| f(g)\|_\infty \| 1 - \tau(\delta_g)\|_\tau \| \delta_g\|_\tau \leq \| 1 - \tau(\delta_g)\| \tau(\delta_g)^{1/2}, \]
where we used that \( \| f\|_\infty \leq 1 \) and that \( \| \delta_g\|_\tau = \tau(\delta_g)^{1/2} \). Now since both \( |1 - \tau(\delta_g)| \leq 1 \) and \( |\tau(\delta_g)|^{1/2} \leq 1 \) we get
\[ (1 - \tau(\delta_g))\tau(\delta_g)^{1/2} \leq (1 - \tau(\delta_g)) \leq 1 - \tau(\delta_g) \leq 1 - \tau(\delta_g) \]
Hence \( \| f\delta_g - \tau(f\delta_g)\delta_g\|_\tau \leq (1 - \tau(\delta_g))^{1/2} \). Lets move on to the second term. Since \( \tau \) is a state we know that \( |\tau(f\chi)| \leq \| f\|_\infty \| \chi\|_\tau \). Using that \( \| f\|_\infty \leq 1 \) and \( \| \chi\|_\tau = \tau(\chi)^{1/2} \) we therefore get
\[ \| \tau(f\chi)\delta_g\|_\tau = |\| \tau(f\chi)\| \| \delta_g\|_\tau \leq \tau(\chi)^{1/2} \tau(\delta_g). \]
We know that \( \tau(\chi) = 1 - \tau(\delta_g) \) because \( \tau \) is a state and again using that \( |\tau(\delta_g)| \leq 1 \) we get
\[ \| \tau(f\chi)\delta_g\|_\tau \leq \tau(\chi)^{1/2} \tau(\delta_g) = (1 - \tau(\delta_g))^{1/2} \tau(\delta_g) \leq (1 - \tau(\delta_g))^{1/2}. \]
Now for that last term. Again since \( \tau \) is a state we know that
\[ \| f\chi - \tau(f)\chi\|_\tau = \| (f - \tau(f))1\chi\|_\tau \leq \| f - \tau(f)1\|_\infty \| \chi\|_\tau, \]
but \( \| f - \tau(f)1\|_\infty \leq \| f\|_\infty + |\tau(f)| \leq 2 \), so we get
\[ \| f\chi - \tau(f)\chi\|_\tau \leq 2\| \chi\|_\tau = 2(1 - \tau(\delta_g))^{1/2}. \]
By now we have estimated all the three terms on the right hand side of (6.3) and summarizing we get
\[ \| f - \tau(f)1\|_\tau = 4(1 - \tau(\delta_g))^{1/2}, \]
which was exactly what we set out to prove. \( \square \)
6.4 Proving the theorem

Let $\mathbb{Z}_2 = \{0, 1\}$ denote the group with two elements. For each $n \in \mathbb{N}_0$ we let $\mathcal{A}_n$ denote the von Neumann algebra $\ell^\infty(\mathbb{Z}_2)$ acting as multiplication operators on the Hilbert space $\mathcal{H}_n := \ell^2(\mathbb{Z}_2)$. If we let $e_n$ denote the projection given by multiplication with the indicator function on $\{0\}$, then $\mathcal{A}_n$ is spanned by the projections $e_n$ and $1 - e_n$. Now define $\xi_n \in \mathcal{H}_n$ by

$$\xi_n(0) = (p_n^{-3})^{1/2} \quad \text{and} \quad \xi_n(1) = (1 - p_n^{-3})^{1/2}.$$  

Then the linear functional $\tau_n : \mathcal{A}_n \rightarrow \mathbb{C}$ given by $x \mapsto (x \xi_n \mid \xi_n)$ is a normalized trace on $\mathcal{A}_n$ with $\tau_n(e_n) = p_n^{-3}$.

For $n \in \mathbb{N}_0$ let denote by $\mathcal{B}_n$ the von Neumann algebra $\ell^\infty(H_n)$ acting as multiplication operators on the Hilbert space $\mathcal{H}_n := \ell^2(H_n)$. If we let $\tau'_n$ denote the trace arising from the normalized counting measure on $H_n$, i.e.

$$\tau'_n(x) = p_n^{-3} \sum_{g \in H_n} x(g)$$

for all $x \in \mathcal{B}_n$, then $\tau'_n$ can also be expressed as $\tau'_n(x) = (x \xi'_n \mid \xi'_n)$ where $\xi'_n = p_n^{-3/2} \sum_{g \in H_n} \delta_g$. There is a natural way of identifying $\mathcal{A}_n$ with a von Neumann subalgebra of $\mathcal{B}_n$ in a trace preserving way. This is done by identifying $e_n$ with $\delta_e$ and $1 - e_n$ with $1 - \delta_e = \sum_{h \in H_n \setminus \{e\}} \delta_h$. From this point forth we will use this identification.

Define an action $\Lambda \curvearrowright \mathcal{B}_n$ by $(\phi^g_n x)(h) = x(g^{-1} \cdot h)$ for all $g \in \Lambda$, $h \in H_n$ and $x \in \mathcal{B}_n$, where $g \cdot h$ still means the natural action of $\Lambda$ on $H_n$. Furthermore, define an action $\Lambda \curvearrowright LK$ by $\sigma_g(L_\eta) = L_{g \cdot \eta}$ for $g \in \Lambda$ and $L_\eta \in LK$, where $g \cdot \eta$ denotes the action of $\Lambda$ on $\ell^2(K)$ given by $(g \cdot \eta)(h) = \eta(g^{-1} \cdot h)$ when $h \in K$. It is not hard to see that these actions are well-defined, and that both $\sigma_g$ and $\theta_g$ are linear maps for all $g \in \Lambda$ since the linear operations are defined pointwise. If $\eta \in \ell^2(H_n)$ for some $n \in \mathbb{N}_0$ and $g \in \Lambda$ then $g \cdot \eta \in \ell^2(H_n)$ since $g^{-1} \cdot h \in H_n$ for all $h \in H_n$, and therefore we may restrict $\sigma$ to an action $\Lambda \curvearrowright LH_n$. We will also denote the restriction $\sigma$.

Lemma 6.11. For each $n \in \mathbb{N}_0$ there is a trace preserving $*$-isomorphism $\alpha_n : \mathcal{B}_n \rightarrow LH_n$ satisfying

$$\alpha_n(e_n) = p_n^{-3} \sum_{h \in H_n} \lambda_h \quad \text{and} \quad \sigma_{g'} \circ \alpha_n = \alpha_n \circ \theta_{p_n^{-1}}$$

for all $g \in \Lambda$ when $g'$ denotes the transpose of $g$.

Proof. Let $n \in \mathbb{N}_0$, and define a map $\Phi_n : LH_n \rightarrow \ell^1(H_n)$ by $\phi_n(x) = x \delta_e$. This map is a well-defined bijection since $H_n$ is finite. It is also a $*$-homomorphism between the $*$-algebras $LH_n$ and $\ell^1(H_n)$, and hence a $*$-isomorphism. Let $\Gamma_n : \ell^1(H_n) \rightarrow C(H_n; \mathbb{C})$ denote the Gelfand transform,
which we know is a *-isomorphism from Remark 6.8. Since $H_n$ is finite, $\ell^\infty(H_n) = C(H_n; \mathbb{C})$ with the same $C^*$-algebraic structure, and if we define the map $\alpha_n$ to be the inverse of the map $\Gamma_n\Phi_n$, then $\alpha_n$ is a *-isomorphism $\mathcal{B}_n \to L\mathcal{H}_n$. We shall see that it satisfies the two requirements of the lemma.

Let us start by showing that $\alpha_n$ is trace preserving. This obviously is equivalent to the fact that $\Gamma_n\Phi_n$ is trace preserving, and since this map is easier to work with, this is what we will show. Let $g \in H_n$, then

$$\tau_n'\Gamma_n\Phi_n(\lambda_g) = \tau_n(\Gamma_n(\delta_g)) = p_n^{-3} \sum_{h \in H_n} \Gamma_n(\delta_g)(h).$$

If we recall from Remark 6.8 how the Gelfand transform works, then we get

$$\tau_n'\Gamma_n\Phi_n(\lambda_g) = p_n^{-3} \sum_{h \in H_n} \exp\left(2\pi ip_n^{-1}(g \mid h)\right).$$

Now by Lemma 6.9 this is zero if $g \neq e$ and one if $g = e$, so it agrees with the quantity $\tau(\lambda_g) = (\lambda_g \delta_e \mid \delta_e)$. Since $\{\lambda_g : g \in H_n\}$ span $L\mathcal{H}_n$, this shows that $\Gamma_n\Phi_n$, and therefore also $\alpha_n$, is trace preserving.

We proceed to show that $\alpha_n(e_n) = p_n^{-3} \sum_{h \in H_n} \lambda_h$, and we use the same strategy as before, i.e. we show that

$$\Gamma_n\Phi_n\left(p_n^{-3} \sum_{h \in H_n} \lambda_h\right) = e_n.$$ 

Now if we evaluate the left hand side in a point $g \in H_n$ then we get

$$\Gamma_n\Phi_n\left(p_n^{-3} \sum_{h \in H_n} \lambda_h\right)(g) = p_n^{-3} \sum_{h \in H_n} \Gamma_n(\delta_h)(g)$$

$$= p_n^{-3} \sum_{h \in H_n} \exp\left(2\pi ip_n^{-1}(h \mid g)\right),$$

so by Lemma 6.9 this is zero if $h \neq e$ and 1 if $h = e$, which exactly shows that $\Gamma_n\Phi_n(p_n^{-3} \sum_{h \in H_n} \lambda_h)(g) = \delta_e = e_n$. Thus $\alpha_n(e_n) = p_n^{-3} \sum_{h \in H_n} \lambda_h$.

Finally we need to show that $\sigma_{g'} \circ \alpha_n = \alpha_n \circ \theta_{g^{-1}}$ for all $g \in H_n$. What we do is to show that

$$\Gamma_n\Phi_n\sigma_{g'} = \theta_{g^{-1}}\Gamma_n\Phi_n$$

for all $g \in \Lambda$. If $g \in \Lambda$ and $h \in H_n$, then $\sigma_{g'}\lambda_h = \lambda_{g'h}$ so we see that

$$(\Gamma_n\Phi_n\sigma_{g'}\lambda_h)(h') = \Gamma_n((\delta_{g'h})\lambda_h)(h') = \sum_{h' \in H_n} \exp\left(2\pi ip_n^{-1}(g' \mid h' \mid h)\right).$$

It is easy to check that $(g' \mid h' \mid h') = (h \mid g \cdot h')$ and so we get

$$(\Gamma_n\Phi_n\sigma_{g'}\lambda_h)(h') = \sum_{h' \in H_n} \exp\left(2\pi ip_n^{-1}(h \mid g \cdot h')\right)$$

$$= (\Gamma_n\Phi_n\lambda_h)(g \cdot h')$$

$$= (\theta_{g^{-1}}\Gamma_n\Phi_n\lambda_h)(h').$$
Now since \( h' \in H_n \) was arbitrary this shows that \( \Gamma_n \Phi_n \sigma g \lambda_h = \theta g^{-1} \Gamma_n \Phi_n \lambda_h \), and since the span of \( \{ \lambda_h : h \in H_n \} \) is the whole of \( LH_n \), we conclude that \( \Gamma_n \Phi_n \sigma g = \theta g^{-1} \Gamma_n \Phi_n \). So we have proved that \( \alpha_n \) has the desired properties.

We now form two large von Neumann algebras out of these smaller von Neumann algebras using the tensor product. Since \( \xi_n \) and \( \xi'_n \) are unit vectors for all \( n \in \mathbb{N}_0 \) we may form the infinite tensor product Hilbert spaces

\[
\mathcal{H} = \bigotimes_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}' = \bigotimes_{n=0}^{\infty} \mathcal{H}'_n.
\]

The von Neumann algebras \( \mathcal{A}_n \) and \( \mathcal{B}_n \) acts on \( \mathcal{H}_n \) and \( \mathcal{H}'_n \) respectively so we may form the von Neumann algebras

\[
\mathcal{A} = \bigotimes_{n=0}^{\infty} \mathcal{A}_n \quad \text{and} \quad \mathcal{B} = \bigotimes_{n=0}^{\infty} \mathcal{B}_n.
\]

Here \( \mathcal{A} \) acts on \( \mathcal{H} \) and \( \mathcal{B} \) acts on \( \mathcal{H}' \). Denote by \( \mathcal{A}^0 \) and \( \mathcal{B}^0 \) the dense subsets

\[
\bigcup_{n=0}^{\infty} \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_n \quad \text{and} \quad \bigcup_{n=0}^{\infty} \mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_n
\]

of \( \mathcal{A} \) and \( \mathcal{B} \) respectively. There is a natural choice of trace on \( \mathcal{A} \), namely

\[
\tau : x \mapsto (x \xi | \xi), \quad \mathcal{A} \to \mathbb{C}.
\]

This map clearly defines an ultraweakly continuous faithful state on \( \mathcal{A} \) since \( \xi \) is a unit vector, and because

\[
\tau(x_0 \otimes \cdots \otimes x_n) = \prod_{k=0}^{n} (x_k \xi_k | \xi_k) = \prod_{k=0}^{n} \tau_k(x_k)
\]

for all \( n \in \mathbb{N}_0 \) and \( x_i \in \mathcal{A}_i \) \((i = 0, \ldots, n)\), we see that the map \( (x, y) \mapsto \tau(xy) - \tau(yx) \) is identically zero on the ultraweakly dense subset \( \mathcal{A}^0 \times \mathcal{A}^0 \) of \( \mathcal{A} \times \mathcal{A} \). By continuity this means that \( \tau \) is a trace on \( \mathcal{A} \). The same type of argument shows that

\[
\tau' : x \mapsto (x \xi' | \xi'), \quad \mathcal{B} \to \mathbb{C}
\]

defines an ultraweakly continuous faithful normalized trace on \( \mathcal{B} \). Note that with these traces the canonical identifications \( \mathcal{A}_n \hookrightarrow \mathcal{A} \) and \( \mathcal{B}_n \hookrightarrow \mathcal{B} \) becomes trace preserving for all \( n \in \mathbb{N}_0 \).

Let us discuss why we can identify \( \mathcal{A} \) with a von Neumann subalgebra of \( \mathcal{B} \). For each \( n \in \mathbb{N}_0 \) we have an identification \( \mathcal{A}_n \hookrightarrow \mathcal{B}_n \), and denoting
this identification by $i_n$, we can consider the linear map $i : \mathcal{A}^n \to \mathcal{B}$ defined on basic tensors by

$$i(x_0 \otimes \cdots \otimes x_n) = i_0(x_0) \otimes \cdots \otimes i_n(x_n)$$

for all $n \in \mathbb{N}_0$ and $x_i \in \mathcal{A}_i$ ($i = 0, \ldots, n$). Since $i_n$ is trace preserving for all $n \in \mathbb{N}_0$, we see that

$$\tau'(i(x_0 \otimes \cdots \otimes x_n)) = \prod_{k=0}^n \tau'_k(i_k(x_k)) = \prod_{k=0}^n \tau_k(x_k) = \tau(x_0 \otimes \cdots \otimes x_n)$$

for all $n \in \mathbb{N}_0$ and $x_i \in \mathcal{A}_i$ ($i = 0, \ldots, n$), which shows that the map $i$ is also trace preserving. It is straightforward to check that $i$ is a $*$-homomorphism since all the maps $i_n$ ($n \in \mathbb{N}_0$) are $*$-homomorphisms. Now by Proposition A.4 the map $i$ extends to an ultraweakly continuous trace preserving $*$-homomorphism on the weak closure of $\mathcal{A}^n$, i.e. a map $\mathcal{A} \to \mathcal{B}$. This map is injective since it preserves a faithful trace, and by Proposition A.3 the image is a von Neumann subalgebra. This justifies the identification of $\mathcal{A}$ with a von Neumann subalgebra of $\mathcal{B}$. From now on we will not distinguish between the two.

For all $n \in \mathbb{N}_0$ we have an isomorphism $\alpha_n : \mathcal{B}_n \to LH_n$ and we will see that we can combine these maps to a map $\alpha : \mathcal{B} \to LK$. Let

$$\alpha : \mathcal{B}^n \to LK$$

be the map defined on basic tensors by

$$\alpha(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = \alpha_0(x_0)\alpha_1(x_1)\cdots\alpha_n(x_n)$$

for all $n \in \mathbb{N}_0$ and $x_i \in \mathcal{A}_i$ ($i = 0, \ldots, n$). We will show that $\alpha$ is a trace preserving $*$-homomorphism. We start with trace preserving. Suppose that $x_i \in \mathcal{A}_i$ ($i = 0, \ldots, n$). Then

$$\tau(\alpha_0(x_0)\cdots\alpha_n(x_n)) = (\alpha_0(x_0)\cdots\alpha_n(x_n)\delta_e)(e)$$

$$= (\alpha_0(x_0)\delta_e \ast \cdots \ast \alpha_n(x_n)\delta_e)(e).$$

Now since the support of $x_i\delta_e$ is contained in $H_i$ ($i = 1, \ldots, n$) and $H_i \cap H_j = \{e\}$ when $i \neq j$, it follows that

$$(\alpha_0(x_0)\delta_e \ast \cdots \ast \alpha_n(x_n)\delta_e)(e) = (\alpha_0(x_0)\delta_e)(e) \cdots (\alpha_n(x_n)\delta_e)(e).$$

The right hand side is exactly $\tau(\alpha_0(x_0)) \cdots \tau(\alpha_n(x_n))$, so we get

$$\tau(\alpha_0(x_0)\cdots\alpha_n(x_n)) = \tau(\alpha_0(x_0)) \cdots \tau(\alpha_n(x_n))$$

$$= \tau'_0(x_0) \cdots \tau'_n(x_n),$$

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but here the right hand side is exactly $\tau'(x_0 \otimes x_1 \otimes \cdots \otimes x_n)$, and we conclude that $\alpha$ is trace preserving. Since the common support of $\alpha_i(x_i)\delta_e$ and $\alpha_j(x_j)\delta_e$ was $\{e\}$ when $i \neq j$, it follows that $\alpha_i(x_i)$ and $\alpha_j(x_j)$ commutes when $i \neq j$. Using this it is easy to show that $\alpha$ is multiplicative and respects the involution, i.e. the $*$-operation. Clearly $\alpha$ is also linear, so $\alpha$ is a trace preserving $*$-homomorphism. By Proposition A.4 we can extend $\alpha$ to a ultraweakly continuous trace preserving $*$-homomorphism $\alpha : \mathcal{B} \rightarrow \mathcal{L}K$, and we will argue that $\alpha$ is in fact a $*$-isomorphism. That $\alpha$ is injective should be clear since it preserves a faithful trace. The image of $\alpha$ is a von Neumann subalgebra of $\mathcal{L}K$ by Proposition A.3, so since it contains $\lambda_g$ for all $g \in K$ the image must be $\mathcal{L}K$, hence $\alpha$ is surjective.

After having combined the maps $\alpha_n$ $(n \in \mathbb{N}_0)$ into one map $\alpha : \mathcal{A} \rightarrow \mathcal{L}K$, we are interested in the actions $\theta^n$ and $\sigma$ defined earlier. We know that the $\sigma_g$ is linear for all $g \in \Lambda$. Let us show that $\sigma_g$ is actually a $*$-isomorphism for all $g \in \Lambda$. Let $g \in \Lambda$, then clearly $\sigma_g$ is bijective since $\sigma_e$ is the identity and $\sigma_0\sigma_h = \sigma_{gh}$ when also $h \in \Lambda$. Hence we only need to show that $\sigma_g$ is multiplicative and preserves the involution. Let us start with the involution. Since

$$ (g \cdot \xi^*)(h) = \xi((-g^{-1} \cdot h)) = \xi(g^{-1} \cdot (-h)) = (g \cdot \xi)^*(h) $$

for all $h \in K$, which shows that $g \cdot \eta^* = (g \cdot \eta)^*$ for all $\eta \in \ell^2(G)$. Because of this it easily follows that $\sigma_g(x^*) = (\sigma_g x)^*$ for all $x \in \mathcal{L}K$, but this exactly shows that $\sigma_g$ preserves the involution. To prove multiplicativity, suppose that $\eta, \eta' \in \ell^2(K)$. Then with the change of variable $h' = g^{-1} \cdot h''$ we see that

$$ (g \cdot (\eta \ast \eta'))(h) = \sum_{h' \in K} \eta(g^{-1} \cdot h - h')\eta'(h') $$

$$ = \sum_{h'' \in K} \eta(g^{-1} \cdot (h - h''))\eta'(g^{-1} \cdot h'') $$

$$ = ((g \cdot \eta) \ast (g \cdot \eta'))(h) $$

for all $h \in K$, which shows that $g \cdot (\eta \ast \eta') = (g \cdot \eta) \ast (g \cdot \eta')$, and it follows that $\sigma_g$ is multiplicative. Note that $\sigma_g$ is trace preserving since $g \cdot e = e$ for all $g \in \Lambda$, which implies that $\sigma_g\eta(e) = \eta(e)$. Because of this $\sigma_g$ is also ultraweakly continuous by Proposition A.4 (since $\mathcal{A}$ is clearly dense in $\mathcal{A}$).

As with $\sigma_g$ the action $\theta^n_g$ is actually a $*$-isomorphism for all $g \in \Lambda$, but in this case it is easier. That $\theta^n_g$ is a $*$-homomorphism follows directly from the fact that all the operations on $\mathcal{B}_n$ are defined pointwise. Now because $\theta_g\theta_h = \theta_{gh}$ for all $g, h \in \Lambda$ and $\theta_e$ is the identity, it follows that $\theta_g$ is bijective and hence a $*$-isomorphism for all $g \in \Lambda$. It is also easy to see that $\theta^n_g$ must be trace preserving since the trace is just the normalized counting measure and $h \mapsto g \cdot h$ is an automorphism of $H_n$ for all $g \in \Lambda$. We want to define an
Clearly implies that action of $\Lambda$ on $\mathcal{B}$ that agrees with $\theta^n$ when restricted to $LH_n$ ($n \in \mathbb{N}_0$). For this, let $g \in \Lambda$ and define a map $\theta_g : \mathcal{B}^o \to \mathcal{B}$ by

$$\theta_g(x_0 \otimes \cdots \otimes x_n) = \theta_g(x_0) \otimes \cdots \otimes \theta_g(x_n)$$

for all $n \in \mathbb{N}_0$ and $x_i \in \mathcal{B}_i$ ($i = 0, \ldots, n$). It is straightforward to check that $\theta_g$ defines a trace preserving $*$-homomorphism, since this is true for $\theta^n_g$ ($n \in \mathbb{N}_0$). By Proposition A.4 we can extend $\theta_g$ to a trace preserving $*$-homomorphism $\theta_g : \mathcal{B} \to \mathcal{B}$, and it must necessarily be injective since it preserves a faithful trace. The identities $\theta_g \theta_h = \theta_{gh}$ and $\theta_e = 1$ holds on $\mathcal{B}^o$ for all $g, h \in \Lambda$, and by continuity this holds on $\mathcal{B}$. This last two identities clearly implies that $\theta_g$ is surjective, so $\theta_g$ is in fact a $*$-isomorphism.

By now we are ready to state the following lemma, which proves certain things about the map $\alpha$ and summarizes some of the important facts from the discussion above.

**Lemma 6.12.** The map $\alpha$ restricts to a trace preserving $*$-isomorphism

$$\alpha : \mathcal{A} \to L\mathcal{G} \cap (L\Lambda)^{\prime}$$

with the property that $\alpha(e_n)$ lies in the center of $L\mathcal{G}_{n+1}$ and is given by the formula $\alpha(e_n) = p_n^{-3} \sum_{h \in H_n} \lambda_h$.

**Proof.** It follows from Lemma 6.5 that $L\mathcal{G} \cap (L\Lambda)^{\prime} = LK \cap (L\Lambda)^{\prime}$, since an element $L_x$ in $L\Lambda$ must satisfy that $x$ is constant on the sets of the form $\{ghg^{-1} : g \in \Lambda\}$ with $h \in G$. Hence, if we show that $LK \cap (L\Lambda)^{\prime} = \alpha(\mathcal{A})$, then $\alpha$ restricts to an isomorphism $\mathcal{A} \to L\mathcal{G} \cap (L\Lambda)^{\prime}$.

Before showing this, let us consider the actions $\theta$ and $\sigma$ on $\mathcal{B}$ and $LK$ respectively. Let $g \in \Lambda$ and $x \in \mathcal{B}_n$ for some $n \in \mathbb{N}_0$. From Lemma 6.11 we know that

$$\sigma_g \alpha(x) = \sigma_g \alpha_n(x) = \alpha_n(\theta_{g^{-1}} x) = \alpha(\theta_{g^{-1}} x),$$

so since $\sigma_g \alpha$ and $\alpha \theta_{g^{-1}}$ are both ultraweakly continuous $*$-homomorphisms and $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ generates $\mathcal{B}^o$, we get that

$$\sigma_g \alpha(x) = \alpha(\theta_{g^{-1}} x) \quad (6.4)$$

for all $x \in \mathcal{B}$ and all $g \in \Lambda$.

We constructed the group $G_0 \subseteq G$ so that $g \cdot h = ghg^{-1}$ for all $g \in \Lambda$ and $h \in K$, which means that for each $h \in K$

$$\{ghg^{-1} : g \in \Lambda\} = \{g \cdot h : g \in \Lambda\},$$

and if $L_0 \in LK$, then this shows that $L_0 \in (L\Lambda)^{\prime}$ if and only if $\eta(h) = \eta(g^{-1} \cdot h) = (g \cdot h)(h)$ for all $h \in K$ and $g \in \Lambda$. Or in other words $L_0 \in (L\Lambda)^{\prime}$ if and only if $L_0 = L_0 \eta = \sigma_g (L_0)$ for all $g \in \Lambda$. We are interested in showing
that \( \alpha(\mathcal{A}) = LG \cap (L \Lambda)' \) so apparently this can be done by showing that \( \sigma_g x = x \) for all \( g \in \Lambda \) if and only if \( x \in \alpha(\mathcal{A}) \), when \( x \in LK \). We already know that \( \alpha \) is an isomorphism \( \mathcal{B} \to LK \), so the identity (6.4) shows that
\[
\{ x \in LK \mid \forall g \in \Lambda : \sigma_g(x) = x \} = \alpha \left( \{ y \in \mathcal{B} \mid \forall g \in \Lambda : \theta_g y = y \} \right).
\]
Thus all we need to prove is that \( \mathcal{A} = \{ y \in \mathcal{B} \mid \forall g \in \Lambda : \theta_g y = y \} \). Given a subset \( C \subseteq \mathcal{B} \) we denote by \( C^\Lambda \) the set
\[
C^\Lambda = \{ x \in C \mid \forall g \in \Lambda : \theta_g x = x \}
\]
and with this notation, what we need to show is that \( \mathcal{B}^\Lambda = \mathcal{A} \). Since \( \theta_g \) is ultraweakly continuous for all \( g \in \Lambda \) it follows that \( \mathcal{B}^\Lambda \) is ultraweakly closed. Since \( \mathcal{B}^o \) is ultraweakly dense in \( \mathcal{B} \) and \( \mathcal{A}^o \) ultraweakly dense in \( \mathcal{A} \), we only need to show that \( (\mathcal{B}^o)^\Lambda = \mathcal{A}^o \), since continuity then ensures that \( \mathcal{B}^\Lambda = \mathcal{A} \).

Fix some \( n \in \mathbb{N}_0 \). Let \( U \) be the unitary from Proposition 5.5 making the two von Neumann algebras \( \mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_n \) and \( \ell^\infty(\mathcal{B}_0 + \cdots + \mathcal{B}_n) \) unitarily equivalent. It is easy to check that
\[
U^* \theta_g U = g \cdot (U^* x U),
\]
where \( g \cdot y \) denotes the function \( (g \cdot y)(h) = y(g^{-1} \cdot h) \) for all \( g \in \Lambda , h \in \mathcal{H}_0 + \cdots + \mathcal{H}_n \) and \( y \in \ell^\infty(\mathcal{H}_0 + \cdots + \mathcal{H}_n) \). The von Neumann algebra \( \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_n \) is spanned by elements of the form
\[
f_0 \otimes \cdots \otimes f_n
\]
where \( f_i = e_i \) or \( f_i = 1 - e_i \) \( (i = 0, \ldots, n) \), and it is straightforward to check that
\[
U^*(f_0 \otimes \cdots \otimes f_n)U = 1_{U_0 \times \cdots \times U_n}
\]
where either \( U_i = \{0\} \) or \( U_i = H_i \setminus \{0\} \) \( (i = 0, \ldots, n) \). Clearly elements of the form (6.5) span the set of elements \( y \in \ell^\infty(\mathcal{H}_0 + \cdots + \mathcal{H}_n) \) satisfying \( g \cdot y = y \), since the sets \( U_0 \times \cdots \times U_n \) are exactly the orbits of the action of \( \Lambda \) on \( \mathcal{H}_0 + \cdots + \mathcal{H}_n \) by Lemma 6.6. From this it follows that
\[
(\mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_n)^\Lambda = \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_n.
\]
Since this holds for all \( n \in \mathbb{N}_0 \) we see that
\[
\bigcup_{n=0}^\infty \overline{\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_n} = \bigcup_{n=0}^\infty (\overline{\mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_n})^\Lambda = \left( \bigcup_{n=0}^\infty \overline{\mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_n} \right)^\Lambda
\]
Now this is actually what we wanted to show since the left hand side is \( \mathcal{A}^o \) and the right hand side is \( (\mathcal{B}^o)^\Lambda \) because \( \mathcal{A} \) and \( \mathcal{B}_i \) are finite dimensional for all \( i \in \mathbb{N}_0 \) so that
\[
\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_n = \mathcal{A}_0 \odot \cdots \odot \mathcal{A}_n \quad \text{and} \quad \overline{\mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_n} = \mathcal{B}_0 \odot \cdots \odot \mathcal{B}_n
\]
for all \( n \in \mathbb{N}_0 \). Thus we have proved that \( B^\Lambda = \mathcal{A} \) and conclude that \( \alpha \) restricts to an isomorphism \( \mathcal{A} \rightarrow LG \cap (LA)' \). This is obviously still trace preserving.

Next we want to show that \( \alpha(e_n) = p_n^{-3} \sum_{h \in H_n} \lambda_h \) for all \( n \), but this follows from Lemma 6.11, since \( \alpha(e_n) = \alpha_n(e_n) \) by construction. Lastly, we will show that \( \lambda_g \alpha(e_n) \lambda_{g^{-1}} = \alpha(e_n) \). We see that

\[
\lambda_g \alpha(e_n) \lambda_{g^{-1}} = \lambda_g \left( p_n^{-3} \sum_{h \in H_n} \lambda_h \right) \lambda_{g^{-1}} = p_n^{-3} \sum_{h \in H_n} \lambda_{ghg^{-1}},
\]

and since the map \( h \mapsto ghg^{-1} \) is an automorphism of \( H_n \), because it is a normal subgroup of \( G_{n+1} \) by Lemma 6.7, it follows that

\[
\lambda_g \alpha(e_n) \lambda_{g^{-1}} = p_n^{-3} \sum_{h \in H_n} \lambda_{ghg^{-1}} = p_n^{-3} \sum_{h' \in H_n} \lambda_{h'} = \alpha(e_n).
\]

This shows that \( \alpha(e_n) \) is in the center of \( LG_{n+1} \).

\[\square\]

**Theorem 6.13.** The group \( G \) is a countable inner amenable i.c.c. group, whose group von Neumann algebra does not have property \( \Gamma \).

**Proof.** That \( G \) is countable i.c.c. was already established in Lemma 6.5, so we must show that \( G \) is inner amenable and that \( LG \) does not have property \( \Gamma \). We start by showing inner amenability, and we will do this by fulfilling the requirement of Theorem 3.6.

For each \( n \in \mathbb{N}_0 \) let \( \xi_n = p_n^{3/2} \alpha(e_n) \delta_e \), where \( \alpha \) is the isomorphism from Lemma 6.12. Since

\[
\|\xi_n\|^2 = p_n^3 \|\alpha(e_n) \delta_e\|^2 = p_n^3 \left\| p_n^{-3} \sum_{g \in H_n} \lambda_g \delta_e \right\|^2 = p_n^{-3} \left\| \sum_{g \in H_n} \delta_g \right\|^2 = 1, \tag{6.6}
\]

the sequence \( (\xi_n)_{n \geq 0} \) is a sequence of unit vectors in \( \ell^2(G) \). Moreover \( \xi_n(e) = (p_n^{-3/2} \sum_{g \in H_n} \lambda_g)(e) = p_n^{-3/2} \), so we see that

\[
\xi_n(e) \longrightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{6.7}
\]

We know from Lemma 6.12 that \( \alpha(e_n) \in \mathcal{Z}(G_{n+1}) \), so whenever \( g \in G_N \) with \( N \leq n + 1 \) it holds that \( \pi_g \xi_n = \xi_n \). This shows that the sequence \( (\pi_g \xi_n - \xi_n)_{n \geq 0} \) is eventually zero for all \( g \in G \), since all \( g \in G \) is in \( G_N \) for some \( N \in \mathbb{N}_0 \). In particular

\[
\|\pi_g \xi_n - \xi_n\|_2 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \tag{6.8}
\]
Now by (6.6), (6.7) and (6.8) the sequence $(\xi_n)_{n \geq 0}$ satisfies the hypothesis of Theorem 3.6, proving inner amenability of $G$.

Now we will see that $LG$ does not have property $\Gamma$; this will be done by contradiction. So suppose that $LG$ has property $\Gamma$. Since $G$ is countable we can choose a sequence of unitaries $x_n \in LG$ ($n \in \mathbb{N}$) with $x_n(e) = \tau(x_n) = 0$ such that

$$\|\lambda_g x_n - x_n \lambda_g\|_\tau \to 0 \quad \text{as} \quad n \to \infty \quad (6.9)$$

for all $g \in G$. What we want is to show that $\tau(x_n) \to 1$ as $n \to \infty$, because this will have the absurd consequence that $0 = 1$, and we will be forced to conclude that $LG$ does not have property $\Gamma$. It is straightforward to check that $\|\pi_g x - x\|_\tau = \|\lambda_g x - x\|_\tau$ for all $g \in G$ and $x \in LG$, so if we let $\hat{\xi}_n = x_n \delta_e$ for all $n \in \mathbb{N}$, then

$$\|\pi_g \hat{\xi}_n - \hat{\xi}_n\|_2 = \|\pi_g x_n - x_n\|_\tau = \|\lambda_g x_n - x_n \lambda_g\|_\tau$$

for all $g \in G$ and $n \in \mathbb{N}$. Because of (6.9) this shows that $(\hat{\xi}_n)_{n \geq 1}$ is a sequence of almost $\pi$-invariant vectors. In particular the restriction $\pi|_\Lambda$ of $\pi$ to $\Lambda$ is a unitary representation of $\Lambda$ and $(\hat{\xi}_n)_{n \geq 1}$ is a sequence of almost $\pi|_\Lambda$-invariant vectors. Let $P$ denote the projection of $\ell^2(G)$ onto the closed subspace of $\pi|_\Lambda$-invariant vectors, and for each $n \in \mathbb{N}$ let $y_n = \mathcal{E}(x_n)$, where $\mathcal{E}$ denotes the conditional expectation of $LG$ onto $LG \cap (\Lambda \Lambda)'$. A vector $\eta \in \ell^2(G)$ is $\pi|_\Lambda$-invariant if and only if $\eta(h) = \eta(g^{-1}hg)$ for all $h \in G$ and $g \in \Lambda$, so if $L_\eta \in LG$, then $\eta$ is $\pi|_\Lambda$-invariant if and only if $L_\eta \in LG \cap (\Lambda \Lambda)'$. This together with Proposition A.2 shows that $P \hat{\xi}_n = \mathcal{E}(x_n) \delta_e$ for all $n \in \mathbb{N}$. Now since $\Lambda$ has property (T) we get by Proposition 6.3 that

$$\|x_n - y_n\|_\tau = \|\xi_n - P \xi_n\|_2 \to 0 \quad \text{as} \quad n \to \infty. \quad (6.10)$$

For each $N \in \mathbb{N}_0$ there is a canonical copy of $\mathbb{Z}$ appearing in the factor $K_N \times \mathbb{Z}$ of $G_{N+1}$ generated by the element $(0, 1)$, denoted by $g_N$. This $g_N$ commutes with $K_N \subset G_{N+1}$, so $\lambda_{g_N}$ commutes with $L K_N$. In particular, $\lambda_{g_N}$ commutes with $\mathcal{E}_{L K_N}(y_n)$ where $\mathcal{E}_{L K_N}$ denotes the conditional expectation of $LG$ onto $L K_N$ with respect to the usual trace on $LG$. Because of this we can write

$$\lambda_{g_N} y_n - y_n \lambda_{g_N} = \lambda_{g_N} (y_n - \mathcal{E}_{L K_N}(y_n)) - (y_n - \mathcal{E}_{L K_N}(y_n)) \lambda_{g_N}.$$  

If $x \in LG$ with $\text{supp}(x \delta_e) \subset C$ for some $C \subset G$, then

$$\text{supp}(\lambda_g x \delta_e) \subset gC \quad \text{and} \quad \text{supp}(x \lambda_g \delta_e) \subset C g \quad (6.11)$$

for all $g \in G$. If we identify $\ell^2(G) \cong \ell^2(G \setminus K_N) \oplus \ell^2(K_N)$ in the natural way, then $\text{supp}(y_n \delta_e - \mathcal{E}_{L K_N}(y_n) \delta_e) \subset G \setminus K_N$ since

$$(y_n - \mathcal{E}_{L K_N}(y_n)) \delta_e = (1 - q_N) y_n \delta_e$$
by Proposition A.2, where $q_N$ denotes the projection onto $\ell^2(K_N)$. Now by Lemma 6.5 we have $LG \cap (LA)' = LK \cap (LA)'$, so in particular $y_n \in LK$, and therefore

$$\text{supp}(y_n \delta_e - E_{LK_N}(y_n) \delta_e) \subset K \setminus K_N.$$ 

The sets $g_N(K \setminus K_N)$ and $(K \setminus K_N)g_N$ are disjoint by construction of $G$, so since the supports of

$$\lambda_{g_N}(y_n - E_{LK_N}(y_n)) \delta_e \quad \text{and} \quad (y_n - E_{LK_N}(y_n)) \lambda_{g_N} \delta_e,$$

are contained in $g_N(K \setminus K_N)$ and $(K \setminus K_N)g_N$ respectively, these two elements must be orthogonal. Now computing the norm of (6.11) we get

$$\| \lambda_{g_N} y_n - y_n \lambda_{g_N} \|_2 = \| \lambda_{g_N} (y_n - E_{LK_N}(y_n)) \delta_e \|_2^2 + \| (y_n - E_{LK_N}(y_n)) \lambda_{g_N} \delta_e \|_2^2$$

$$= 2 \| y_n - E_{LK_N}(y_n) \|_F^2. \quad (6.12)$$

Now the left hand side is the square of the trace norm of $\lambda_{g_N} y_n - y_n \lambda_{g_N}$, but this can be estimate by

$$\| \lambda_{g_N} y_n - y_n \lambda_{g_N} \|_\tau \leq \| \lambda_{g_N} (y_n - x_n) \|_\tau + \| \lambda_{g_N} x_n - x_n \lambda_{g_N} \|_\tau$$

$$+ \| (x_n - y_n) \lambda_{g_N} \|_\tau$$

Since we have shown that the first term on the right hand side converges to 0 as $n \to \infty$ and the second term does the same by assumption, we get that

$$\| \lambda_{g_N} y_n - y_n \lambda_{g_N} \|_\tau \to 0 \quad \text{as} \quad n \to \infty. \quad (6.13)$$

for all $N \in \mathbb{N}_0$. Fix some $N \in \mathbb{N}_0$. The projections $1 - e_k \ (k \in \mathbb{N})$ all commute, so the product $\prod_{k=N}^{N+n} (1 - e_k) \ (n \in \mathbb{N})$ forms a decreasing sequence of projections inside $\mathcal{A}$. Thus is is strong operator convergent to the projection $f_N = \prod_{k=N}^{\infty} (1 - e_k)$, with 7

$$\| f_N \xi \|_2^2 = \lim_{n \to \infty} \left\| \prod_{k=N}^{N+n} (1 - e_k) \xi \right\|_2^2 = \lim_{n \to \infty} \prod_{k=N}^{N+n} \| (1 - e_k) \xi_k \|^2.$$ 

Using that $\tau'(f_N) = \| f_N \xi' \|_2^2$ and $\tau'_k(1 - e_k) = \| (1 - e_k) \xi_k' \|^2$ since both $f_N$ and $1 - e_k$ are projections, we get

$$\tau'(f_N) = \lim_{n \to \infty} \prod_{k=N}^{N+n} \tau'_k(1 - e_k) = \lim_{n \to \infty} \prod_{k=N}^{N+n} (1 - p_k^{-3}) = \prod_{k=N}^{\infty} (1 - p_k^{-3}).$$

7Remember that $\xi'$ was the distinguished vector from the infinite tensor product Hilbert space $\delta'$ and $\tau'$ the trace on $\mathcal{A}$ given by $x \mapsto (x \xi' \mid \xi')$.
Suppose that \( n \in \mathbb{N}_0 \) and that \( x \) is in the unit ball of \( \mathcal{A}_N \otimes \cdots \otimes \mathcal{A}_{N+n} \). By Proposition 5.5 we can identify \( \mathcal{A}_N \otimes \cdots \otimes \mathcal{A}_{N+n} \) and \( \ell^\infty(\mathbb{Z}^{n+1}_2) \) in a trace preserving way, such that \((1 - e_N) \otimes \cdots \otimes (1 - e_{N+n})\) corresponds to the indicator function \( \chi_n \) on \((1, \ldots, 1) \in \mathbb{Z}^{n+1}_2\). By Lemma 6.10 we get that
\[
\|x - \tau'(x) 1\|_{\tau'} \leq 4(1 - \tau'(\chi_n))^{1/2} = 4 \left(1 - \prod_{k=N}^{N+n} (1 - p_k^{-3}) \right)^{1/2}.
\]

Since \( 0 \leq 1 - p_i^{-3} < 1 \) for all \( i \in \mathbb{N} \) we know that \( \tau'(f_N) \leq \prod_{k=N}^{N+n} (1 - p_k^{-3}) \) and therefore
\[
\|x - \tau'(x) 1\|_{\tau'} \leq 4(1 - \tau'(f_N))^{1/2}.
\]

We know that \( \bigcup_{k=0}^\infty \mathcal{A}_N \otimes \cdots \otimes \mathcal{A}_{N+k} \) is strong operator dense in \( \bigotimes_{k=0}^\infty \mathcal{A}_k \), so by Kaplansky's Density Theorem [Zhu, Theorem 19.5] the unit ball of \( \bigcup_{k=0}^\infty \mathcal{A}_N \otimes \cdots \otimes \mathcal{A}_{N+k} \) is strong operator dense in the unit ball of \( \bigotimes_{k=0}^\infty \mathcal{A}_k \).

Now since the trace norm \( \| \cdot \|_{\tau'} \) is strong operator continuous it follows that
\[
\|x - \tau'(x) 1\|_{\tau'} \leq 4(1 - \tau'(f_N))^{1/2}
\]
holds for all \( x \) in the unit ball of \( \bigotimes_{k=0}^\infty \mathcal{A}_k \). By construction \( y_n \) commutes with \( LA \) for all \( n \in \mathbb{N} \) and we will see that \( \mathcal{E}_{LK_N}(y_n) \) commutes with \( LA \) for all \( n \in \mathbb{N} \) as well. Let \( \eta_n := y_n \delta_e \). Then \( y_n \) commutes with \( LA \) if and only if \( \eta_n \) is constant on sets of the form \( \{ h g h^{-1} : g \in \Lambda \} \) when \( h \in K \).

Let \( \eta_n' := \mathcal{E}_{LK_N}(y_n) \delta_e \). Then by Proposition A.2 \( \eta_n' = \mathcal{E}_{LK_N}(y_n) \delta_e \) is just the projection of \( \eta_n \) onto \( L^2(K_N) \), i.e. \( \eta_n \) multiplied with the indicator function on \( K_N \). Since \( g K_N g^{-1} = K_N \) it follows that \( \eta_n' \) is also constant on sets of the form \( \{ h g h^{-1} : g \in \Lambda \} \) when \( h \in K \); it is just zero on some of the sets where \( \eta_n \) might have been non-zero. Hence \( \mathcal{E}_{LK_N}(y_n) \) commutes with \( LA \) for all \( n \in \mathbb{N} \) as well. In other words \( \mathcal{E}_{LK_N}(y_n) \in \mathcal{L}(G) \cap (\mathcal{L}(\Lambda)^\dagger \circ \mathcal{L}(\Lambda)') \), so by Lemma 6.12 we can choose some \( a_n \in \mathcal{A} \) with \( \alpha(a_n) = \mathcal{E}_{LK_N}(y_n) \). Since both \( \mathcal{E}_{LK_N} \) and \( y_n \) is of norm 1 we know that \( \| \mathcal{E}_{LK_N}(y_n) \| \leq 1 \). Now since \( \alpha \) is an isomorphism \( \alpha \) is also an isometry, so we know that \( \| a_n \| \leq 1 \) as well. It is not hard to see from the construction of \( \alpha \) that \( \alpha \) maps \( \bigotimes_{k=0}^\infty \mathcal{A}_k \) onto \( \mathcal{L}(K_N) \) and therefore it must be the case that \( a_n \in \bigotimes_{k=0}^\infty \mathcal{A}_k \). Because of this it follows that
\[
\| a_n - \tau'(a_n) 1\|_{\tau'} \leq 4(1 - \tau'(f_N))^{1/2}
\]
for all \( n \in \mathbb{N} \). If we recall that both \( \alpha \) and \( \mathcal{E}_{LK_N} \) are trace preserving we get
\[
\tau'(a_n) = \mathcal{E}_{LK_N}(y_n) = \tau(y_n),
\]
and furthermore
\[
\| a_n - \tau'(a_n) 1\|_{\tau'} = \| \alpha(a_n - \tau(y_n) 1\|_{\tau} = \| \mathcal{E}_{LK_N}(y_n) - \tau(y_n) 1\|_{\tau}.
\]

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Remember that \( y_n \in \mathcal{L}(K) \) so \( \text{supp} \eta_n \subseteq K \)
To summarize we now know that for all \( n, N \in \mathbb{N} \) the inequality
\[
\| E_{\mathcal{L}K_N}(y_n) - \tau(y_n)1 \|_\tau \leq 4(1 - \tau(f_N))^{1/2}.
\]
holds. Let us examine the limit as \( N \to \infty \) of the right hand side. By [Rud, Theorem 15.5] the product \( \prod_{k=0}^{\infty}(1 - p_k^{-3}) \) is strictly positive since the sequence \( p_1^{-3}, p_2^{-3}, \ldots \) is summable with \( 0 \leq p_k^{-3} < 1 \) for all \( k \in \mathbb{N}_0 \). Now that the limit is positive clearly has the consequence
\[
\tau(f_N) = \prod_{k=N}^{\infty}(1 - p_k^{-3}) \to 1 \quad \text{as} \quad N \to \infty,
\]
which implies that
\[
\sup_{n \in \mathbb{N}} \| E_{\mathcal{L}K_N}(y_n) - \tau(y_n)1 \|_\tau \to 0 \quad \text{as} \quad N \to \infty. \quad (6.14)
\]
Now we are almost done. Let \( \varepsilon > 0 \). We know that \( \| \tau(y_n)1 - \tau(x_n)1 \|_\tau = |\tau(y_n - x_n)| \) and by Cauchy-Schwarz \( |\tau(y_n - x_n)| \leq \| y_n - x_n \|_\tau \), so by (6.10) and (6.13) we can choose \( n_0 \in \mathbb{N} \) so that
\[
\| y_n - y_n \|_\tau, \quad \| y_n - E_{\mathcal{L}K_N}(y_n) \|_\tau \quad \text{and} \quad \| \tau(y_n)1 - \tau(x_n)1 \|_\tau
\]
are all strictly less than \( \frac{\varepsilon}{4} \) for \( n \geq n_0 \). By (6.14) we can choose some \( N \in \mathbb{N} \) such that \( \| E_{\mathcal{L}K_N}(y_n) - \tau(y_n)1 \|_\tau < \frac{\varepsilon}{4} \) for all \( n \in \mathbb{N} \), and finally by the triangle inequality we get
\[
\| x_n - \tau(x_n)1 \|_\tau \leq \| x_n - y_n \|_\tau + \| y_n - E_{\mathcal{L}K_N}(y_n) \|_\tau + \| E_{\mathcal{L}K_N}(y_n) - \tau(y_n)1 \|_\tau + \| \tau(y_n)1 - \tau(x_n)1 \|_\tau < \varepsilon
\]
for all \( n \geq n_0 \). This shows that \( \| x_n - \tau(x_n)1 \|_\tau \to 0 \) as \( n \to \infty \), and since
\[
|1 - \tau(x_n)| = \| x_n \|_\tau - \| \tau(x_n)1 \|_\tau \leq \| x_n - \tau(x_n)1 \|_\tau
\]
it follows that \( \tau(x_n) \to 1 \) as \( n \to \infty \), yielding the desired contradiction. We are therefore forced to conclude that \( \mathcal{L}G \) does not have property \( \Gamma \).

**Concluding remark** This concludes the proof from [Vae], and thus disproves that a the group von Neumann algebra of a countable discrete inner amenable \( i.c.c. \) group always has property \( \Gamma \). As mentioned in Lemma 6.5 the group \( G \) is not finitely generated, so in principle it is still possible that inner amenability of the group implies property \( \Gamma \) of the group von Neumann algebra, if we in addition to the other demands require the group to be finitely generated.
A Various results

There are a few results that we need Section 6, which we do not prove. These results concern the conditional expectation, \( \ast \)-homomorphisms between von Neumann algebras and the Gelfand transform on \( \ell^1(G) \) for some finite group \( G \).

**Theorem A.1.** If \( \mathcal{M} \) is a von Neumann algebra with a faithful normal trace \( \tau \) and \( \mathcal{N} \) is a von Neumann subalgebra, then there exists a linear operator \( E : \mathcal{M} \to \mathcal{N} \) of norm one, with the properties:

(i) It is trace preserving, i.e. \( \tau \circ E = \tau \);

(ii) It is idempotent, i.e. \( E^2 = E \);

(iii) For all \( x \in \mathcal{M} \) and \( y \in \mathcal{N} \), \( \tau(E(x)y) = \tau(xy) \).

This map is called the conditional expectation of \( \mathcal{M} \) onto \( \mathcal{N} \) with respect to \( \tau \). It is sometimes denoted \( \) if more than one conditional expectation is considered.

One way to think of the conditional expectation is as a projection in the GNS-construction of \( \mathcal{M} \) with respect to \( \tau \). If we form the GNS constructions \( L^2(\mathcal{M}, \tau) \) of \( \mathcal{M} \) with respect to \( \tau \), then \( E \) (or rather its extension to \( L^2(\mathcal{M}, \tau) \)) will be the orthogonal projection of \( L^2(\mathcal{M}, \tau) \) onto the closed subspace \( L^2(\mathcal{N}, \tau) \). A special case of this is stated in the following lemma.

**Proposition A.2.** Let \( G \) be a discrete group and let \( \mathcal{N} \) be a von Neumann subalgebra of \( L^G \). Let \( E : L^G \to \mathcal{N} \) denote the conditional expectation of \( L^G \) onto \( \mathcal{N} \) with respect to the standard trace on \( L^G \), and let \( P \) denote the orthogonal projection of \( \ell^2(G) \) onto the closure of \( \mathcal{N} \delta_e \). Then for all \( x \in LG \) we get \( P(x\delta_e) = E(x)\delta_e \). If \( \mathcal{N} = LH \) for some subgroup \( H \) of \( G \), then the closure of \( \mathcal{N} \delta_e \) is equal to \( \ell^2(H) \).

**Proposition A.3.** The image of a von Neumann algebra under a ultraweakly continuous \( \ast \)-homomorphism is itself a von Neumann algebra.

**Proposition A.4.** Suppose that \( \mathcal{N} \) and \( \mathcal{M} \) are von Neumann algebras with faithful normal traces \( \tau \) and \( \tilde{\tau} \) respectively. If \( \mathcal{N}_0 \) is a weak operator dense \( \ast \)-subalgebra and \( \varphi : \mathcal{N}_0 \to \mathcal{M} \) is a trace preserving \( \ast \)-homomorphism, then \( \varphi \) extends to an ultraweakly continuous trace preserving \( \ast \)-homomorphism \( \mathcal{N} \to \mathcal{M} \).

**Proposition A.5.** For a finite abelian group \( G \), consider the Banach \( \ast \)-algebra \( \ell^1(G) \). If \( \hat{G} \) denotes the dual group, then the maximal ideal space of \( \ell^1(G) \) may be identified with \( \hat{G} \) and the Gelfand transform \( \Gamma : \ell^1(G) \to C(\hat{G}; \mathbb{C}) \) is a \( \ast \)-isomorphism. With this maximal ideal space the Gelfand transform is given by \( \Gamma(\delta_g)(\varphi) = \varphi(g) \) for \( g \in G \) and \( \varphi \in \hat{G} \).
References


