E-Theory for $C^*$-Algebras over Topological Spaces

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Abstract

We define $E$-theory for separable $C^*$-algebras over second countable sober spaces and prove that this satisfies excision and has the universal property, as it very well should. We prove that the $E$-theory over a second countable space may be approximated by $E$-theory over finite spaces. We use this to give a very effective criterion for determining when morphisms in the $E$-theory category $\mathcal{E}(X)$ are invertible. We also define filtrated $K$-theory and give simple UCT results using the $E$-theoretic bootstrap class. Finally we apply our finite approximation to give a classification result.

Resumé (på dansk)

Vi definerer $E$-teori for separable $C^*$-algeber over andentællelige sobre rum og viser at dette opfylder udskæring og har den universale egenskab, netop som det burde. Vi viser at $E$-teori over andentællelige rum kan approksimeres af $E$-teori over endelige rum. Vi benytter dette til at give et effektivt kriterium for at afgøre hvornår morfier i $E$-teorikategorien $\mathcal{E}(X)$ er invertible. Vi definerer også filtreret $K$-teori og giver simple UCT resultater ved at gøre brug af den $E$-teoretiske bootstrapklasse. Til sidst benytter vi vores endelige approksimation til at give et klassifikationsresultat.
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>vi</td>
</tr>
<tr>
<td>Prerequisites</td>
<td>vi</td>
</tr>
<tr>
<td>Acknowledgement</td>
<td>vi</td>
</tr>
</tbody>
</table>

### 1 Introduction

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
</tbody>
</table>

### 2 $C^*$-algebras over Topological Spaces

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 The Basics</td>
<td>3</td>
</tr>
<tr>
<td>2.2 Restriction to Sober Spaces</td>
<td>5</td>
</tr>
<tr>
<td>2.3 Basic Constructions</td>
<td>7</td>
</tr>
<tr>
<td>2.4 Adjoint Functors</td>
<td>10</td>
</tr>
</tbody>
</table>

### 3 $E$-Theory

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Asymptotic Morphisms</td>
<td>15</td>
</tr>
<tr>
<td>3.2 Composition Product of Asymptotic Morphisms</td>
<td>18</td>
</tr>
<tr>
<td>3.3 The Basics of $E$-Theory</td>
<td>24</td>
</tr>
<tr>
<td>3.4 Relations to $K$-Theory - Part I</td>
<td>28</td>
</tr>
<tr>
<td>3.5 The Connes-Higson Construction</td>
<td>31</td>
</tr>
<tr>
<td>3.6 Excision</td>
<td>32</td>
</tr>
<tr>
<td>3.7 The Universal Property</td>
<td>37</td>
</tr>
</tbody>
</table>

### 4 $E$-Theory over Finite Spaces

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 The Canonical Filtration</td>
<td>43</td>
</tr>
<tr>
<td>4.2 Relations to $K$-Theory - Part II</td>
<td>45</td>
</tr>
<tr>
<td>4.3 $E$-Theory as Homotopy Theory</td>
<td>47</td>
</tr>
<tr>
<td>4.4 Approximation by Finite Spaces</td>
<td>50</td>
</tr>
</tbody>
</table>

### 5 Filtrated $K$-Theory

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 $E(X)$ as a Triangulated Category</td>
<td>56</td>
</tr>
<tr>
<td>5.2 Ideals in Triangulated Categories</td>
<td>61</td>
</tr>
<tr>
<td>5.3 Projective Objects and Derived Functors</td>
<td>63</td>
</tr>
<tr>
<td>5.4 Getting Enough Projective Objects</td>
<td>66</td>
</tr>
<tr>
<td>5.5 The Universal Property of Filtrated $K$-Theory</td>
<td>70</td>
</tr>
<tr>
<td>5.6 A Universal Coefficient Theorem</td>
<td>72</td>
</tr>
<tr>
<td>5.7 The $E$-Theoretic Bootstrap Class</td>
<td>74</td>
</tr>
<tr>
<td>5.8 $KK$-Theory and Applications</td>
<td>77</td>
</tr>
</tbody>
</table>

### A Bott Periodicity

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1 The Toeplitz Algebra</td>
<td>80</td>
</tr>
<tr>
<td>A.2 The Periodicity Theorem in $C^\ast \text{alg}(X)$</td>
<td>82</td>
</tr>
</tbody>
</table>

### B Triangulated Categories

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
</table>

### C Notation

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.1 Notation</td>
<td>88</td>
</tr>
</tbody>
</table>
Preface

The main purpose of this thesis is to give a detailed introduction to $E$-theory for $C^*$-algebras over topological spaces as given in [7], and to prove some theorems for this $E$-theory. In the very last section we apply some of these theorems to obtain a classification result. No new results are given in this thesis.

Prerequisites

The reader of this thesis is expected to be familiar with the basics of $K$-theory for operator algebra. However, it is not required that the reader has any prior knowledge of either $E$-theory nor $KK$-theory. If the reader is already familiar with the $E$-theory constructed by Higson and Connes in [4], we suggest that the reader skips Section 3 with exception of Section 3.7. The constructions done are very similar to the original constructions and some of these are very technical.

If the reader is not familiar with Joachim Cuntz’ proof of Bott periodicity, we have included it in the appendix as Section A. We have, furthermore, adapted it so that it holds for $C^*$-algebras over topological spaces.

It will be a great advantage when reading Sections 4.3 and 4.4 that the reader has knowledge of homotopy theory for topological spaces. However, this is not required unless you want to go into detail with Lemma 4.4.2.

In order to understand Section 5, the reader is expected to be familiar with the basics of homological algebra, including basic category theory and the construction of derived functors. If the reader is only familiar with homological algebra in module categories, it will be reassuring to know that this theory converts to homological algebra in abelian categories without any effort. We will be doing homological algebra in triangulated categories. For the reader not familiar with triangulated categories we have written a small section on triangulated categories, Section B. We should remark that the version of the octahedral axiom which we use is not any of the standard versions from [21].

As a final comment, we should mention that in Section C all standard notation is mentioned. Hence if the reader is not sure of a notational meaning, they should try looking it up in the appendix.

Acknowledgement

I would like to thank Sara Arklint for the long discussion on the category $\mathcal{NT}$ which we had. I’m sorry we almost missed lunch. I would also like to thank Rasmus Bentmann and Rohan Lean for helping me get to the bottom of how $\mathcal{NT}$ is defined.\footnote{This is discussed in Remark 5.4.5.} Applause goes out to my friends and family for actually leaving me alone when I asked them to. I realise that you wanted to be there for me, but I just get a lot more work done when I am not disturbed every second moment. Also, I would like to thank my brother, Sebastian, for updating my Mp3-player with a bunch of great music which I had never heard of until you introduced me to it. Because of you this writing process has been a lot more bearable. A
special thanks goes out to the Ethiopian ancestors of today’s Oromo people, who were believed to have been the first to recognize the energizing effect of the coffee bean plant. Without coffee, this thesis would probably not have exceeded 20 pages.

My final and greatest gratitude goes out to my advisor Ryszard Nest for introducing me to this wonderful topic and for answering all of my, sometimes stupid, questions. This thesis would not have been possible without you.

Jamie Gabe  
Copenhagen, March 2012
1 Introduction

This thesis is within the area of classification of $C^*$-algebras. More precisely, within Elliott’s program to classify certain $C^*$-algebras by means of $K$-theoretic data. Often when doing classification, one would work with $KK$-theory. However, $KK$-theory has the disadvantage that it does not satisfy excision.

This was the motivation for Nigel Higson to construct $E$-theory which does satisfy excision. In [11], Higson proved the existence of $E$-theory by abstract category theory, in terms of a functor $C^*_{\text{sep}} \to \mathcal{E}$ which was the universal half-exact, stable, homotopy functor. However, Higson only proved the existence of such a functor, but did not manage to construct the functor, and thus $E$-theory was more or less impossible to actually apply. But shortly after, in [4], Higson together with Alain Connes managed to construct such an $E$-theory functor. This construction was, as is the case for us, done only for separable $C^*$-algebras by considering asymptotic morphisms between such $C^*$-algebras.

Roughly speaking, an asymptotic morphism is a family of maps between $C^*$-algebras, indexed by $[0, \infty)$, such that it looks like a $\ast$-homomorphism when going to $\infty$. In [23] Efton Park and Jody Trout define a generalisation of $E$-theory for $C_0(X)$-algebras. This generalisation seems reasonable, since it satisfies the same universal property as the original $E$-theory. In [7] this was generalised even further to separable $C^*$-algebras over topological spaces, more precisely over second countable sober spaces. We intend to reconstruct this latter variant of $E$-theory from the start, leaving out no details. As a part of the construction we, of course, show that this $E$-theory satisfies excision and has the universal property.

Roughly speaking a $C^*$-algebra over a topological space $X$ is a $C^*$-algebra $A$ such that we for every open subset $U$ of $X$ associate an ideal of $A$ denoted $A(U)$. These should behave nicely with respect to the lattice structure of $\mathcal{O}(X)$. It should be noted that we do not have any Hausdorff assumptions on our space $X$. As an application of the universal property of $E$-theory over topological spaces we find that adjoint functors between categories of separable $C^*$-algebras over topological spaces, descend to adjoint functors between the corresponding $E$-theory, assuming they behave rather nicely. We use this to establish quite a few isomorphisms of $E$-groups.

After constructing this new version of $E$-theory we of course intend to apply it. Our first main application is as follows. If $X$ is a second countable sober space we can construct an approximation of finite spaces $(X_n)_{n \in \mathbb{N}}$ of $X$. There is a canonical forgetful functor $\mathcal{E}^*_{\text{sep}}(X) \to \mathcal{E}^*_{\text{sep}}(X_n)$ which allows us to consider $C^*$-algebras over $X$ as $C^*$-algebras over $X_n$ for every $n$. Our first main theorem is that we get a natural short exact sequence

$$\lim_{\leftarrow} E_{s+1}(X_n; A, B) \to E_s(X; A, B) \to \lim_{\leftarrow} E_s(X_n, A, B)$$

for all separable $C^*$-algebras over $X$, $A$ and $B$. An important application of this short exact sequence is our second main result. It says that for any second countable sober space $X$, a morphism $A \to B$ in the $E$-theory category $\mathcal{E}(X)$ is
an isomorphism if and only if it induces an invertible morphism $A(U) \to B(U)$ in $E$ for every $U \in \mathcal{O}(X)$.

When this is done we move our attention to category theory. We prove that $\mathcal{E}(X)$ is (equivalent to) a triangulated category. We use this to construct filtrated $K$-theory for $E$-theory over finite sober spaces, as is done for $KK$-theory in [18]. We should note that the methods are a lot simpler in $E$-theory, since $E$-theory satisfies excision. Using this we develop UCT results and in particular we construct an $E$-theoretic bootstrap class $B_E(X)$ which helps us express the UCT results. By applying our second main theorem above, we also get a powerful result for elements in the bootstrap class. To elaborate, let $X$ be any second countable sober space and $A$ and $B$ be $C^*$-algebras over $X$ in the $E$-theoretic bootstrap class. Then a morphism $A \to B$ in $\mathcal{E}(X)$ is an isomorphism if and only if the induced map $K_*(A(U)) \to K_*(B(U))$ is an isomorphism for every $U \in \mathcal{O}(X)$.

At the end of the thesis we will discuss how one can apply $E$-theory in order to get classification results. We end the thesis by applying the above main theorems in order to get a classification which is due to Marius Dadarlat and Ralf Meyer in [7]. It says that if $A$ is a separable, nuclear $C^*$-algebra with Hausdorff primitive ideal space $X$, and if every ideal in $A$ is $E$-contractible then

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong C_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}.$$
2 $C^*$-algebras over Topological Spaces

Throughout this entire thesis we will be working with $C^*$-algebras over topological spaces. These are a generalisation of extensions of $C^*$-algebras, $C_0(X)$-algebras and $C^*$-bundles. Roughly speaking a $C^*$-algebra over a topological space is a $C^*$-algebra together with some information about its ideals related to the open subsets of $X$.

We will define these and try to give the reader an intuitive idea of what we are working with. We will also show that it essentially is no loss of generality to only consider $C^*$-algebras over sober spaces. We will do a lot of basic constructions with $C^*$-algebras over $X$. Amongst others we construct cylinders which allows us to define homotopies, and we construct pull-backs. At the end of this section we construct functors between categories of $C^*$-algebras over topological spaces and prove certain adjunction relations. This will be a nice tool once we have the universal property of $E$-theory.

2.1 The Basics

In this section we define $C^*$-algebras over topological spaces and give a few easy examples of such.

Let us first make one thing clear. In this thesis all ideals are assumed to be two-sided and closed.

Recall (e.g. [24], §3.13 and §4.1) that an ideal $p$ in a $C^*$-algebra $A$ is called primitive if it is the kernel of an irreducible representation $A \to B(H)$. We let $\text{Prim}(A)$\(^2\) denote the space of all primitive ideals in $A$, which comes naturally equipped with a topology called the Jacobsen topology.\(^3\) There is a lattice isomorphism between the complete lattice $\mathcal{O}(\text{Prim}(A))$ of open subsets of $\text{Prim}(A)$ and the complete lattice $\mathcal{I}(A)$ of ideals in $A$ given by

$$\mathcal{I}(A) \ni J \mapsto \{p \in \text{Prim}(A) \mid J \nsubseteq p\} \in \mathcal{O}(\text{Prim}(A)).$$

Hence we will in general make no distinction of $\mathcal{O}(\text{Prim}(A))$ and $\mathcal{I}(A)$. We are now ready to define $C^*$-algebras over topological spaces.

Definition 2.1.1. Let $X$ be a topological space. A $C^*$-algebra over $X$ is a $C^*$-algebra $A$ together with a continuous map $\psi : \text{Prim}(A) \to X$.

For any open subset $U$ of $X$, we let $A(U)$ denote the ideal $\psi^{-1}(U) \in \mathcal{O}(\text{Prim}(A)) = \mathcal{I}(A)$.

For any closed subset $S$ of $X$, let $A(S)$ denote the quotient $A/A(X \setminus S)$. For any element $a \in A$ we let $\|a\|_S$ denote the norm of the image of element in the quotient.

More generally, for a locally closed set $Y \subseteq X$, i.e. $Y = U_1 \setminus U_2$ for two open sets $U_2 \subseteq U_1$ of $X$, let $A(Y)$ denote the quotient $A(U_1)/A(U_2)$.\(^4\)

\(^2\)Sometimes denoted $\bar{A}$ or $\text{Pr}(A)$ in the literature.

\(^3\)Sometimes called the hull-kernel topology in the literature.

\(^4\)One can easily show that the quotient does not depend on the choice of $U_1$ and $U_2$. See [19] Lemma 2.15 for more details.
Let $A$ and $B$ be $C^*$-algebras over $X$. A $*$-homomorphism $\phi : A \to B$ is called $X$-equivariant or a $*$-homomorphism over $X$ if $\phi(A(U)) \subseteq B(U)$ for any open subset $U$ of $X$.

We let $\mathcal{C}^*\text{alg}(X)$ denote the category with objects being $C^*$-algebras over $X$ and morphisms the $X$-equivariant $*$-homomorphisms. Moreover, we let $\mathcal{C}^*\text{sep}(X)$ denote the full subcategory for which all the $C^*$-algebras over $X$ are separable.

**Example 2.1.2.** Let $\star$ be the one-point topological space. Then $\mathcal{C}^*\text{alg}(\star)$ is isomorphic the category $\mathcal{C}^*\text{alg}$ of $C^*$-algebras with $*$-homomorphisms. Hence we will make no distinction between these two categories.

**Example 2.1.3.** Let $X$ denote the two-point topological space with (non-Hausdorff) topology $\mathcal{O}(X) = \{\emptyset, \{1\}, \{1, 2\} \}$. Let $A$ and $B$ be $C^*$-algebras over $X$ and $\phi : A \to B$ be an $X$-equivariant $*$-homomorphism. This gives us a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & A(\{1\}) & \longrightarrow & A & \longrightarrow & A(\{2\}) & \longrightarrow & 0 \\
& & \downarrow{\phi} & & \downarrow{\phi} & & \downarrow{\phi} & & \\
0 & \longrightarrow & B(\{1\}) & \longrightarrow & B & \longrightarrow & B(\{2\}) & \longrightarrow & 0
\end{array}
\]

with exact rows. Moreover, one easily sees that given such a commutative diagram, the $C^*$-algebras over $X$, $A$ and $B$, and the $X$-equivariant $*$-homomorphism $\phi$ can be determined uniquely. Hence $\mathcal{C}^*\text{alg}(X)$ is isomorphic to the category of $C^*$-algebra extensions.

**Example 2.1.4.** Let $X = X_1 \sqcup X_2$ be a disjoint union of topological spaces. Then for any $C^*$-algebra over $X$, $A$, the underlying $C^*$-algebra $A(X)$ decomposes as $A(X_1) \oplus A(X_2)$. Since $\text{Prim}(A(X_1) \oplus A(X_2)) \cong \text{Prim}(A(X_1)) \times \text{Prim}(A(X_2))$ canonically, it follows that $A(X_i)$ has the structure of a $C^*$-algebra over $X_i$ for $i = 1, 2$. This yields an equivalence of the categories $\mathcal{C}^*\text{alg}(X_1 \sqcup X_2)$ and $\mathcal{C}^*\text{alg}(X_1) \times \mathcal{C}^*\text{alg}(X_2)$.

**Example 2.1.5.** Let $X$ be a locally compact Hausdorff space and $(A, \psi)$ be a $C^*$-algebra over $X$. There is a natural $*$-homomorphism

\[
\psi^*: C_0(X) \to C_0(\text{Prim}(A)), \quad f \mapsto f \circ \psi.
\]

By the Dauns-Hofmann Theorem we may identify $C_0(\text{Prim}(A))$ with $ZM(A)$, the center of the multiplier algebra of $A$. Doing this $(A, \psi)$ induces a $C_0(X)$-algebra. It is proven in [19] that this functor yields an isomorphism of the categories $\mathcal{C}^*\text{alg}(X)$ and $\mathcal{C}^*\text{alg}(C_0(X))$, the latter being the category of $C_0(X)$-algebras. Combining this with the main result of [22], we also get that the category $\mathcal{C}^*\text{alg}(X)$ is equivalent to the category of upper semi-continuous $C^*$-bundles.

For the reader who is already familiar with $E$-theory for $C_0(X)$-algebras or for $C^*$-bundles, such as E. Park and J. Trout’s $RE$-theory from [23], it might
be reassuring to know that these $E$-theories are equivalent to the one we will define. This follows since they satisfy the same universal property (Theorem 3.7.1).

2.2 Restriction to Sober Spaces

In this section we explain, as is done in [19], that it is essentially no loss of generality only to consider $C^*$-algebras over sober spaces. We then prove that these are easier to define than $C^*$-algebras over general topological spaces. Lastly we define what quasi $C^*$-algebras over sober spaces are, and give a couple of examples of such. In order to give these examples we state an extended version of the Bartle-Graves selection theorem for $C^*$-algebras, which will play an important part when working with asymptotic morphisms.

Definition 2.2.1. A topological space $X$ is called sober if every irreducible closed subset of $X$, i.e. a closed subset which is not the union of any two proper closed subsets, is the closure of exactly one singleton $\{x\}$.

Given the lattice $\mathcal{O}(X)$ for a sober space $X$, we can always obtain the space $X$ and its topology, since the irreducible closed subsets of $X$ are in one-to-one correspondence with the points of $X$.

We should note that Hausdorff spaces are sober, and sober spaces are $T_0$ (also known as Kolmogorov). However, being sober has no immediate relation to being $T_1$.

Let $X$ be a non-sober space and let $\hat{X}$ be the set of all irreducible closed subsets of $X$. For any closed subset $S$ of $X$ we let $\hat{S}$ denote the set of all $A \in \hat{X}$ such that $A \subseteq S$. One easily verifies that the collection of sets $\hat{S}$ for closed subsets $S$ of $X$, contains $\emptyset$ and $\hat{X}$ and is closed under finite unions and arbitrary intersections. Hence this collection induces the closed sets for a topology on $\hat{X}$, and with this topology $\hat{X}$ is a sober space. We call $\hat{X}$ the sobrification of $X$.

Let $\iota : X \to \hat{X}$ be the canonical map $\iota(x) = \{x\}$. By how we constructed the closed subsets of $\hat{X}$ it clearly follows that $\iota$ is continuous, and induces a lattice isomorphism $\iota^* : \mathcal{O}(\hat{X}) \to \mathcal{O}(X)$. Hence $\iota$ induces a functor $\hat{i} : C^*\text{alg}(X) \to C^*\text{alg}(\hat{X})$ by mapping $(A, \psi)$ to $(A, \iota \psi)$. This functor is fully and faithful since the equivariant $*$-homomorphisms only depend on the lattices $\mathcal{O}(X) \cong \mathcal{O}(\hat{X})$.

Since what we care about when working with $C^*$-algebras over topological spaces is the structure induced by the map $\mathcal{O}(X) \to I(A)$, it is essentially no loss of generality to assume that $X$ is sober.

We will from now on always assume that all topological spaces are sober.

We will, however, still write sentences like 'Let $X$ be a sober space' when we find it fitting. But if nothing is mentioned, the space $X$ is assumed to be sober.

One of the nice properties of sober spaces, is that $C^*$-algebras over sober spaces are easier to define than over general topological spaces. This follows from the proposition below.

\footnote{Most of the time, all we are doing is replacing the word 'topological' with the word 'sober'.}
Proposition 2.2.2. Let $X$ be a sober space. There is a one-to-one correspondence between continuous maps $\psi : \text{Prim}(A) \to X$ and lattice maps $\psi^* : \mathcal{O}(X) \to \mathcal{I}(A)$ which respect finite infima, arbitrary suprema and where $\psi^*(0) = 0$ and $\psi^*(X) = A$.

Proof. It is well known from general topology that if $\psi : \text{Prim}(A) \to X$ is continuous then the induced map $\psi^* : \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(A)) = \mathcal{I}(A)$ satisfies the above conditions.

Let $\psi^* : \mathcal{O}(X) \to \mathcal{I}(A) = \mathcal{O}(\text{Prim}(A))$ satisfy the above conditions. For $p \in \text{Prim}(A)$ we define $U_p$ to be the union of all $U \in \mathcal{O}(X)$ such that $p \notin \psi^*(U)$. Since $\psi^*$ respects suprema, $p \notin \psi^*(U_p)$ and $U_p$ is the largest element of $\mathcal{O}(X)$ satisfying this. Hence $A_p := X \setminus U_p$ is the smallest closed set such that $p \notin \psi^*(X \setminus A_p)$. Note that $A_p$ is non-empty since $p \in \psi^*(X)$. Since $\psi^*$ respects finite infima the minimality of $A_p$ implies that $A_p$ is irreducible. Hence there exists a unique element $\psi(p)$ in $X$, such that $A_p = \{\psi(p)\}$.

This gives a map $\psi : \text{Prim}(A) \to X$. Moreover, for any $U$ in $\mathcal{O}(X)$ we note that $\psi(p) \notin U$ if and only if $A_p \cap U = \emptyset$ if and only if $p \notin \psi^*(U)$. Hence $\psi^{-1}(U) = \psi^*(U)$ and thus $\psi$ is a continuous map which generates $\psi^*$. In other words our correspondence $\psi \mapsto \psi^*$ is surjective.

Since $X$ is sober, and in particular $T_0$, any two distinct continuous maps $\psi_1, \psi_2 : \text{Prim}(A) \to X$ induce distinct maps $\psi_1^*, \psi_2^* : \mathcal{O}(X) \to \mathcal{I}(A)$. Hence the correspondence is injective. \qed

At times it will be necessary to work with something weaker than $C^*$-algebras over sober spaces.

Definition 2.2.3. Let $X$ be a sober space. A quasi $C^*$-algebra over $X$ is a $C^*$-algebra $A$ together with a lattice map $\psi^* : \mathcal{O}(X) \to \mathcal{I}(A)$ which respects finite infima and finite suprema and where $\psi^*(0) = 0$ and $\psi^*(X) = A$.

Before giving examples of quasi $C^*$-algebras over sober spaces, we will state a theorem known as the Bartle-Graves selection theorem for $C^*$-algebras. The usual Bartle-Graves theorem works with Banach spaces. This extended version is due to Terry A. Loring in [15]. The proof of the theorem is omitted.

Theorem 2.2.4 (Bartle-Graves selection theorem). Let $\pi : A \to B$ be a surjective $*$-homomorphism of $C^*$-algebras. For every $M > 1$ there exists a (not necessarily linear) section $\sigma : B \to A$ of $\pi$, i.e. $\pi \sigma = 1_B$, such that

(i) $\sigma$ is continuous,
(ii) $\sigma(\lambda b) = \lambda \sigma(b)$,
(iii) $\sigma(b^*) = \sigma(b)^*$,
(iv) $\|\sigma(b)\| \leq M\|b\|$,

for every $b \in B$ and $\lambda \in \mathbb{C}$. 

6
The 'ordinary' Bartle-Graves selection theorem for Banach spaces only contains item (i) above. One of the main advantages of this extended version, is item (iv), which essentially tells us that $\sigma$ is bounded on bounded sets. We will make use of this in the following example where we introduce two types of quasi $C^*$-algebras over sober spaces, which will play an important role when working with asymptotic morphisms.

**Example 2.2.5.** Let $A$ be a $C^*$-algebra over $X$ and $T = [0, \infty)$. Then the $C^*$-algebras $C_b(T, A)$ and $C_b(T, A)/C_0(T, A)$ together with the maps

$$\mathcal{O}(X) \ni U \mapsto C_b(T, A(U)), \quad \mathcal{O}(X) \ni U \mapsto \frac{C_b(T, A(U)) + C_0(T, A)}{C_0(T, A)}$$

are quasi $C^*$-algebras over $X$ but not necessarily $C^*$-algebras over $X$. We will only prove that $C_b(T, A)$ is a quasi $C^*$-algebra over $X$.

The only thing which is not obvious is that

$$C_b(T, A(U_1 \cup U_2)) \subseteq C_b(T, A(U_1)) + C_b(T, A(U_2))$$

for $U_1, U_2 \in \mathcal{O}(X)$. Let $f : T \to A(U_1) + A(U_2)$ be a bounded continuous function. By the Bartle-Graves selection theorem there exists a continuous section $\sigma$ of the quotient map $A(U_1) \to \frac{A(U_1) + A(U_2)}{A(U_1) \cap A(U_2)}$ which is bounded on bounded sets. Let $f_1$ denote the composite function

$$T \xrightarrow{f} A(U_1) + A(U_2) \to \frac{A(U_1) + A(U_2)}{A(U_2)} \cong \frac{A(U_1)}{A(U_1) \cap A(U_2)} \xrightarrow{\sigma} A(U_1).$$

Since $\sigma$ is continuous and bounded on bounded sets, $f_1$ is a bounded continuous function and so is $f_2 = f - f_1$. Since $f_i \in C_b(T, A(U_i))$ for $i = 1, 2$ it follows that $C_b(T, A)$ is a quasi $C^*$-algebra over $X$.

### 2.3 Basic Constructions

One of the most basic constructions of $C^*$-algebras is the **direct sum**. If $(A_\alpha)$ is a family of $C^*$-algebras over $X$, then we obviously get a $C^*$-algebra over $X \bigoplus A_\alpha$ by the map

$$\mathcal{O}(X) \ni U \mapsto \bigoplus A_\alpha(U) \in I(\bigoplus A_\alpha).$$

Similarly we will construct other $C^*$-algebras over $X$.

**Definition 2.3.1.** An extension of $C^*$-algebras over $X$ is a sequence $0 \to J \to A \to B \to 0$ in $C^*\text{alg}(X)$ such that

$$0 \to J(U) \to A(U) \to B(U) \to 0$$

is an extension of $C^*$-algebras for every $U$ in $\mathcal{O}(X)$.

---

6Whenever we say direct sum we are talking about $c_0$-direct sums.
Lemma 2.3.2. Let $A$ be a $C^*$-algebra over $X$ and $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be an extension of $C^*$-algebras. Then there is a natural extension of $C^*$-algebras over $X$, $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$, where $J$ and $B$ are $C^*$-algebras over $X$ by the maps

$$\mathcal{O}(X) \ni U \mapsto J \cdot A(U) \in \mathcal{I}(J), \quad \mathcal{O}(X) \ni U \mapsto \pi(A(U)) \in \mathcal{I}(B).$$

Moreover, every extension of $C^*$-algebras over $X$ arises in this way.

This also holds if we restrict ourselves to quasi $C^*$-algebras over $X$.

Proof. It is trivial to check that this construction gives an extension of $C^*$-algebras over $X$ which is natural. If $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ is a $C^*$-algebra over $X$ then $B(U) = \pi(A(U))$ by definition. Since $B(U)$ embeds into $B(X)$, $J(U)$ is the kernel of $\pi|_{A(U)}$ which is $J \cdot A(U)$. Clearly restricting to quasi $C^*$-algebras over $X$ changes nothing. \qed

Lemma 2.3.3. Let $X$ be a sober space and $B$ be a $C^*$-algebra. Then there are induced functors $- \otimes_m B : \mathcal{C}^*\text{alg}(X) \rightarrow \mathcal{C}^*\text{alg}(X)$, where $\otimes_m$ denotes either the maximal or the spatial (also called minimal) tensor product. Define $A \otimes_m B$ to be the $C^*$-algebra over $X$ defined by the map $\psi^* : \mathcal{O}(X) \rightarrow \mathcal{I}(A \otimes_m B)$ given by $\psi^*(U) = A(U) \otimes_m B$. Moreover, the functor $- \otimes_m B$ is exact, i.e. it maps extensions to extensions.

Proof. The map $\psi^*$ is well defined since the functor $- \otimes_m B : \mathcal{C}^*\text{alg} \rightarrow \mathcal{C}^*\text{alg}$ preserves ideals and clearly $A \otimes_m B$ is a quasi $C^*$-algebra over $X$. Let $(U_\alpha)$ be a family of open subsets of $X$ and let $U := \bigcup U_\alpha$. Approximating any element in $A(U) \otimes_m B$ with an element of $A(U) \odot B$,\footnote{We denote by $A \odot B$ the algebraic tensor product of $A$ and $B$.} and this element with one in $(\sum A(U_\alpha)) \odot B = \sum A(U_\alpha) \odot B$ gives that

$$A(U) \otimes_m B \subseteq \sum_{\alpha} A(U_\alpha) \otimes_m B$$

and since the other inclusion is trivial $A \otimes_m B$ is $C^*$-algebra over $X$. Since the obvious morphisms are well defined the functor is exact by Lemma 2.3.2. \qed

Remark 2.3.4. In [19] they make a more general construction of the spatial tensor product for separable $C^*$-algebras over topological spaces. In fact, if $A$ and $B$ are separable $C^*$-algebras, there is a canonical continuous map

$$\text{Prim}(A \otimes B) \rightarrow \text{Prim}(A) \times \text{Prim}(B)$$
which is described in greater detail in [2] IV.3.2.24. This induces a bifunctor
\[ \otimes : \mathcal{C}^{\text{sep}}(X) \times \mathcal{C}^{\text{sep}}(Y) \to \mathcal{C}^{\text{sep}}(X \times Y) \]
in the obvious way. Note that this definition works even if \( X \) and \( Y \) are not sober. If \( X \) is sober and \( Y = \star \) then the definition coincides with our construction in Lemma 2.3.3.

**Example 2.3.5.** Lemma 2.3.3 allows us to take suspensions, cones and cylinders and to stabilize \( C^* \)-algebras over \( X \). Moreover, any extension \( 0 \to J \to A \to B \to 0 \) of \( C^* \)-algebras, induces natural transformations
\[ - \otimes_{\text{max}} J \Rightarrow - \otimes_{\text{max}} A \Rightarrow - \otimes_{\text{max}} B \]
which induce extensions of \( C^* \)-algebras over \( X \). By this we mean that if \( D \) is a \( C^* \)-algebra over \( X \), then
\[ 0 \to D \otimes_{\text{max}} J \to D \otimes_{\text{max}} A \to D \otimes_{\text{max}} B \to 0 \]
is an extension of \( C^* \)-algebras over \( X \), which follows from Lemma 2.3.2.

**Remark 2.3.6.** Since we can define cylinders, we can also define homotopies of \( C^* \)-algebras over \( X \). This is done exactly as in the non-equivariant case. Similarly we may define homotopy equivalences, and define what it means for a \( C^* \)-algebra over \( X \) to be contractible.

We should warn the reader that homotopies of the underlying \( C^* \)-algebra may not in general be converted to a homotopy of \( C^* \)-algebras over \( X \). As an example it is well-known that the \( C^* \)-algebra \( C^0([0,1]) \) is contractible. However, \( C^0([0,1]) \) has a canonical structure of a \( C^* \)-algebra over the compact Hausdorff space \([0,1] \), and this \( C^* \)-algebra over \([0,1] \) is not contractible.

Recall that if \( \pi : A \to B \) and \( \phi : D \to B \) are \( * \)-homomorphisms then \( A \oplus_B D := \{(a,d) \in A \oplus D \mid \pi(a) = \phi(d)\} \) is the pull-back. If \( 0 \to J \to A \xrightarrow{\pi} B \to 0 \) is an extension of \( C^* \)-algebras then we get the pull-back diagram which is the commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \to & J & \to & A \oplus_B D & \to & D & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & J & \to & A & \xrightarrow{\pi} & B & \to & 0.
\end{array}
\]

**Lemma 2.3.7.** Let \( X \) be a second countable sober space, let \( 0 \to J \to A \xrightarrow{\pi} B \to 0 \) be an extension of \( C^* \)-algebras over \( X \) and \( \phi : D \to B \) be an \( X \)-equivariant \( * \)-homomorphisms. Then \( E := A \oplus_B D \) is a \( C^* \)-algebra over \( X \) given by the map
\[ \mathcal{O}(X) \ni U \mapsto E \cap (A(U) \oplus D(U)) \in \mathcal{I}(E). \]
Moreover, the statement is still true if \( A \) and \( B \) are only quasi \( C^* \)-algebras over \( X \).

\[ ^{8} \text{All we are actually saying, is that not every } * \text{-homomorphism is an } X \text{-equivariant } * \text{-homomorphism.} \]
Proof. Note that $E$ is almost obviously a quasi $C^*$-algebra over $X$. The only thing not obvious is that $E(U_1 \cup U_2) = E(U_1) + E(U_2)$, more specifically that $E(U_1 \cup U_2) \subseteq E(U_1) + E(U_2)$. For $(a, d) \in E(U_1 \cup U_2)$ decompose $d \in D(U_1 \cup U_2) = D(U_1) + D(U_2)$ into $d_1 + d_2$ with $d_i$ in $D(U_i)$. Then pick $a_i$ such that $\pi(a_i) = \phi(d_i)$. Then $a - a_1 - a_2$ is in the kernel of $\pi$ and is thus in $J(U_1) + J(U_2)$.

Hence decompose $a_1 + a_2 - a = x_1 + x_2$ for $x_i \in J(U_i)$. Then $(a_i - x_i, b_i) \in E(U_i)$ and $(a_1 - x_1, b_1) + (a_2 - x_2, b_2) = (a, d)$.

It remains to show that $E(\bigcup U_n) = \sum E(U_n)$. Note that since $X$ is second countable it suffices to consider countable families of open sets. We may assume that $(U_n)$ is an increasing sequence of open sets. For any open subset $U$ of $X$ we know that $0 \to J(U) \to E(U) \to D(U) \to 0$ is exact. Let $U := \bigcup U_n$. Since $J$ and $D$ are $C^*$-algebras over $X$ we get that

$$J(U) = \sum J(U_n) = \bigcup J(U_n) = \lim_{\longrightarrow} J(U_n)$$

and similarly $D(U) = \lim_{\longleftarrow} D(U_n)$. The inductive limit functor is exact and thus we get that

$$0 \to J(U) \to \lim_{\longrightarrow} E(U_n) \to D(U) \to 0$$

is an extension and by the five lemma the canonical inclusion $\lim E(U_n) \to E(U)$ is an isomorphism. Hence $E(U) = \lim E(U_n) = \sum(E(U_n))$ and thus $E$ is a $C^*$-algebra over $X$. One should note that we only used that $A$ and $B$ were quasi $C^*$-algebras over $X$. \qed

**Example 2.3.8.** Lemma 2.3.7 allows us to take mapping cones and mapping cylinders of $X$-equivariant $*$-homomorphisms.

### 2.4 Adjoint Functors

In this section we will construct functors between categories of $C^*$-algebras over sober spaces, and show that these satisfy certain adjunction relations. This will be a powerful tool once we have the universal property of $E$-theory.

Let $X$ and $Y$ be sober spaces and $f : Y \to X$ be a continuous map. This gives rise to a functor $f_* : \mathcal{C}^*\text{alg}(Y) \to \mathcal{C}^*\text{alg}(X)$ given by $f_*(A, \psi) = (A, f \circ \psi)$. In particular if $Y$ is a subspace of $X$ and $f$ is the inclusion, we call the corresponing functor for the extension functor from $Y$ to $X$ and denote it by $i_Y^!$.

If $Y \in \mathbb{S}(X)$, i.e. $Y$ is a locally closed subset of $X$, we define the restriction functor of $X$ to $Y$ to be the functor $r_Y^! : \mathcal{C}^*\text{alg}(X) \to \mathcal{C}^*\text{alg}(Y)$, where $r_Y^!(A) = A(Y)$ is given by

$$\mathbb{O}(Y) \ni U \mapsto A(U) \in \mathbb{I}(A(Y)).$$

Recall that if $Y = U_1 \setminus U_2$ for open subsets $U_2 \subseteq U_1$ of $X$, then $A(Y) = A(U_1) / A(U_2)$. Moreover, if $U \in \mathbb{O}(Y)$ then there exists a $V \in \mathbb{O}(X)$ such that $U = V \cap Y$ and thus

$$A(U) = A(V \cap Y) = A((V \cap U_1) \setminus (V \cap U_2)) = A(V \cap U_1) / A(V \cap U_2)$$

Note that we could have just as well made this construction with $X$ and $Y$ being non-sober spaces.
which is why the above definition makes sense. Note that when \( Y \in \mathcal{L}C(X) \) then \( r_Y^X \circ i_Y^X = 1_Y \). Here \( 1_Y \) denotes the identity functor on \( \mathcal{C}^\ast \text{alg}(Y) \).

**Lemma 2.4.1.** Let \( X \) be a sober space and \( Y \subseteq X \).

(i) If \( Y \) is open then \( i_Y^X \) is left adjoint to \( r_Y^X \), i.e.

\[
\text{Hom}_X(i_Y^X(A), B) \cong \text{Hom}_Y(A, r_Y^X(B))
\]

is a natural bijection, for \( A \in \text{Ob}(\mathcal{C}^\ast \text{alg}(Y)) \), \( B \in \text{Ob}(\mathcal{C}^\ast \text{alg}(X)) \).

(ii) If \( Y \) is closed then \( i_Y^X \) is right adjoint to \( r_Y^X \), i.e.

\[
\text{Hom}_X(A, i_Y^X(B)) \cong \text{Hom}_Y(r_Y^X(A), B)
\]

is a natural bijection, \( A \in \text{Ob}(\mathcal{C}^\ast \text{alg}(X)) \), \( B \in \text{Ob}(\mathcal{C}^\ast \text{alg}(Y)) \).

**Proof.** In both cases the map \( \text{Hom}_X \to \text{Hom}_Y \) will be given by applying \( r_Y^X \) and using that \( r_Y^X \circ i_Y^X = 1_Y \). Hence naturality will come automatically. Let \( Y \) be open and \( \phi : i_Y^X(A) \to B \) be an \( X \)-equivariant \(*\)-homomorphism. The underlying \(*\)-homomorphism of \( r_Y^X(\phi) \) is \( \phi : A(Y) \to r_Y^X(B)(Y) = B(Y) \). If \( \psi : A \to r_Y^X(B) \) is a \( Y \)-equivariant \(*\)-homomorphism, the underlying \(*\)-homomorphism \( \psi : A(Y) \to r_Y^X(B)(Y) = B(Y) \) satisfies

\[
\psi(A(U \cap Y)) \subseteq r_Y^X(B)(U \cap Y) = B(U \cap Y) \subseteq B(U)
\]

for \( U \in \mathcal{O}(X) \) and since \( A(U \cap Y) = i_Y^X(A)(U) \) this induces an \( X \)-equivariant \(*\)-homomorphism \( \psi' : i_Y^X(A) \to B \). Since the underlying \(*\)-homomorphism does not change by either of the above maps, this proves (i).

Let \( Y \) be closed and \( V := X \setminus Y \). If \( \phi : A \to i_Y^X(B) \) we note that \( \phi(A(V)) \subseteq i_Y^X(B)(V) = 0 \). The underlying \(*\)-homomorphism of \( r_Y^X(\phi) \) is \( \phi : A(Y) = A(X)/A(V) \to B(Y) \). If \( \psi : r_Y^X(A) \to B \) define the \(*\)-homomorphism \( \psi' : A(X) \to B(Y) \) as the composite \( A(X) \to A(X)/A(V) \xrightarrow{\phi} B(Y) = i_Y^X(B)(X) \). To see that this is an \( X \)-equivariant \(*\)-homomorphism \( A \to i_Y^X(B) \), let \( U \in \mathcal{O}(X) \) and observe that

\[
\psi'(A(U)) = \psi(A(U)/(A(U) \cdot A(V))) = \psi(A(U)/A(U \cap V)) = \psi(A(U \cap Y)) \subseteq B(U \cap Y) = i_Y^X(B)(U).
\]

It is straight forward to check that the two induced maps of Hom-sets are each others inverses, which proves (ii). \( \square \)

Let \( x \in X \). The inclusion \( \{x\} \subseteq X \) induces an extension functor \( i_x : \mathcal{C}^\ast \text{alg} \cong \mathcal{C}^\ast \text{alg}({\{x\}}) \to \mathcal{C}^\ast \text{alg}(X) \). Note that if \( Y \subseteq X \) is locally closed then

\[
i_x(A)(Y) = \begin{cases} A & \text{if } x \in Y \\ 0 & \text{otherwise}. \end{cases}
\]

This functor will play a large role later on, when studying finite spaces.
Let $Y$ be a locally closed subset of $X$. Then there is an obvious induced functor $\text{ev}_Y : \mathcal{C}^* \mathfrak{alg}(X) \to \mathcal{C}^* \mathfrak{alg}$, called the evaluation functor in $Y$, given by $\text{ev}_Y(A) = A(Y)$.

Lastly let $S \subseteq X$ be any subset. Then there is an induced functor $\mathcal{F}_S : \mathcal{C}^* \mathfrak{alg}(X) \to \mathcal{C}^* \mathfrak{alg}$, called the filtration functor around $S$, given by $\mathcal{F}_S(A) = \bigcap_{S \subseteq U \in \mathcal{O}(X)} A(U)$. Note that for an $X$-equivariant *-homomorphism $\phi : A \to B$

$$\phi(\mathcal{F}_S(A)) \subseteq \phi(A(U)) \subseteq B(U)$$

for every $U \in \mathcal{O}(X)$ such that $S \subseteq U$. Hence $\phi(\mathcal{F}_S(A)) \subseteq \mathcal{F}_S(B)$ and thus $\mathcal{F}_S$ is well-defined. If $S$ consists only of one point $x$ we denote $\mathcal{F}_S$ by $\mathcal{F}_x$. Observe that if $S$ has a minimal open neighborhood $U_S$ then $\mathcal{F}_S = \text{ev}_{U_S}$.

The following lemma is an easy consequence of Lemma 2.4.1.

**Lemma 2.4.2.** Let $X$ be a sober space and let $x \in X$.

(i) Then $i_x$ is right adjoint to $\text{ev}_{\{x\}}$, i.e.

$$\text{Hom}_X(A, i_x(B)) \cong \text{Hom}(A(\{x\}), B)$$

is a natural bijection, for $A \in \text{Ob}(\mathcal{C}^* \mathfrak{alg}(X))$, $B \in \text{Ob}(\mathcal{C}^* \mathfrak{alg})$.

(ii) Then $i_x$ is left adjoint to $\mathcal{F}_x$, i.e.

$$\text{Hom}_X(i_x(A), B) \cong \text{Hom}(A, \mathcal{F}_x(B))$$

is a natural bijection, for $A \in \text{Ob}(\mathcal{C}^* \mathfrak{alg})$, $B \in \text{Ob}(\mathcal{C}^* \mathfrak{alg}(X))$.

In particular, if $x$ has a minimal open neighborhood $U_x$ then

$$\text{Hom}_X(i_x(A), B) \cong \text{Hom}(A, B(U_x))$$

is a natural bijection, for $A \in \text{Ob}(\mathcal{C}^* \mathfrak{alg})$, $B \in \text{Ob}(\mathcal{C}^* \mathfrak{alg}(X))$.

**Proof.** Let $C := \{x\}$. Any open subset $V \subseteq C$ contains $x$ and thus $(i^C_x B)(V) = B$. Hence $\text{Hom}_C(A(i^C_x B)) \cong \text{Hom}(A(C), B)$ naturally for $A \in \text{Ob}(\mathcal{C}^* \mathfrak{sep}(C))$. By observing that $i^C_x \circ i^C_x = i^C_x$, Lemma 2.4.1 implies

$$\text{Hom}_X(A, i^C_x(B)) \cong \text{Hom}_C(i^C_x(A), i^C_x(B)) \cong \text{Hom}(A(C), B)$$

where the bijections are natural.

To prove (ii) let $\mathcal{U}_x$ denote the set of all open subsets of $X$ containing $x$. An $X$-equivariant *-homomorphism $i_x A \to B$ restricts to *-homomorphisms $A = (i_x A)(U) \to B(U)$ for every $U \in \mathcal{U}_x$. Hence this induces a *-homomorphism $A \to \bigcap_{U \in \mathcal{U}_x} B(U) = \mathcal{F}_x(B)$, Moreover, given such a *-homomorphism there is an induced $X$-equivariant *-homomorphism $i_x A \to B$ which gives the other adjunction. \[\Box\]
Recall (e.g. [12] §II.7) that \( F \) being left adjoint to \( G \) is equivalent to the existence of natural transformations

\[
\varepsilon : 1 \to GF, \quad \delta : FG \to 1,
\]
called the unit and the counit, such that \( \delta_{F(A)}F(\varepsilon_A) = 1_{F(A)} \) and \( G(\delta_B)\varepsilon_{G(B)} = 1_{G(B)} \). Moreover, \( \varepsilon_A : A \to GF(A) \) is the morphism corresponding to \( 1_{F(A)} \) by the adjunction and \( \delta_B : FG(B) \to B \) is the morphism corresponding to \( 1_{G(B)} \).

**Lemma 2.4.3.** Let \( X \) be a compact Hausdorff space. The functor \( C(X,-) : \mathcal{C}^*\text{alg} \to \mathcal{C}^*\text{alg}(X), \) where \( C(X) \) is the canonical \( C^* \)-algebra over \( X \), is left adjoint to the forgetful functor \( F : \mathcal{C}^*\text{alg}(X) \to \mathcal{C}^*\text{alg}, \) i.e.

\[
\text{Hom}_X(C(X,A),B) \cong \text{Hom}(A,B(X))
\]
is a natural bijection, for \( A \in \text{Ob}(\mathcal{C}^*\text{alg}), \) \( B \in \text{Ob}(\mathcal{C}^*\text{alg}(X)) \).

**Proof.** We will do the prove by constructing a unit and a counit for the functors. The unit \( \varepsilon : 1 \to FC(X,-) \) will be the natural transformation where \( \varepsilon_A : A \to C(X,A) \) is given by \( \varepsilon_A(a) = 1 \otimes a \).\(^{10}\)

The counit \( \delta : C(X,-)F \to 1_X \) will be slightly harder to define. Let \( (B,\psi) \) be a \( C^* \)-algebra over \( X \). By the Dauns-Hofmann Theorem we may identify \( C_b(\text{Prim}(B)) \) with \( ZM(B) \), the centralizer of the multiplier algebra of \( B \). Clearly there is an induced \( * \)-homomorphism \( \psi^* : C(X) \to C_b(\text{Prim}(B)) \cong ZM(B) \) given by \( \psi^*(f) = f \circ \psi \). We define

\[
\delta_B := \psi^* \times 1_{B(X)} : C(X) \otimes B(X) \to B,
\]
i.e. \( \delta_B(f \otimes b) = \psi^*(f)b \)

which is a well-defined \( * \)-homomorphism since the ranges of \( \psi^* \) and \( 1_{B(X)} \) commute in \( M(B) \). To see that this is \( X \)-equivariant let \( U \in \mathcal{O}(X) \). Recall that the ideal \( B(U) \) corresponds to the open subset \( \{ p \mid B(U) \not\subseteq p \} \) of \( \text{Prim}(B) \). Hence if \( f \in C_0(U) \) then \( f(\psi(p)) = 0 \) when \( B(U) \subseteq p \). If \( f \otimes b \in C_0(U) \otimes B(X) \), the Dauns-Hofmann Theorem gives that

\[
\|\delta_B(f \otimes b)\|_{B/p} = \|\psi^*(f)b\|_{B/p} = \|f(\psi(p))b\|_{B/p} = 0,
\]
for every \( B(U) \subseteq p \), which implies that

\[
\delta_B(f \otimes b) \in \bigcap_{B(U) \subseteq p} p = B(U).
\]

Hence \( \delta_B \) is \( X \)-equivariant. It remains to show that \( \delta_{C(X,A)}C(X,\varepsilon_A) = 1_{C(X,A)} \) and \( F(\delta_B)\varepsilon_{F(B)} = 1_{F(B)} \) where the latter is immediate since

\[
F(\delta_B)\varepsilon_{F(B)}(b) = \delta_B(1 \otimes b) = \psi^*(1)b = b.
\]

\(^{10}\)As usual we denote by \( f \otimes a \) for \( f \in C(X) \) and \( a \in A \), the function \( (f \otimes a)(x) = f(x)a \). In particular, \( 1 \otimes a \) is the constant function \( a \).
Given a $C^*$-algebra $A$, let $\psi : \text{Prim}(C(X, A)) \to X$ be the continuous map such that $\psi^{-1}(U) = C_0(U, A)$ for $U \in \mathcal{O}(X)$ and let $f \otimes a \in C(X, A)$. Note that

$$\delta_{C(X, A)} C(X, \varepsilon_A)(f \otimes a) = \delta_{C(X, A)}(f \otimes 1 \otimes a) = \psi^*(f)(1 \otimes a).$$

Since $X$ is Hausdorff, and thus every irreducible closed subset is a singleton, it follows that $\psi(p) = x$ if and only if $C_0(X \setminus \{x\}, A) \subseteq p$. If $x \in X$ and $C_0(X \setminus \{x\}, A) \subseteq p$ the Dauns-Hofmann Theorem gives

$$[\psi^*(f)(1 \otimes a)]_{C(X, A)/p} = [1 \cdot f(x) \otimes a]_{C(X, A)/p}.$$

Hence

$$[\psi^*(f)(1 \otimes a)]_{C(x, A)} = [1 \cdot f(x) \otimes a]_{C(x, A)},$$

where $C(x, A) := C(X, A)/C_0(X \setminus \{x\}, A)$. Thus

$$\|\psi^*(f)(1 \otimes a) - f \otimes a\| = \sup_{x \in X} \|1 \cdot f(x) \otimes a - f \otimes a\|_{C(x, A)} = \sup_{x \in X} \|f(x)a - f(x)a\| = 0,$$

which implies that $\delta_{C(X, A)} C(X, \varepsilon_A) = 1_{C(X, A)}$. \hfill \Box

Remark 2.4.4. Note that everything done in this section could be restricted to functors between the full subcategories $\mathcal{C}^*\text{sep}(X)$ of separable $C^*$-algebras over $X$. However, in Lemma 2.4.3 we would also have to require that $X$ should be second countable.
3 \(E\)-Theory

In this section we define the basics of \(E\)-theory. The construction is done similarly as the original construction of Connes and Higson in [4] with the extra assumption that our asymptotic morphisms should be \textit{approximately} \(X\)-equivariant. Certain parts, especially the composition of asymptotic morphisms, becomes rather technical. We prove that our version of \(E\)-theory has all the desired properties, such as excision and the universal property. In the end we prove that certain functors of categories of \(C^*\)-algebras over topological spaces descend to the \(E\)-theory category, and that when doing this we preserve adjunctions. Thus Section 2.4 induces several adjunctions of \(E\)-theory categories.

### 3.1 Asymptotic Morphisms

In this section we define asymptotic morphisms and what it means for an asymptotic morphism to be approximately \(X\)-equivariant. Moreover, we define homotopies for (approximately \(X\)-equivariant) asymptotic morphisms. The definitions from this section are all from [7].

From now on we let \(T\) denote the interval \([0, \infty)\). At times it will ease notation significantly to replace \(T\) with some other space homeomorphic to \(T\), such as \((0, 1]\). In these special cases we will redefine \(T\), but in general \(T = [0, \infty)\).

**Definition 3.1.1.** Let \(A\) and \(B\) be \(C^*\)-algebras over \(X\). An asymptotic morphism from \(A\) to \(B\) is a map \(\phi : A \to \mathcal{C}_b(T, B)\) such that the composite

\[ A \to \mathcal{C}_b(T, B) \to B_\infty := \mathcal{C}_b(T, B) / \mathcal{C}_0(T, B) , \]

which we denote by \(\dot{\phi}\), is a \(\ast\)-homomorphism.

If \(\phi_0\) and \(\phi_1\) are asymptotic morphisms from \(A\) to \(B\) we say that \(\phi_0\) and \(\phi_1\) are \textit{equivalent} if \(\dot{\phi}_0 = \dot{\phi}_1\).

We say that an asymptotic morphism \(\phi\) is \textit{uniformly continuous} if \(\phi : A \to C_0(T, B)\) is continuous.

**Remark 3.1.2.** Let \((\phi_t)_{t \in T}\) be a family of maps from \(A\) to \(B\) satisfying

- \(t \mapsto \phi_t(a)\) is bounded and continuous for every \(a\) in \(A\),
- \(\|\phi_t(a + \lambda b^*) - \phi_t(a) - \lambda \phi_t(b)^*\| \to 0\) for \(t \to \infty\) for any \(a, b \in A\) and \(\lambda \in \mathbb{C}\),
- \(\|\phi_t(ab) - \phi_t(a)\phi_t(b)\| \to 0\) for any \(a, b \in A\).

Then \((\phi_t)\) induces an asymptotic morphism and moreover any asymptotic morphism induces such a family of maps. This yields a one-to-one correspondence between asymptotic morphisms and families \((\phi_t)\) with the given properties. Hence we will not distinct between these two definitions.\(^{11}\)

\(^{11}\)Hopefully without confusing the reader more than necessary.
We will often use the notation \((\phi_t) : A \rightarrow B\) for an asymptotic morphism \((\phi_t)\) from \(A\) to \(B\).

**Lemma 3.1.3.** Every asymptotic morphism is equivalent to uniformly continuous asymptotic morphism.

**Proof.** By the Bartle-Graves selection theorem the quotient map \(C_b(T, B) \twoheadrightarrow B^\infty\) has a continuous section \(\sigma : B^\infty \rightarrow C_b(T, B)\). Hence \(\phi : A \rightarrow C_b(T, B)\) is equivalent to \(\sigma \phi\) which is obviously continuous. \qed

**Definition 3.1.4.** Let \((\phi_t)\) be an asymptotic morphism from \(A\) to \(B\). We say that \((\phi_t)\) is approximately \(X\)-equivariant if for any open subset \(U\) of \(X\),

\[
\|\phi_t(a)\|_{X \setminus U} \to 0, \quad \text{for every } a \in A(U).
\]

Recall that \(\|b\|_{X \setminus U}\) is the norm of \(b \in B(X)\) in the quotient \(B(X \setminus U) = B(X)/B(U)\).

**Remark 3.1.5.** If \(\phi\) is equivalent to an approximately \(X\)-equivariant asymptotic morphism then \(\phi\) is also approximately \(X\)-equivariant.

**Lemma 3.1.6.** Let \(\phi\) be an asymptotic morphism from \(A\) to \(B\). The following are equivalent

(i) \(\phi\) is approximately \(X\)-equivariant,

(ii) \(\dot{\phi}\) is \(X\)-equivariant,

(iii) For every open subset \(U\) of \(X\),

\[
\phi(A(U)) \subseteq C_b(T, B(U)) + C_0(T, B),
\]

(iv) For every closed subset \(S\) of \(X\),

\[
\limsup_{t \to \infty} \|\phi_t(a)\|_S \leq \|a\|_S, \quad \text{for every } a \in A(X).
\]

**Proof.** We will prove \((ii) \Leftrightarrow (iii) \Leftrightarrow (i) \Leftrightarrow (iv)\). Note that \((ii) \Leftrightarrow (iii) \Rightarrow (i) \Leftrightarrow (iv)\) is obvious.

For \((i) \Rightarrow (iii)\) let \(U\) be an open subset of \(X\), let \(S := X \setminus U\) and let \(\pi_S : C_b(T, B) \rightarrow C_b(T, B(S))\) be the quotient map. Since \((i)\) is (clearly) equivalent to \(\pi_S(\phi(A(U))) \subseteq C_0(T, B(S))\) it suffices to prove that

\[
\pi_S^{-1}(C_0(T, B(S))) \subseteq C_b(T, B(U)) + C_0(T, B).
\]

By the Bartle-Graves selection theorem pick a continuous section \(\sigma\) of the quotient map \(B \twoheadrightarrow B(S)\) such that \(\|\sigma(b)\| \leq 2\|b\|\) for \(b \in B(S)\). Any \(f\) in \(C_b(T, B)\) can be decomposed to \(f = g + h\) with \(g := f - \sigma \circ \pi_S(f)\) and \(h := \sigma \circ \pi_S(f)\), for which \(g \in C_b(T, B(U))\). If \(\pi_S(f) \in C_0(T, B(S))\) then \(h \in C_0(T, B)\).
For (iv) ⇒ (i) let again \( U \in \mathcal{O}(X) \), \( S := X \setminus U \) and let \( \varepsilon > 0 \) and \( a \in A \) be given. We decompose \( a = a_1 + a_2 \) with \( a_1 \in A(U) \) and \( \|a_2\| < \|a\|_S + \varepsilon \). Since \( \lim \|\phi_t(a_1)\|_S = 0 \) we get
\[
\limsup \|\phi_t(a)\|_S \leq \limsup \|\phi_t(a_1)\|_S + \limsup \|\phi_t(a_2)\|_S \leq \limsup \|\phi_t(a_2)\|_S = \|\dot{\phi}(a_2)\| \leq \|a_2\| < \|a\|_S + \varepsilon.
\]

\( \Box \)

Just as with \( X \)-equivariant \(*\)-homomorphisms we define homotopies of approximately \( X \)-equivariant asymptotic morphisms.

**Definition 3.1.7.** Let \( \phi_0 \) and \( \phi_1 \) be asymptotic morphisms from \( A \) to \( B \). We say that \( \phi_0 \) and \( \phi_1 \) are homotopic, written \( \phi_0 \simeq \phi_1 \), if there exists an asymptotic morphism \( \Phi \) from \( A \) to \( IB := C([0,1], B) \) such that \( \phi_0 = ev_0 \Phi \) and \( \phi_1 = ev_1 \Phi \).

If \( \phi_0 \) and \( \phi_1 \) are approximately \( X \)-equivariant we require that \( \Phi \) is approximately \( X \)-equivariant.

We denote by \([[[A,B]]]_X \) the set of homotopy classes of approximately \( X \)-equivariant asymptotic morphisms from \( A \) to \( B \). The homotopy class of an approximately \( X \)-equivariant asymptotic morphism \( \phi \) is denoted \([[[\phi]]] \).

If \( \Phi \) is a homotopy of asymptotic morphisms we will very often denote \( ev_s \Phi \) by \( \Phi_s \) for \( s \in [0,1] \). Hence we might write \( (\Phi_s) \) for such a homotopy.

Often we will denote the homotopy class of an asymptotic morphism \( \phi_t \) by \([[[\phi_t]]] \) instead of the somewhat clumsy notation \([[[\phi_t]]] \). Hence if \( \phi_t = \phi \) then \([[[\phi_t]]] = [[[\phi]]] \).

**Lemma 3.1.8.** Equivalent approximately \( X \)-equivariant asymptotic morphisms are homotopic. Moreover, if the asymptotic morphisms are uniformly continuous then the homotopy can be chosen to be uniformly continuous.

**Proof.** Let \( \phi_0 \) and \( \phi_1 \) be equivalent and let \( T = (0,1] \). Then \( \Phi_t : A \to IB \) for \( t \in T \) given by
\[
\Phi_{s,t}(a) = \phi_{0,s-t}(a) - \phi_{1,s-t}(a) + \phi_{1,t}(a)
\]

for \( a \in A(X) \) and \( s \in [0,1] \), gives the desired homotopy. The least obvious condition from Remark 3.1.2 is the approximate multiplicity so we will prove this. Since \( \phi_0 \) and \( \phi_1 \) are equivalent, \( \sup_s \|\phi_{0,s-t}(a) - \phi_{1,s-t}(a)\| \to 0 \) for \( t \to 0 \) for every \( a \in A(X) \). Hence
\[
\|\Phi_t(a)\Phi_t(b) - \Phi_t(ab)\| \\
\leq \sup_s \|\phi_{0,s-t}(a) - \phi_{1,s-t}(a)\| (\|\phi_{0,s-t}(b) - \phi_{1,s-t}(b)\| \\
+ \sup_s \|\phi_{0,s-t}(a) - \phi_{1,s-t}(a)\| \|\phi_{1,t}(b) - \phi_{1,s-t}(b)\| \\
+ \sup_s \|\phi_{1,t}(a)\phi_{1,t}(b) - \phi_{1,t}(ab)\| + \sup_s \|\phi_{0,s-t}(ab) - \phi_{1,s-t}(ab)\| \\
\to 0,
\]

for \( t \to 0 \) for every \( a, b \in A(X) \). Similarly \( \Phi \) is approximately \( X \)-equivariant. If \( \phi_0 \) and \( \phi_1 \) are continuous then so is \( \Phi \).

\( \Box \)
3.2 Composition Product of Asymptotic Morphisms

As the title indicates, the goal of this section is to construct a composition product of asymptotic morphisms. However, the title is slightly misleading in the sense that we will not construct a composition product of asymptotic morphisms, but rather a composition product of homotopy classes of asymptotic morphisms. This construction will be rather technical. The main idea of the construction is, that if we compose two asymptotic morphisms in a way so that the second asymptotic morphism grows really fast, then the composition will almost be an asymptotic morphism.

When this is done, we will define tensor products of asymptotic morphisms.

We start with a lemma from [7]. The proof is inspired by ([3], Chapter 2, Appendix B, Lemma 3) and ([7] Lemma 2.16).

Lemma 3.2.1. Let X be a second countable sober space, A, B and C be separable C*-algebras over X and let \( \phi : A \to C_0(T,B) \) and \( \psi : B \to C_0(T,C) \) be uniformly continuous approximately X-equivariant asymptotic morphisms. Let \( A_0 \) be any \( \sigma \)-compact dense *-subalgebra of A. Then there exists an increasing, continuous map \( r_0 : T \to T \) such that for any increasing, continuous map \( r : T \to T \) with \( r \geq r_0 \), there is an approximately X-equivariant asymptotic morphism \( \theta : A \to C_0(T,C) \) such that

\[
\| \theta_i(a) - \psi_{r(t)}(a) \| \to 0, \quad \text{for every } a \in A_0.
\]

Proof. Let \( (U_n)_{n \in \mathbb{N}} \) be a basis for \( X \) and let \( (a_{ij})_{i,j \in \mathbb{N}} \) be given such that \( (a_{ij})_{j \in \mathbb{N}} \) is a dense sequence in \( A(U_i) \) for each \( i \). Our goal is to construct \( r_0 \) such that for \( r \geq r_0 \),

1. \( (\psi_{r(t)} \circ \phi_i) \) is an asymptotic morphism from \( A_0 \) to \( C \), i.e. it satisfies the conditions of Remark 3.1.2 with \( A_0 \) in stead of \( A \),
2. \( \| \psi_{r(t)}(a_{ij}) \|_{X \setminus U_i} \to 0 \) for every \( i,j \).

If these conditions are satisfied then \( (\psi_{r(t)} \circ \phi_i) \) induces a *-homomorphism \( A_0 \to C_\infty \) which extends to a *-homomorphism \( \hat{\theta} \) from \( A \to C_\infty \). Now \( \hat{\theta} \) lifts to an asymptotic morphism \( \theta \) from \( A \) to \( C \) which is approximately X-equivariant by (ii).

We start by constructing \( r_{0_0} \) satisfying (i). Choose an increasing sequence of compact subsets \( (K_n) \) of \( A \) such that \( K_n + K_n \subseteq K_{n+1} \), \( K_n K_n \subseteq K_{n+1} \), \( \lambda K_n = K_n \) for \( \lambda \in \mathbb{C}, |\lambda| \leq 1 \) and \( A_0 = \bigcup K_n \). Since \( \phi \) is continuous we can choose a strictly increasing sequence \( (t_n) \) in \( T \), going to \( \infty \), such that for \( t \geq t_n \)

\[
\| \phi_t(a + \lambda b^*) - \phi_t(a) - \lambda \phi_t(b)^* \| < \frac{1}{n}, \quad \text{for } a, b \in K_n, |\lambda| \leq 1,
\]

\[
\| \phi_t(ab) - \phi_t(a) \cdot \phi_t(b) \| < \frac{1}{n}, \quad \text{for } a, b \in K_n,
\]

\[
\| \phi_t(a) \| < \| a \| + \frac{1}{n}, \quad \text{for } a \in K_n.
\]

\[\]
Now define compact subsets $K'_n$ of $B$, such that
\[
\{ \phi_t(a) \mid a \in K_{n+2}, t \leq t_{n+2} \} \subseteq K'_n
\]
and such that $K'_n + K'_n \subseteq K'_{n+1}$, $K'_n K'_n \subseteq K'_{n+1}$ and $\lambda K'_n = K'_n$ for $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. Pick an increasing sequence $(t'_n)$ in $T$ with the same properties as $(t_n)$, but with respect to $\psi$ and $(K'_n)$. We claim that any continuous increasing function $r_{00} : T \to T$ such that $r_{00}(t_n) \geq t'_{n+1}$, satisfies (i).

Let $r \geq r_{00}$. We will only show that
\[
\| \psi_{r(t)}(\phi_t(ab)) - \psi_{r(t)}(\phi_t(a)) \cdot \psi_{r(t)}(\phi_t(b)) \| \to 0
\]
for $a, b \in A_0$. The rest is either similar or obvious.\(^{13}\) Let $a, b \in A_0$, pick $n$ such that $a, b \in K_n$ and let $t \in [t_n, t_{n+1})$. Since $\phi_t(a), \phi_t(b) \in K'_n$, we have
\[
\| \psi_{r(t)}(\phi_t(a) \cdot \phi_t(b)) - \psi_{r(t)}(\phi_t(a)) \cdot \psi_{r(t)}(\phi_t(b)) \| < \frac{1}{n}.
\]
Hence
\[
\| \psi_{r(t)}(\phi_t(ab)) - \psi_{r(t)}(\phi_t(a)) \cdot \psi_{r(t)}(\phi_t(b)) \| \leq \| \psi_{r(t)}(\phi_t(ab)) - \psi_{r(t)}(\phi_t(a) \cdot \phi_t(b)) \| + \frac{1}{n}.
\]
(1)
Note that $\phi_t(a) \cdot \phi_t(b) \in K'_n \subseteq K'_{n+1}$ and $\phi_t(ab) - \phi_t(a) \cdot \phi_t(b) \in K'_{n+1}$. Since $r(t) \geq t'_{n+1}$ we get that
\[
\| \psi_{r(t)}(\phi_t(ab)) - \psi_{r(t)}(\phi_t(a) \cdot \phi_t(b)) - \psi_{r(t)}(\phi_t(a) - \phi_t(a) \cdot \phi_t(b)) \| < \frac{1}{n}.
\]
Combining this with (1) above yields
\[
\| \psi_{r(t)}(\phi_t(ab)) - \psi_{r(t)}(\phi_t(a)) \cdot \psi_{r(t)}(\phi_t(b)) \| < \| \psi_{r(t)}(\phi_t(ab)) - \phi_t(a) \cdot \phi_t(b) \| + \frac{2}{n}
\]
\[
< \| \phi_t(ab) - \phi_t(a) \cdot \phi_t(b) \| + \frac{3}{n}
\]
\[
< \frac{4}{n}.
\]
Since $a, b \in K_m$ for $m \geq n$, the result follows. Hence $r_{00}$ satisfies (i).

Our next goal is to find $r_0 \geq r_{00}$ satisfying (ii) which will finish the proof. By Lemma 3.1.6 we may decompose $\phi(a_{ij}) = f_{ij} + g_{ij}$ with $f_{ij} \in C_b(T, B(U_i))$ and $g_{ij} \in C_0(T, B)$ for each $i$ and $j$. Define the compact sets
\[
M_n := \bigcup_{i,j=1}^n f_{ij}([1,n+1]) \cup g_{ij}([1,n+1]) \subseteq B,
\]
\[
N_m := \bigcup_{j=1}^n f_{ij}([1,n+1]) \subseteq B(U_i).
\]
\(^{13}\)We should warn the reader of certain technicalities regarding the approximate scalar multiplication. We only have nice conditions for $|\lambda| \leq 1$, so we should restrict to this case. A way of doing this is to let $k \geq |\lambda|$, write $\lambda = \frac{1}{k} + \cdots + \frac{1}{k}$ and then use the approximate addition.
Pick an increasing sequence \( (s_n) \) in \( T \) such that for \( t \geq s_n \)
\[
\| \psi_t(a + b) - \psi_t(a) - \psi_t(b) \| < \frac{1}{n}, \quad \text{for } a, b \in M_n
\]
\[
\| \psi_t(a) \| < \| a \| + \frac{1}{n}, \quad \text{for } a \in M_n.
\]

Since \( \psi \) is approximately \( X \)-equivariant and continuous we may pick increasing sequences \( (r_{i,n}) \) in \( T \) for each \( i \), such that
\[
\| \psi_t(a) \|_{X \setminus U_i} < \frac{1}{n}, \quad \text{for all } a \in N_{i,m} \text{ and } t \geq r_{i,n}.
\]

Choose an increasing continuous function \( r_0 : T \to T \) such that \( r_0 \geq r_{00} \) and
\[r_0(n) \geq \max\{s_n, r_{1,n}, \ldots, r_{n,n}\} \] for \( n \geq 1 \). Let \( r \geq r_0 \) and \( i, j \in \mathbb{N} \). For \( m \geq i, j \) and \( m \leq t < m + 1 \) we get
\[
\| \psi_{r(t)}(\phi_0(a_{ij})) \|_{X \setminus U_i} = \| \psi_{r(t)}(f_{ij}(t) + g_{ij}(t)) \|_{X \setminus U_i}
\]
\[
< \| \psi_{r(t)}(f_{ij}(t)) \|_{X \setminus U_i} + \| \psi_{r(t)}(g_{ij}(t)) \|_{X \setminus U_i} + \frac{1}{m}
\]
\[
< \| \psi_{r(t)}(g_{ij}(t)) \|_{X \setminus U_i} + \frac{2}{m}
\]
\[
< \| g_{ij}(t) \| + \frac{3}{m}.
\]

Since \( m \) was chosen arbitrarily large and \( g_{ij}(t) \to 0 \) for \( t \to \infty \) we get that \( r_0 \) satisfies (ii).

We can now prove that there is a composition product. The associativity part of the proof is greatly inspired by ([27], Theorem 1.12).

**Proposition 3.2.2.** The construction in Lemma 3.2.1 induces a composition product
\[=[[A, B]]_X \times [[[B, C]]_X \to [[[A, C]]_X].\]

**Proof.** Let \( \phi : A \to C_b(T, B) \) and \( \psi : B \to C_b(T, C) \) be approximately \( X \)-equivariant asymptotic morphisms. By Lemma 3.1.3 we may assume that \( \phi \) and \( \psi \) are uniformly continuous, and thus induce \( \theta \) from Lemma 3.2.1. We will start by proving that the homotopy class of \( \theta \) depends neither on \( A_0 \) nor \( r_0 \) but only on the homotopy classes of \( \phi \) and \( \psi \). The construction does not depend on \( A_0 \) since any \( * \)-subalgebra of \( A \) generated by two \( \sigma \)-compact \( * \)-subalgebras is again \( \sigma \)-compact.

Let \( r_{00}, r_{01} : T \to T \) such that both satisfy the condition of \( r_0 \) in Lemma 3.2.1, and let \( r_0 \geq r_{00}, r_1 \geq r_{01} \) be continuous and increasing. Pick \( \theta_0 \) and \( \theta_1 \) as in the lemma. Then \((\psi_{s_{r_0(t)}} + (1-s)_{r_0(t)} \circ \phi_t)\) for \( s \in [0,1] \) on \( A_0 \) extends to a homotopy from \( \theta_0 \) to \( \theta_1 \). Hence the construction does not depend on choice of \( r_0 \).

Let \( \sigma \) be a continuous section of the quotient map \( \pi : C_b(T, B) \to B_{\infty} \). If \( \phi_0 \) and \( \phi_1 \) are homotopic then by Lemma 3.1.8 there are uniformly continuous homotopies \( \phi_0 \simeq \sigma \phi_0 \) and \( \sigma \phi_1 \simeq \phi_1 \). Moreover, if \( \Phi \) is the homotopy \( \phi_0 \simeq \phi_1 \) then the map \( a \mapsto \sigma \pi \circ \Phi(a) \) is a uniformly continuous homotopy \( \sigma \phi_0 \simeq \sigma \phi_1 \). Hence we may assume that \( \Phi : \phi_0 \simeq \phi_1 \) and \( \Psi : \psi_0 \simeq \psi_1 \) are uniformly continuous.
Construct the asymptotic morphisms \( \theta_0, \theta_1 \) and \( \Theta \) from the above uniformly continuous approximately \( X \)-equivariant asymptotic morphisms and pick an increasing continuous function \( r : T \to T \) which satisfies the condition from Lemma 3.2.1 for each of the maps. Now
\[
\| \Theta_{t,s}(a) - \theta_{0,t}(a) \| \leq \| \Theta_{t,s} - \Psi_{r(t)} \Phi_t(a) \| + \| \theta_{0,t} - \psi_{0,r(t)} \phi_0(a) \| \to 0
\]
for each \( a \in A_0 \). Hence \( \Theta_0 \) (resp. \( \Theta_1 \)) is equivalent to \( \theta_0 \) (resp. \( \theta_1 \)) and thus \( \theta_0 \approx \theta_1 \) by Lemma 3.1.8.

It remains to show that the product is associative. Let \( A_0 \) be a \( \sigma \)-compact dense \(*\)-subalgebra and observe that \( \bigcup_{t \in T} \phi_t(A_0) \) is \( \sigma \)-compact. Hence we may pick a \( \sigma \)-compact dense \(*\)-subalgebra \( B_0 \) of \( B \) which contains \( \bigcup \phi_t(A_0) \). Let \( (\rho_t) : C \to D \) be yet another uniformly continuous approximately \( X \)-equivariant asymptotic morphism. Using the construction from Lemma 3.2.1, pick \( r \) and \( (\theta_t) \) such that \( \theta_t \) is equivalent to \( (\psi_{r(t)} \phi_t) \) on \( A_0 \) and such that \( r \) satisfies the same as the corresponding \( r_0 \) satisfied in the proof of Lemma 3.2.1 with respect to the compact set \( K_n \) in \( A_0 \) and \( (t_n) \).

Define increasing sequences of compact sets \( (K'_n) \) in \( B_0 \) and \( (K''_n) \) in \( C \) such that
\[
\{ \phi_t(a) \ | \ a \in K_{n+2}, t \leq t_{n+2} \} \subseteq K'_n
\]
\[
\{ \psi_{r(t)}(b) \ | \ b \in K'_{n+2}, r(t) \leq r(t_{n+2}) \} \subseteq K''_n
\]
and such that \( K'_n + K''_n \subseteq K'_n+1 \), etc. (both for \( (K'_n) \) and \( (K''_n) \)) as in the proof of Lemma 3.2.1. Moreover, we want \( \bigcup K'_n = B_0 \). Now we may define an increasing continuous function \( w : T \to T \) such that

- \( (\rho_{w(t)} \psi_{r(t)}) \) is equivalent to a uniformly continuous approximately \( X \)-equivariant asymptotic morphism \( (\eta_t) : B \to D \) on \( B_0 \),
- \( (\rho_{w(t)} \theta_t) \) extends from \( A_0 \) to an approximately \( X \)-equivariant asymptotic morphism \( A \to D \),
- for \( t \geq t_n \), \( a, b \in K''_n \) the norms
  \[
  \| \rho_{w(t)}(ab) - \rho_{w(t)}(a)\rho_{w(t)}(b) \| < \frac{1}{n}
  \]
  and similarly for the other possible norms as in the proof of Lemma 3.2.1.

We claim that
\[
\lim_{t \to \infty} \inf_{s \geq t} \| \rho_{w(s)} \psi_{r(s)} \phi_t(ab) - \rho_{w(s)} \psi_{r(s)} \phi_t(a) \cdot \rho_{w(s)} \psi_{r(s)} \phi_t(b) \| = 0 \quad (2)
\]
for \( a, b \in A_0 \) and similarly for the other possible norms. Let \( a, b \in A_0 \), \( n \in \mathbb{N} \) such that \( a, b \in K_n \), \( t_n \leq t < t_{n+1} \) and \( s \geq t \). Since \( \psi_{r(s)} \phi_t(ab) \in K''_N \) for some \( N \geq n \) and similarly for the other elements, an argument as in the proof of
Lemma 3.2.1 gives that
\[
\|\rho_w(s)\psi_{r(s)}\phi_t(ab) - \rho_w(s)\psi_{r(s)}\phi_t(a) \cdot \rho_w(s)\psi_{r(s)}\phi_t(b)\|
\leq \|\rho_w(s)\psi_{r(s)}\phi_t(ab) - \rho_w(s)\psi_{r(s)}\phi_t(a) \cdot \psi_{r(s)}\phi_t(b)\|
\]  
- \rho_w(s)(\psi_{r(s)}\phi_t(ab) - \psi_{r(s)}\phi_t(a) \cdot \psi_{r(s)}\phi_t(b))
\leq \|\rho_w(s)(\psi_{r(s)}\phi_t(ab) - \psi_{r(s)}\phi_t(a) \cdot \psi_{r(s)}\phi_t(b))\|
\leq \|\rho_w(s)(\psi_{r(s)}\phi_t(a) \cdot \psi_{r(s)}\phi_t(b)) - \rho_w(s)(\psi_{r(s)}\phi_t(a) \cdot \rho_w(s)\psi_{r(s)}\phi_t(b))\|
\leq \frac{1}{n} + \|\psi_{r(s)}\phi_t(ab) - \psi_{r(s)}\phi_t(a) \cdot \psi_{r(s)}\phi_t(b)\| + \frac{1}{n} + \frac{1}{n}
\leq \frac{3}{n} + \frac{4}{n}
\]
where the last inequality follows from the proof of Lemma 3.2.1. Since \( N \geq n \), and \( n \) can be chosen arbitrarily large, (2) follows. Similarly, the same condition follows for the other possible norms.

Lastly, pick \( v \) such that \( v(t) \geq t \) and such that \( (\eta_{v(t)}\phi_t) \) extends from \( A_0 \) to an approximately \( X \)-equivariant asymptotic morphism. By how we constructed \( w \)
\[
(\rho_{w(s)}(v(t) + (1-s)t)\psi_{r(s-v(t)+(1-s)t)}\phi_t), \text{ for } s \in [0, 1]
\]
extends from \( A_0 \) to an approximately \( X \)-equivariant asymptotic morphism from \( A \) to \( ID \). Now, for \( s = 0 \), this is equivalent to \( (\rho_{w(t)}\theta_t) \) on \( A_0 \), which is the construction of \( [[[\rho]]([[\psi]])[\phi]] \), and for \( s = 1 \), it is equivalent to \( \eta_{v(t)}\phi_t \) on \( A_0 \) which is the construction of \( [[[\rho]]([[\psi]])[\phi]] \). Hence the composition product is associative.

\( \square \)

Remark 3.2.3. Let \((\phi_t) : A \to B \) be an approximately \( X \)-equivariant asymptotic morphism and \( \psi : B \to C \) be an \( X \)-equivariant \(*\)-homomorphism. We get a canonically induced approximately \( X \)-equivariant asymptotic morphism from \( B \) to \( C \) by letting \( \psi_t(a) = \psi(a) \). By construction \( [[[\rho]]][[\psi]] = [[[\psi]]][[\phi]] \) if \( \phi \) is uniformly continuous. If \( \phi \) is not uniformly continuous pick a homotopy \( \Phi \) from \( \phi \) to a uniformly continuous asymptotic morphism \( \phi' \). Composing \( \Phi \) with \( \psi \) gives a homotopy from \( \psi \circ \phi \) to \( \psi \circ \phi' \) and thus \( [[[\rho]]][[\psi]] = [[[\psi]]][[\phi]] \), even if \( \phi \) is not uniformly continuous.

Similarly let \((\psi_t) : B \to C \) be an approximately \( X \)-equivariant asymptotic morphism and \( \phi : A \to B \) be an \( X \)-equivariant \(*\)-homomorphism. Note that for \( r \) increasing and continuous such that \( r(t) \geq t \), \((\psi_{r(t)}\phi) \) is an approximately \( X \)-equivariant asymptotic morphism. Since the composition does not depend on the choice of \( r \), we get that \( [[[\rho]]][[\phi]] = [[[\psi_t]]][[\phi]] \), again even if \( \psi \) is not uniformly continuous by the same argument as above.

Let \( A, C \) be a \( C^* \)-algebra over \( X \) and \( B, D \) be \( C^* \)-algebras. There is a canonical \( X \)-equivariant \(*\)-homomorphism of quasi \( C^* \)-algebras over \( X \), \( C_\infty \otimes_{\max} D_\infty \to (C \otimes_{\max} D)_\infty \), given by
\[
[f] \otimes [g] \mapsto [t \mapsto f(t) \otimes g(t)].
\]
Let $\phi : A \to C_b(T, C)$ be an approximately $X$-equivariant asymptotic morphisms and $\psi : B \to D$ be an asymptotic morphism. Define, up to equivalence, the approximately $X$-equivariant asymptotic morphism $\phi \otimes \psi : A \otimes_{\text{max}} B \to C_b(T, C \otimes_{\text{max}} D)$ by the composite

$$A \otimes_{\text{max}} B \xrightarrow{\phi \otimes \psi} C_\infty \otimes_{\text{max}} D_\infty \xrightarrow{\sigma} (C \otimes_{\text{max}} D)_\infty \xrightarrow{\text{ID}} C_b(T, C \otimes_{\text{max}} D),$$

where $\sigma$ is a section of the quotient map.

The homotopy class of $\phi \otimes \psi$ depends only on the homotopy class of $\phi$ and the asymptotic morphism induced by $\psi$. To see this let $\Phi : A \to C_b(T, IC)$ be a homotopy. Identifying $IC \otimes_{\text{max}} D$ with $I(C \otimes_{\text{max}} D)$ we see that $\Phi \otimes \psi$ is a homotopy from $\Phi_0 \otimes \psi$ to $\Phi_1 \otimes \psi$. Similarly a homotopy $\Psi : B \to ID$ induces a homotopy $\phi \otimes \Psi_0$ to $\phi \otimes \Psi_1$.

The reason for introducing these tensor products in this section is the following lemma, which we will end this section by.

**Lemma 3.2.4.** Let $A \to C_b(T, C)$ be an approximately $X$-equivariant asymptotic morphism and $\psi : B \to D$ an asymptotic morphism of $C^*$-algebras. Then

$$[[\phi \otimes 1]][[1 \otimes \psi]] = [[\phi \otimes \psi]] = [[1 \otimes \psi]][[\phi \otimes 1]].$$

**Proof.** The proof is somewhat similar to part (i) in the proof of Lemma 3.2.1. Equivariance plays no important part, and thus it suffices to prove only one of the equalities.

Observe, that for any element $\sum_{a_k}^n a_k \otimes b_k$ in $A \otimes B$, we get that

$$\| (\phi \otimes \psi)_t(\sum_{a_k}^n a_k \otimes b_k) - \sum_{a_k}^n \phi_t(a_k) \otimes \psi(b_k) \| \to 0.$$

Let $(a_k)$ and $(b_k)$ be dense sequences in $A$ and $B$ respectively such that $\text{span}\{a_k \otimes b_k \mid k \geq 0\}$ is dense in $A \otimes_{\text{max}} B$, and choose $\phi \otimes 1$ and $1 \otimes \psi$ uniformly continuous. Define the compact sets

$$K_0 = \{ \lambda(a_0 \otimes b_0) \mid |\lambda| \leq 1\}$$

$$K_n = K_{n-1} + K_{n-1} + \{ \lambda(a_n \otimes b_n) \mid |\lambda| \leq 1\}, \text{ for } n \geq 1$$

and note that $\bigcup K_n = \text{span}\{a_k \otimes b_k \mid k \geq 0\}$. Moreover, we observe that every element in $K_n$ can be written as $\sum_{a_k}^n \lambda_k a_k \otimes b_k$ with $|\lambda_k| \leq 2^n$. Let $(t_n)$ be an increasing sequence in $T$ such that

$$\| (1 \otimes \psi)_t(\sum_{a_k}^n \lambda_k a_k \otimes b_k) - \sum_{a_k}^n \lambda_k a_k \otimes \psi_t(b_k) \| < \frac{1}{n}$$

for $\sum_{a_k}^n \lambda_k(a_k \otimes b_k) \in K_n$ and $t \geq t_n$. Defining the set

$$K'_n = \{ (1 \otimes \psi)_t(x) \mid x \in K_n, t \leq t_n + 1 \}$$

$$+ \{ \sum_{a_k}^n \lambda_k(a_k \otimes b_k) \mid \sum_{a_k}^n \lambda_k(a_k \otimes b_k) \in K_n, t \leq t_n + 1 \}$$

for $\sum_{a_k}^n \lambda_k(a_k \otimes b_k) \in K_n$ and $t \geq t_n$. Defining the set
and doing an argument similar to that of part (i) in the proof of Lemma 3.2.1 we get that
\[ \| (\phi_r \otimes \psi)_t(x) - (\phi \otimes 1)_{r(t)}(1 \otimes \psi)_t(x) \| \to 0, \]
for \( x \) in some suitably dense \( \sigma \)-compact *-subalgebra of \( A \otimes B \subseteq A \otimes_{\text{max}} B \) intersected with \( \text{span}\{ a_k \otimes b_k \mid k \geq 0 \} \). Here \( \phi_r \) is the asymptotic morphism \((\phi_{r(t)})\) which is homotopic to \( \phi \) by Remark 3.2.3.

3.3 The Basics of \( E \)-Theory

As indicated in the sections title, we will in this section define the basics of \( E \)-theory. We will first prove that the homotopy classes of approximately equivariant asymptotic morphisms come equipped with an abelian group structure. We use this to define the \( E \)-groups. Combining these with the composition from the previous section we define the \( E \)-theory category \( \mathcal{E}(X) \). At the end of this section we prove that \( \mathcal{E}(X) \) is an additive category which has countable coproducts.

**Lemma 3.3.1.** Let \( X \) be a sober space, and let \( A \) and \( B \) be \( C^* \)-algebras over \( X \). The composition \( + \) on \([[[A, \Sigma B \otimes K]]_X\) given by
\[ [[\phi]] + [[\phi']] = [[\text{diag}(\phi, \phi')]] \]
where we identify \( M_2(K) \) with \( K \), turns \([[[A, \Sigma B \otimes K]]_X\) into an abelian group.

**Proof.** We start by showing that the composition is well-defined. Clearly this does not depend on representing elements from the homotopy classes, since homotopies pass to matrices. Recall that any isomorphism \( M_2(K) \to K \) is given by \((a_{ij}) \mapsto \sum_{ij} V_i a_{ij} V_j^*\) where \( V_1, V_2 \) are isometries such that \( V_1 V_1^* + V_2 V_2^* = 1 \). Given any other pair of isometries \( V_1', V_2' \) such that \( V_1' V_1'^* + V_2' V_2'^* = 1 \), there is a unitary \( U \in \mathcal{M}(K) \) such that \( U V_i = V_i' \) for \( i = 1, 2 \). Since the unitary group \( \mathcal{U}(\mathcal{M}(K)) \) is path connected there is a continuous path from \( U \) to \( 1 \). Hence any two isomorphisms \( M_2(K) \to K \) are homotopic, and thus the construction does not depend on choice of isomorphism and is thus well-defined. By interchanging \( V_1 \) and \( V_2 \) this also gives that \([[[\phi]] + [[\phi']]] = [[[\phi']]] + [[[\phi]]]\). This could also be seen by the standard rotation trick.

Let \( K(H) \) be the compact operators on \( H \) with orthonormal basis \((\xi_n)_{n \in \mathbb{N}}\), and let \( e_{ij} \) denote the \( ij \)'th matrix units of \( K(H) \), i.e. \( e_{ij} \xi_j = \xi_i \) and \( \ker e_{ij} = \{ \xi_j \} \). There is a canonical isomorphism \( M_2(K(H)) \to K(H) \) which maps \( \text{diag}(0, e_{ij}) \) to \( e_{2i, 2j} = V e_{ij} V^* \) where \( V \) is the isometry \( V \xi_n = \xi_{2n} \). Define a strong operator continuous path of isometries \( V_s \) by
\[ V_s \xi_n = \cos s \cdot \xi_n + \sin s \cdot \xi_{2n}, \quad s \in [0, \frac{\pi}{2}] \]
which connects \( 1 \) and \( V \). Then the asymptotic morphism \( \Phi : A \to C_b(T, I \Sigma B \otimes K) \) given by
\[ \Phi_{s,t}(a) = (1 \otimes V_s) \phi_t(a) (1 \otimes V_s^*) \]
is a homotopy from $\phi$ to $\text{diag}(0, \phi)$ under the above isomorphism. In particular $[[0]]$ is the neutral element of the composition.

To see that the composition is associative use the standard rotation to get that

$$\left( \begin{array}{cc} \text{diag}(\phi, \psi) & 0 \\ 0 & \text{diag}(0, \eta) \end{array} \right) \simeq \left( \begin{array}{cc} \text{diag}(\phi, 0) & 0 \\ 0 & \text{diag}(\psi, \eta) \end{array} \right),$$

and use that $[[0]]$ is the neutral element. It remains to show that every element has an inverse.

Let $\phi'_t : A \to C_0(\mathbb{R}, B \otimes K)$ be an approximately $X$-equivariant asymptotic morphism. Clearly $\phi'$ is homotopic to an approximately $X$-equivariant asymptotic morphism $\phi$ such that $\text{supp}(\phi_t(a)) \subseteq (-\infty, -1]$ for every $a \in A$ and $t \in T$. Let $\overline{\phi}$ denote the asymptotic morphism $\overline{\phi}_t(a)(x) = \phi_t(a)(-x)$ for every $a \in A$, $t \in T$ and $x \in \mathbb{R}$. Define $U : [0, 1] \to C_0(\mathbb{R}, M_2(U\mathcal{M}(B \otimes K)))$ by

$$U_s(x) = \begin{cases} 1, & \text{for } x \leq 0 \\ V_{sx}, & \text{for } 0 < x < 1 \\ V_s, & \text{for } 1 \leq x \end{cases}$$

where $V_s$ is the standard rotation matrix $V_s = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$. Then

$$U_s \text{diag}(\phi_t, \overline{\phi}_t) U_s^*$$

is a homotopy from $\text{diag}(\phi_t, \overline{\phi}_t)$ to $\text{diag}(\phi_t + \overline{\phi}_t, 0)$ which is null homotopic since $\phi_t(a) + \overline{\phi}_t(a)$ is an even function for every $a$ and $t$. The latter observation follows from the homotopy

$$\Phi_{s,t}(a)(x) = \begin{cases} \phi_t(a)(x-s) & \text{for } x \leq 0 \\ \overline{\phi}_t(a)(x+s) & \text{for } x > 0 \end{cases}, \quad \text{for } s \in [0, \infty].$$

Remark 3.3.2. As seen in the proof of Lemma 3.3.1, there are other equivalent definitions of the group addition. We will describe some of these.

Using the idea from proving that the composition was well-defined, let $V'_1, V'_2 \in \mathcal{M}(K)$ be isometries such that $V'_1 V'_1^* + V'_2 V'_2^* = 1$. Define $V_1 := 1_{\Sigma B} \otimes V'_1, V_2 := 1_{\Sigma B} \otimes V'_2 \in \mathcal{M}(\Sigma B \otimes K)$. Then for any approximately $X$-equivariant asymptotic morphisms $\phi$ and $\phi'$ from $A$ to $\Sigma B \otimes K$ we get that

$$[[\phi_1]] + [[\phi_2]] = [[V_1 \phi_1 V_1^* + V_2 \phi_2 V_2^*]],$$

where we use the notation that $(V \phi V^*)_t(a) = V \phi_t(a) V^*$. An other way of realising the group addition is as when proving that $\overline{\phi}$ is the inverse of $\phi$. Let $\phi_1$ and $\phi_2$ be approximately $X$-equivariant asymptotic morphisms which we may choose up to homotopy such that $\text{supp}(\phi_{1,t}(a)) \subseteq (-\infty, -1]$ and $\text{supp}(\phi_{2,t}(a)) \subseteq [1, \infty)$. Then it was proven in Lemma 3.3.1 that $[[\phi_1]] + [[\phi_2]] = [[\phi_1 + \phi_2]].$
We may also view this as follows. Observe that

$$C_b(T, \Sigma B \otimes K) \oplus C_b(T, \Sigma B \otimes K) \cong C_b(T, C_0((-\infty, 0) \cup (0, \infty), B \otimes K)),$$

which is an ideal of $C_b(T, \Sigma B \otimes K)$ by the canonical embedding. Thus if $\phi_1$ and $\phi_2$ are approximately $X$-equivariant asymptotic morphisms from $A$ to $\Sigma B \otimes K$ we may define an approximately $X$-equivariant asymptotic morphism $\phi_1 \vee \phi_2$ by the composite

$$A \xrightarrow{(\phi_1, \phi_2)} C_b(T, \Sigma B \otimes K) \oplus C_b(T, \Sigma B \otimes K) \to C_b(T, \Sigma B \otimes K).$$

Then $[[\phi_1]] + [\phi_2] = [[\phi_1 \vee \phi_2]]$ by the same argument as above.

**Definition 3.3.3.** Let $X$ be a second countable sober space and let $A$ and $B$ be separable $C^*$-algebras over $X$. We define

$$E_0(X; A, B) := \{(\Sigma A \otimes K, \Sigma B \otimes K)\}_X, \quad E_1(X; A, B) := E_0(X; A, \Sigma B)$$

which are abelian groups by Lemma 3.3.1.

We let $\mathcal{E}(X)$ denote the category with objects being all separable $C^*$-algebras over $X$ and morphism sets being the abelian groups $E_0(X; A, B)$.

If two $C^*$-algebras over $X$ are isomorphic in $\mathcal{E}(X)$ we say that they are $E(X)$-equivalent. Similarly, if a $C^*$-algebra over $X$ is $E(X)$-equivalent to 0, we say that it is $E(X)$-contractible.

Also if $X = \star$ is the one point topological space, we denote $E_0(A, B)$ and $\mathcal{E}(X)$ simply by $E_0(A, B)$ and $\mathcal{E}$ respectively.

Due to the definition above we will from now on make the following restrictions.

*From now on we will always assume that every topological space $X$ is second countable and sober, and that every $C^*$-algebra over $X$ is separable.*

There is a canonical functor $\mathcal{E}^*_{\text{sep}}(X) \to \mathcal{E}(X)$ which maps an $X$-equivariant $\ast$-homomorphism $\pi : A \to B$ to the class

$$[\pi] := \{[\Sigma \pi \otimes 1_K]\} \in E_0(X; A, B),$$

where we consider $\Sigma \pi \otimes 1_K$ as a constant asymptotic morphism.

The reader might be wondering why we make this distinction in notation between classes of asymptotic morphisms and classes of asymptotic morphisms induced by $X$-equivariant $\ast$-homomorphisms. Other authors rarely make a distinction in notation but there are several reasons to do this. One is to ease the notation when asymptotic morphisms are induced by $\ast$-homomorphisms, which is often the case. An other reason, and perhaps the most important, is to help us avoid making mistakes such as ([1], Proposition 25.6.2) and ([7], Lemma 2.26). In the proofs the authors represent a class in $E_0(X; A, B)$ by an asymptotic morphism from $A$ to $B$ which we may not since we need to suspend
and stabilize our $C^*$-algebras.\footnote{Both results have been corrected in the form of Lemma 3.7.2. Though this result is not as 'nice' as the corresponding ones of [1] and [7], it serves the exact same purpose and allows us to prove the universal property of $E$-theory.} Hopefully our notation will help us, such that no such mistakes will occur.

**Proposition 3.3.4.** The category $\mathcal{E}(X)$ is additive.

**Proof.** Recall that a category is additive if it has a zero object, any two objects have a product and the morphism sets are abelian groups such that the composition product is bilinear with respect to the group composition.

The zero $C^*$-algebra over $X$ is clearly a zero object. We wish to show that $(B_1 \oplus B_2, [\pi_i])$ is the product of the objects $B_1$ and $B_2$, where $\pi_i : B_1 \oplus B_2 \to B_i$ is the canonical projection for $i = 1, 2$. Let $[[\phi_i]] \in E_0(X; A, B_i)$ for $i = 1, 2$. We have

\[
[\iota_1][[\phi_1]] + [\iota_2][[\phi_2]] = \left[\left(\Sigma \otimes 1_K\right) \phi_1\right] + \left[\left(\Sigma \otimes 1_K\right) \phi_2\right] = \left[\left(\Sigma \otimes 1_K\right) \phi_1 + \left(\Sigma \otimes 1_K\right) \phi_2\right]
\]

where $\iota_i : B_i \to B_1 \oplus B_2$ are the canonical inclusions for $i = 1, 2$. The first equality follows from Remark 3.2.3. To see that the second equality holds consider the path of unitaries $(1, V_s) \in U(M((\Sigma B_1 \otimes K) \oplus (\Sigma B_2 \otimes K)))$ where $V_s$ is the rotation matrix, and the homotopy

\[
\Phi_{s,t}(a) = (1, V_s) \text{ diag}(\iota_1 \phi_1, t(a), \iota_2 \phi_2, t(a)) (1, V_s^*)
\]

which connects $\text{diag}(\iota_1 \phi_1, \iota_2 \phi_2)$ and $\text{diag}(\iota_1 \phi_1 + \iota_2 \phi_2, 0)$. Now by Remark 3.2.3

\[
[\pi_i][[\phi_1]] + [\iota_2][[\phi_2]] = \left[\left(\Sigma \otimes 1_K\right) \phi_1 + \left(\Sigma \otimes 1_K\right) \phi_2\right] = [[\phi_i]]
\]

for $i = 1, 2$. To see that this factorization is unique let $[[\psi]] \in E_0(X; A, B_1 \oplus B_2)$ such that $[[\phi_i]] = [\pi_i][[\psi]]$. We get that

\[
[[\psi]] = \left[\left(\Sigma \otimes 1_K\right) \psi + \left(\Sigma \otimes 1_K\right) \psi\right]
\]

\[
= \left[\left(\Sigma \otimes 1_K\right) \psi\right] + \left[\left(\Sigma \otimes 1_K\right) \psi\right] = \left[\left(\Sigma \otimes 1_K\right) \psi\right]
\]

which implies that $B_1 \oplus B_2$ has the universal property.

It remains to show that the composition product is bilinear. If $[[\phi]] \in E_0(X; A, B) \text{ and } [[\psi_1]], [[\psi_2]] \in E_0(X; B, C)$ with $\phi$, $\psi_1$ and $\psi_2$ uniformly continuous, then pick a $\sigma$-compact dense $*$-subalgebra $A_0$ of $\Sigma A \otimes K$ and an increasing continuous function $r : T \to T$ such that

\[
(\psi_1, r(t) \phi_1), \quad (\psi_2, r(t) \phi_2), \quad (\psi_1 \lor \psi_2, r(t) \phi_1)
\]

are asymptotic morphisms from $A_0$ to $\Sigma C \otimes K$ which extend to approximately $X$-equivariant asymptotic morphisms. Here $\psi_1 \lor \psi_2$ is as defined in Remark 3.3.2. Since $\psi_1, \phi \lor \psi_2, \phi = (\psi_1 \lor \psi_2), \phi$ on $A_0$ it follows that

\[
[[\psi_1]][[\phi]] + [[\psi_2]][[\phi]] = \left([[\psi_1]] + [[\psi_2]]\right)[[\phi]].
\]
Now let $[[\phi_1]], [[\phi_2]] \in E_0(X; A, B)$ and $[[\psi]] \in E_0(X; B, C)$ with $\phi_1, \phi_2$ and $\psi$ uniformly continuous. Observe that the composite

$$(\Sigma B \otimes K) \oplus (\Sigma B \otimes K) \cong C_0(({-\infty, 0}) \cup (0, \infty), B \otimes K) \hookrightarrow \Sigma B \otimes K \xrightarrow{\psi} \Sigma C \otimes K$$

is homotopic to the composite

$$(\Sigma B \otimes K) \oplus (\Sigma B \otimes K) \xrightarrow{(\psi_1, \psi_1)} (\Sigma C \otimes K) \oplus (\Sigma C \otimes K) \hookrightarrow \Sigma C \otimes K,$$

since the inclusions $C_0((-\infty, 0)) \to C_0(\mathbb{R})$ and $C_0((0, \infty)) \to C_0(\mathbb{R})$, are homotopic to the canonical isomorphisms. Let $A_0$ and $r$ be defined similarly as in the above. Then $(\psi_r), ((\phi_1 \vee \phi_2)_i)$ is homotopic to $(\psi_r, \phi_1 \vee \psi_r, \phi_2, \vee r)$ on $A_0$, and this homotopy extends to a homotopy on $\Sigma A \otimes K$ such that

$$[[\psi]]([[\phi_1]] + [[\phi_2]]) = [[\psi]]([[\phi_1]]) + [[\psi]]([[\phi_2]])$$

Recall that additive categories have finite coproducts which are, as objects, equal to the products. One could hope that a countable direct sum of $C^*$-algebras over $X$ induces a countable coproduct in $\mathcal{E}(X)$, and this is in fact the case.

**Proposition 3.3.5.** The category $\mathcal{E}(X)$ has countable coproducts.

**Proof.** Let $B, A_1, A_2, \ldots$ be $C^*$-algebras over $X$ and let $(\phi_i) : \Sigma A_i \otimes K \to \Sigma B \otimes K$ be approximately $X$-equivariant asymptotic morphisms for each $i \in \mathbb{N}$. We may assume that $\phi_i(0) = 0$ for each $i$. Let $A_0$ denote the dense $*$-subalgebra of $\Sigma \bigoplus A_i \otimes K$ of elements $(a_i)$ where only finitely many entries are non-zero. This induces an asymptotic morphism $(\phi_i)$ from $A_0$ to $\Sigma B \otimes K \otimes K$ given by

$$\phi_i(a_i) = \text{diag}(\phi^1_i(a_1), \phi^2_i(a_2), \ldots).$$

This induces a $*$-homomorphism $A_0 \to (\Sigma B \otimes K \otimes K)_\infty$, and thus $(\phi_i)$ extends to an approximately $X$-equivariant asymptotic morphism from $\Sigma \bigoplus A_i \otimes K$ to $\Sigma B \otimes K \otimes K \cong \Sigma B \otimes K$. Moreover, $[[\phi_i]] = [[\phi^0]]$ by Remark 3.3.2.3, where $i : A_i \to \bigoplus A_i$ is the canonical inclusion. Hence $(\bigoplus A_i, [i_i])$ is a coproduct in $\mathcal{E}(X)$ of $A_1, A_2, \ldots$. □

### 3.4 Relations to $K$-Theory - Part I

Until now we have not seen any attempts of trying to calculate the $E$-groups of $C^*$-algebras over $X$. However, in this section we will show that $E$-theory is a generalization of $K$-theory. In fact, the main goal of this section is to show that for any $C^*$-algebra $B$, $E_*(\mathbb{C}, B)$ is naturally isomorphic to $K_*(B)$. In particular, this allows us to compute some $E$-groups by calculating the $K$-theory.

We should note, as we will see in Section 5.4, that this serves a much deeper purpose in implying that the category $\mathcal{E}$ has enough $K$-projective objects.

The results from this section are mostly inspired by similar results from [9].
Lemma 3.4.1. For any separable $C^*$-algebra $A$ the functors

$$E_0(A, -), [[\Sigma A, \Sigma - \otimes \mathbb{K}]] : C^*_{\text{sep}} \to \mathfrak{Ab}$$

are naturally isomorphic.

Proof. The canonical $*$-homomorphism $\Sigma A \to \Sigma A \otimes \mathbb{K}$ given by $a \mapsto a \otimes e_{11}$ induces a group homomorphism $E_0(A, B) \to [[\Sigma A, \Sigma B \otimes \mathbb{K}]]$. It has an inverse given by the composite

$$[[\Sigma A, \Sigma B \otimes \mathbb{K}]] \to [[\Sigma A \otimes \mathbb{K}, \Sigma B \otimes \mathbb{K} \otimes \mathbb{K}]] \to [[\Sigma A \otimes \mathbb{K}, \Sigma B \otimes \mathbb{K}]],$$

where the first map is given by tensoring with $1_{\mathbb{K}}$ and the second is induced by the $*$-isomorphism $p : \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ with inverse homotopic to $1_{\mathbb{K}} \otimes e_{11}$ (and $e_{11} \otimes 1_{\mathbb{K}}$). Checking that these maps are each others inverses is routine and thus omitted. Naturality follows from bilinearity of the composition product.

Lemma 3.4.2. Let $B$ be a separable $C^*$-algebra. Any asymptotic morphism $(\phi_t) : C_0(\mathbb{R}) \to B$ is homotopic to a constant $*$-homomorphism. Moreover, any two such $*$-homomorphisms are homotopic in $C^*_{\text{sep}}$.

Proof. Recall that $C_0(\mathbb{R})$ is the universal $C^*$-algebra generated by an element $a$ such that

$$a^* a + a + a^* = aa^* + a + a^* = 0$$

(i.e. $1 + a$ is a unitary in the unitization). Define the set

$$B_0 = \{ b \in B \mid \|b^* b + b + b^*\| < 1, \|b b^* + b + b^*\| < 1 \}.$$ 

There is a retraction $r : B_0 \to B_0$ given by

$$r(b) = (1 + b)|1 + b|^{-1} - 1,$$

defined by functional calculus in the unitization of $B$. To see that $r$ takes values in $B$, and not in the unitization, observe that

$$|1 + b|^{-1} = \sqrt{1 + b^* b + b + b^*}^{-1} = \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(1 - 2n)(n!)^2 (4^n)} (b^* b + b + b^*)^n \right)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k \left( \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(1 - 2n)(n!)^2 (4^n)} (b^* b + b + b^*)^n \right)^k,$$

and that this latter sum is an element of $B$. This retraction is the identity on elements of $B_0$ for which the norms in the definition of $B_0$ vanish. Moreover, $\|b - r(b)\| \to 0$ if the norms in the definition of $B_0$ tend to zero. Let $(\phi_t) : C_0(\mathbb{R}) \to B$ be an asymptotic morphism and $f(x) = 2i(x - i)^{-1} \in C_0(\mathbb{R})$. Note that $f$ generates $C_0(\mathbb{R})$ and $1 + f$ corresponds to the unitary $z \in C(\mathbb{T})$ when
mapping by the ∗-isomorphism induced by the Cayley transform. Let \( N \) be
given such that \( \phi_{t}(f) \in B_{0} \) for \( t \geq N \) and define an element in \( C_{b}(T, B) \) by

\[
\psi_{t}(f) = \begin{cases} 
  r(\phi_{t}(f)), & \text{for } t \geq N \\
  r(\phi_{N}(f)), & \text{for } t < N .
\end{cases}
\]

By the universal property of \( C_{0}(\mathbb{R}) \) there is a (unique) ∗-homomorphism \( \psi : C_{0}(\mathbb{R}) \rightarrow C_{b}(T, B) \) such that \( \psi_{t}(f) = \psi(f) \). Since \( \phi(f) = \psi(f) \) it follows by
universality of \( C_{0}(\mathbb{R}) \) that \( \hat{\phi} = \psi \) and thus \( \phi \) and \( \psi \) are equivalent. Our desired
result now follows by observing that \( \psi \) is canonically homotopic to the constant ∗-homomorphism \( \psi_{0} \).

If we have chosen two such ∗-homomorphisms \( \pi_{0}, \pi_{1} \) and let \( \Phi : C_{0}(\mathbb{R}) \rightarrow I B \)
be a homotopy from \( \pi_{0} \) to \( \pi_{1} \), then the same construction as above gives a
homotopy of ∗-homomorphisms from \( \pi_{0} \) to \( \pi_{1} \).

With these two lemmas at our aid, we can now prove that \( E \)-theory gener-
alis \( K \)-theory.

**Theorem 3.4.3.** The functors

\[
E_{*}(\mathbb{C}, -), K_{*} : \mathcal{C}^{*}\text{sep} \rightarrow \mathfrak{A}_{0}^{\mathbb{Z}/2}
\]

are naturally isomorphic.

**Proof.** We will prove the equivalent statement, that the functors \( E_{0}(\mathbb{C}, -) \) and
\( K_{1}(\Sigma-) \) are naturally isomorphic. The result then follows since \( K_{1}(\Sigma-) \cong K_{0} \)
and

\[
E_{1}(\mathbb{C}, -) \cong E_{0}(\mathbb{C}, \Sigma-) \cong K_{1}(\Sigma^{2}-) \cong K_{1}.
\]

Recall that any element of \( K_{1}(\Sigma B) \) is represented by a unitary \( u \in (\Sigma B \otimes K)^{\sim} \)
such that \( u - 1 \in \Sigma B \otimes K \). By Lemmas 3.4.1 and 3.4.2 we may represent
an element \([\pi]\) in \( E_{0}(\mathbb{C}, B) \) by a ∗-homomorphism \( \pi : C_{0}(\mathbb{R}) \rightarrow \Sigma B \otimes K \).

We define a map \( E_{0}(\mathbb{C}, B) \rightarrow K_{1}(\Sigma B) \) by mapping \([\pi]\) to \([1 + \pi(f)]\) where
\( f(x) = 2i(x - i)^{-1} \in C_{0}(\mathbb{R}) \). This is well-defined by Lemma 3.4.2 and is a group
homomorphism since

\[
[1 + \text{diag}(\pi_{0}(f), \pi_{1}(f))] = [\text{diag}(1 + \pi_{0}(f), 1 + \pi_{1}(f))] \in K_{1}(\Sigma B).
\]

We claim that this is an isomorphism which we will prove by constructing an
inverse.

Let \( u \in (\Sigma B \otimes K)^{\sim} \) be a unitary such that \( u - 1 \in \Sigma B \otimes K \). By the universal
property of \( C_{0}(\mathbb{R}) \) there is a ∗-homomorphism \( \phi_{u} : C_{0}(\mathbb{R}) \rightarrow \Sigma B \otimes K \) such
that \( \phi_{u}(f) = u - 1 \) where \( f(x) = 2i(x - i)^{-1} \). This induces a map \( K_{1}(\Sigma B) \rightarrow
E_{0}(\mathbb{C}, B) \) by mapping \([u]\) to the class of the constant asymptotic morphism \([\phi_{u}]\)
in \([C_{0}(\mathbb{R}), \Sigma B \otimes K]\) and then applying the natural isomorphism in Lemma 3.4.1.

This map is well-defined since a path of unitaries in \( I (\Sigma B \otimes K) \) from \( u_{0} \) to \( u_{1} \)
induces a homotopy of ∗-homomorphisms \( C_{0}(\mathbb{R}) \rightarrow I \Sigma B \otimes K \) from \( \phi_{u_{0}} \) to \( \phi_{u_{1}} \).

It is straight forward to verify that these two maps are each others inverses.

\( \square \)
3.5 The Connes-Higson Construction

To every extension $0 \to J \to A \to B \to 0$ of $C^*$-algebras over $X$ we may associate an approximately $X$-equivariant asymptotic morphism from $\Sigma B$ to $J$, uniquely determined up to homotopy. This asymptotic morphism is called the Connes-Higson construction, and is due to Alain Connes and Nigel Higson in their original paper [4].

Lemma 3.5.1. Let $J$ be an ideal in the separable $C^*$-algebra $A$. Then there exists a contractive continuous approximate unit $(e_t)_{t \in T}$ in $J$ which is quasi-central in $A$. Moreover, for every $f \in C_0((0,1])$ such an approximate unit satisfies

- $t \mapsto f(e_t)$ is continuous,
- $\|f(e_t)a - af(e_t)\| \to 0$ for every $a$ in $A$,
- $f(e_t)x \to f(1)x$ for every $x$ in $J$.

Proof. Let $(e_n)_{n \in \mathbb{N}}$ be a contractive approximate unit in $J$ which is quasi-central in $A$. For $n < t < n + 1$, defining $e_t := (t - n)e_{n+1} + (n + 1 - t)e_n$ gives the desired approximate unit.

We will prove the three conditions for polynomials $p$ in $C_0((0,1])$. Approximating $f$ by polynomials will yield the desired results. Note that $t \mapsto p(e_t)$ clearly is continuous. Since $(e_t)$ is quasi-central $\|e_t^*a - ae_t^*\| \to 0$ for every $k \in \mathbb{N}$ and $a$ in $A$ and thus $\|p(e_t)a - ap(e_t)\| \to 0$. Finally note that $\|e_t^*x - x\| \to 0$ for every $k \in \mathbb{N}$ and $x$ in $J$. Hence for $p(s) = \sum_{k=1}^n c_k s^k$ we get that

$$\|p(e_t)x - p(1)x\| \leq \sum_{k=1}^n |c_k| \|e_t^kx - x\| \to 0.$$  

Proposition 3.5.2. Let $0 \to J \to A \to B \to 0$ be an extension of $C^*$-algebras over $X$. Then there is an induced approximately $X$-equivariant asymptotic morphism from $\Sigma B$ to $J$, uniquely determined up to homotopy. This is called the Connes-Higson construction.

Proof. In this construction we redefine the suspension functor $\Sigma$ to be tensoring with $C_0((0,1))$. Let $(e_t)_{t \in T}$ be a contractive continuous approximate unit in $J$ which is quasi-central in $A$, and define a $\ast$-homomorphism $\phi' : \Sigma A \to J_\infty$ by

$$\phi'(f \otimes a) = [t \mapsto f(e_t)a], \quad \text{for } f \otimes a \in \Sigma A = C_0((0,1)) \otimes A.$$  

Lemma 3.5.1 implies that $\phi'$ is well-defined and a $\ast$-homomorphism, and that $\phi'$ restricted to $\Sigma J$ is zero. Hence there is an induced $\ast$-homomorphism $\phi : \Sigma B \to J_\infty$ which lifts to an asymptotic morphism $\phi$ from $\Sigma B$ to $J$.

Let $U$ be an open subset of $X$. Since $J(U) = J \cdot A(U)$ it follows that

$$\hat{\phi}(\Sigma B(U)) = \phi'((\Sigma A(U)) \subseteq \frac{C_0(T,J(U)) + C_0(T,J)}{C_0(T,J)}$$  

31
and thus $\phi$ is approximately $X$-equivariant by Lemma 3.1.6.

Clearly the homotopy class of $\phi$ is independent of the lift of $\dot{\phi}$. Let $(e^0_t)$ and $(e^1_t)$ are suitable approximate units with corresponding asymptotic morphisms $\phi^0$ and $\phi^1$. Then $(s \mapsto se^1_t + (1 - s)e^0_t)_{t \in T}$ is a contractive continuous approximate unit of $IJ$ which is quasi-central in $IA$. Hence there is an induced approximately $X$-equivariant asymptotic morphism $\Phi'$ from $\Sigma IB$ to $IJ$. Composing with the canonical embedding $\Sigma B \hookrightarrow \Sigma IB$ we get an approximately $X$-equivariant asymptotic morphism $\Phi$ from $\Sigma B$ to $IJ$ such that $ev_0 \Phi$ (resp. $ev_1 \Phi$) is equivalent to $\phi^0$ (resp. $\phi^1$). Hence the uniqueness holds. \qed

3.6 Excision

The construction of the $E$-groups induces a bifunctor

$$E_0(X; -, -) : \mathcal{C}^{\text{sep}}(X)^{\text{op}} \times \mathcal{C}^{\text{sep}}(X) \to \mathbb{Ab}.$$  

The goal of this section is to prove that this bifunctor satisfies excision in each variable, i.e. for every extension $0 \to J \to A \to B \to 0$ in $\mathcal{C}^{\text{sep}}(X)$ and every separable $C^*$-algebra over $X$, $D$, we get long exact sequences

$$\cdots \to E_0(X; D, \Sigma B) \to E_0(X; D, J) \to E_0(X; D, A) \to E_0(X; D, B)$$

and

$$E_0(X; B, D) \to E_0(X; A, D) \to E_0(X; J, D) \to E_0(X; \Sigma B, D) \to \cdots.$$  

Since the bifunctor is homotopy invariant in each variable, this is equivalent to proving that the bifunctor is half-exact in each variable. Moreover, since the bifunctor is stable in each variable this will imply that the bifunctor satisfies Bott periodicity and induces six-term exact sequences in each variable.

In proving this we also come across another interesting result, that suspension induces a natural isomorphism $E_0(X; A, B) \cong E_0(X; \Sigma A, \Sigma B)$.

Excision was the main reason for the creation of $E$-theory by Connes and Higson in [4]. To quote the article, they call it 'an improvement and a simplification' of Kasparov's $KK$-theory, which does not satisfy excision.\textsuperscript{15}

The results in this section are mostly inspired by similar results in [6], [9] and [27].

Lemma 3.6.1. Consider the commutative diagram of $C^*$-algebras over $X$ with exact rows

$$
\begin{array}{cccccc}
0 & \to & J & \to & A & \xrightarrow{\pi} B & \to & 0 \\
\downarrow{i} & & \downarrow{j} & & \downarrow{j} & & \downarrow{j} \\
0 & \to & J' & \to & A' & \to & B' & \to & 0
\end{array}
$$

and assume that $i$ and $j$ are injective. Let $\gamma$ (resp. $\gamma'$) denote the Connes-Higson construction of the upper (resp. lower) row. If either $i$ or $j$ is an isomorphism then $[[\gamma', \Sigma j]] = [[i \gamma]]$.

\textsuperscript{15}Most operator algebraists would probably disagree that $E$-theory is an improvement of $KK$-theory, but I think that very few would disagree to the fact that it is a simplification.
Proposition 3.6.3. Let theorem and the kernel is Σ.

Proof. Note that the middle vertical arrow is also injective by a simple diagram chase argument. We prove this for \( j \) an isomorphism. The result for \( i \) is similar and simpler. For simplicity we identify \( B \) and \( B' \) and consider \( A \) as a subalgebra of \( A' \). Let \((e_t)\) (resp. \((e'_t)\)) be a continuous approximate unit of \( J \) (resp. \( J' \)), quasi-central in \( A \) (resp. \( A' \)), and let \( \sigma \) be a section of \( \pi \). Then

\[
\gamma^t \Sigma_j (f \otimes b) = f(e'_t) \sigma(b), \quad i \gamma^t (f \otimes b) = f(i(e_t)) \sigma(b),
\]

for \( f \in C_0((0,1)) \) and \( b \in B \). Now

\[
\Phi_{s,t}(f \otimes b) = f((1-s) \cdot e'_t + s \cdot i(e_t)) \sigma(b)
\]
gives a homotopy from \((\gamma^t \Sigma_j)\) to \((i \gamma^t)\).

Lemma 3.6.2. If \( \gamma \) is the Connes-Higson construction of the extension of \( C^* \)-algebras over \( X \), \( 0 \to \Sigma A \to CA \to A \to 0 \), then \([\gamma]\) = \([id_{\Sigma A}]\).

Proof. For convenience we let \( T = [3, \infty) \). Recall that if \((e_t)\) is an approximate unit of \( C_0((0,1)) \), quasi-central in \( C_0([0,1]) \), and \((e'_t)\) is an approximate unit of \( A \), then \((e_t \otimes e'_t)\) is an approximate unit of \( \Sigma A \), quasi-central in \( CA \). For any \( a \in A \) the function \( x \mapsto (1-x)a \) is a lift to \( CA \) and thus \( \gamma^t(f \otimes a) = f(e_t \otimes e'_t)(x) \cdot (1-x)a \). Approximating \( f \) by polynomials we see that

\[
\|f(e_t \otimes e'_t)(x) \cdot (1-x)a - f(e_t(x)) \cdot (1-x)a\| \to 0.
\]

Hence \( \gamma \) is equivalent to \( \gamma' \otimes 1_A \) where \( \gamma' \) is the Connes-Higson construction of \( 0 \to C_0((0,1)) \to C_0((0,1)) \to \mathbb{C} \to 0 \). It remains to show that \( \gamma' \) is homotopic to \( 1_{C_0((0,1))} \).

Define the functions

\[
e_t(x) = \begin{cases} tx, & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{for } \frac{1}{2} \leq x \leq 1 - \frac{1}{2}, \\ t(1-x), & \text{for } 1 - \frac{1}{2} \leq x \leq 1 \end{cases}, \quad g_t(x) = \begin{cases} tx, & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}.
\]

We easily see that \( \|f(e_t(x)) \cdot (1-x) - f(g_t(x))\| \to 0 \) and since \( f \mapsto f(g_t) \) is homotopic to \( 1_{C_0((0,1))} \) we are done.

Recall that the mapping cone \( C_\pi \) of an \( X \)-equivariant \(*\)-homomorphism \( \pi : A \to B \) is the pull-back \( A \oplus_B CB \). Due to the pull-back diagram, there are canonical \( X \)-equivariant \(*\)-homomorphisms \( \Sigma B \to C_\pi \) and \( J \to C_\pi \) where \( J = \ker \pi \). Moreover, there is a canonical \( X \)-equivariant \(*\)-homomorphism \( CA \to C_\pi \) given by \( a \mapsto (a(0), \pi \circ a) \). This is surjective by the Bartle-Graves selection theorem and the kernel is \( \Sigma J \).

Proposition 3.6.3. Let \( 0 \to J \to A \xrightarrow{\pi} B \to 0 \) be an extension of \( C^* \)-algebras over \( X \), let \( i : J \to C_\pi \) be the canonical injection and let \( \gamma \) be the Connes-Higson construction of \( 0 \to \Sigma J \to CA \to C_\pi \to 0 \). Then \([\gamma]\) = \([1_{\Sigma J}]\) and \([\Sigma i][[\gamma]] = [[1_{\Sigma J}]\).
Proof. We will show that \([\gamma_t \Sigma i] = [1_{\Sigma J}]\) and \([\Sigma i \gamma_t] = [1_{\Sigma C_\pi}]\). Since \(\Sigma i\) is a \(*\)-homomorphism this will imply the result by Remark 3.2.3.

Consider the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & \Sigma J & \rightarrow & C J & \rightarrow & J & \rightarrow & 0 \\
| & | & \downarrow & & \downarrow & & \downarrow i \\
0 & \rightarrow & \Sigma J & \rightarrow & C A & \rightarrow & C_\pi & \rightarrow & 0 \\
| & \downarrow \Sigma i & | & \downarrow j & | & \downarrow & & \downarrow \\
0 & \rightarrow & \Sigma C_\pi & \rightarrow & C C_\pi & \rightarrow & C_\pi & \rightarrow & 0
\end{array}
\]

where \(j(a) = (x \mapsto (a(x), y \mapsto \pi(a(x + y - xy))))\). By lemmas 3.6.1 and 3.6.2 the top half of the diagram implies that \([\gamma_t \Sigma i] = [1_{\Sigma J}]\) and the bottom half gives \([\Sigma i \gamma_t] = [1_{\Sigma C_\pi}]\). \(\Box\)

Corollary 3.6.4. The \(X\)-equivariant \(*\)-homomorphism \(i : J \rightarrow C_\pi\) induces an isomorphism in \(\mathcal{C}(X)\).

Proposition 3.6.5. Let \(\pi : A \rightarrow B\) be a surjective \(X\)-equivariant \(*\)-homomorphism, \(\alpha : C_\pi \rightarrow A\) be the canonical projection and \(D\) be a \(C^*\)-algebra over \(X\). Then

\([D, C_\pi]_X \xrightarrow{\alpha_*} [D, A]_X \xrightarrow{\pi_*} [D, B]_X\)

is an exact sequence of pointed sets.

Proof. Clearly \(\pi_* \alpha_* = 0\) since \(\pi \alpha\) factors through \(CB\) which is contractible. Let \(\phi\) be an approximately \(X\)-equivariant asymptotic morphism from \(D\) to \(A\) such that \(\pi \circ \phi\) is null-homotopic. Let \(\Phi\) be such a homotopy from \(D\) to \(CB\). Then \((\phi, \Phi)\) gives an approximately \(X\)-equivariant asymptotic morphism from \(D\) to \(A \oplus_B CB = C_\pi\) and \(\alpha_*[(\phi, \Phi)] = [\phi]\). \(\Box\)

From Proposition 3.6.3 we immediately get the following corollary.

Corollary 3.6.6. Let \(D\) be a \(C^*\)-algebra over \(X\). The functor \([D, \Sigma -]_X\) is a half-exact covariant functor from \(\mathcal{C}^{*}\text{sep}(X)\) to the category of pointed sets. In particular, since it is homotopy invariant, it satisfies excision.

Let \(0 \rightarrow J \rightarrow A \xrightarrow{\gamma} B \rightarrow 0\) be an extension in \(\mathcal{C}^{*}\text{sep}(X)\). The connecting map \([D, \Sigma^2 B]_X \rightarrow [D, \Sigma J]_X\) is given by composing the induced map of \(J : \Sigma^2 B \rightarrow \Sigma C_\pi\) with the inverse of the induced map of \(\Sigma J : \Sigma J \rightarrow \Sigma C_\pi\). Since \(\Sigma i_*\) is multiplication by \([\Sigma i]\), Proposition 3.6.3 implies that the connecting map is multiplication by \([\gamma][\Sigma j]\). \(\Box\)

Lemma 3.6.7. Let \(A\) be a \(C^*\)-algebra over \(X\), \(\eta\) be the Connes-Higson construction of the reduced Toeplitz extension \(0 \rightarrow K \rightarrow \mathcal{T}_0 \xrightarrow{\gamma} C_0(\mathbb{R}) \rightarrow 0\), \(\gamma\) be the Connes-Higson construction of \(0 \rightarrow \Sigma K \otimes A \rightarrow CT_0 \otimes A \rightarrow C_\pi \otimes A \rightarrow 0\) and \(j : \Sigma^2 A \rightarrow C_\pi \otimes A\) be the inclusion. Then

\([\gamma][\Sigma j] = [\Sigma \eta \otimes 1_A]\)

where \(\eta(g \otimes h) = \eta(\overline{g} \otimes h)\) with \(\overline{g}(x) = g(-x)\).
such that $\rho$ stability.

by Lemma 3.6.7 that this isomorphism $D$, stable, homotopy functor and thus Bott periodicity holds. By construction of the suspension induces a natural group isomorphism $\Sigma X$.

Clearly $\Sigma\phi$ induces a group homomorphism and naturality follows from Lemma 3.2.4. By Lemma 3.2.4 we get

$$[[1_{\Sigma^2C} \otimes \phi]] [[\eta_B]] = [[\eta_A]] [[1_{\Sigma} \otimes \phi]]$$

for any approximately $X$-equivariant asymptotic morphism $\phi : \Sigma A \otimes K \to C_0(T, \Sigma B \otimes K)$. In particular, $[[\Sigma^2 \phi]] = [[\eta_A]] [[1_{\Sigma} \otimes \phi]] [[\eta_B]]^{-1}$ which implies

**Proposition 3.6.9.** Suspension induces a natural group isomorphism

$$E_0(X; A, B) \cong E_0(X; \Sigma A, \Sigma B).$$

**Proof.** Clearly $\Sigma$ induces a group homomorphism and naturality follows from Lemma 3.2.4. By Lemma 3.2.4 we get

$$[[1_{\Sigma^2C} \otimes \phi]] [[\eta_B]] = [[\eta_A]] [[1_{\Sigma} \otimes \phi]]$$

for any approximately $X$-equivariant asymptotic morphism $\phi : \Sigma A \otimes K \to C_0(T, \Sigma B \otimes K)$. In particular, $[[\Sigma^2 \phi]] = [[\eta_A]] [[1_{\Sigma} \otimes \phi]] [[\eta_B]]^{-1}$ which implies
that $\Sigma^2$ induces an isomorphism. Hence $\Sigma$ is injective. That $\Sigma$ is surjective follows from commutativity of the following diagram

$$
\begin{array}{c}
E_0(X; \Sigma A, \Sigma B) \xrightarrow{\Sigma} E_0(X; \Sigma^2 A, \Sigma^2 B) \xrightarrow{[\eta]^*} E_0(X; A, B) \\
\downarrow \Sigma \downarrow \Sigma \\
E_0(X; \Sigma A, \Sigma B) \xrightarrow{\Sigma^2} E_0(X; \Sigma^3 A, \Sigma^3 B) \xrightarrow{[\eta]^*} E_0(X; \Sigma A, \Sigma B).
\end{array}
$$

Remark 3.6.10. By the above proposition and Bott periodicity in the second variable, we observe that

$$E_1(X; A, B) = E_0(X; A, \Sigma B) \cong E_0(X; \Sigma A, \Sigma^2 B) \cong E_0(X; \Sigma A, B)$$

where the isomorphisms are natural. Hence we will not make any distinction between these groups.

We are now ready to prove that $E$-theory satisfies excision.

**Theorem 3.6.11.** The bifunctor

$$E_0(X; -, -) : \mathcal{C}^* \text{sep}(X)^{op} \times \mathcal{C}^* \text{sep}(X) \to \text{Ab}$$

is a half-exact, stable, homotopy functor in each variable.

**Proof.** We already know the result in the second variable and thus only need half-exactness in the first variable. Let $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$ be an extension in $\mathcal{C}^* \text{sep}(X)$, let $\alpha : C_\pi \to A$ be the projection, $\beta : \Sigma A \to C_\alpha$ be the inclusion and $\tau : \Sigma B \to C_\alpha$ be the $X$-equivariant homomorphism $\tau(f \otimes b) = ((0, \tilde{f} \otimes b), 0)$, where $\tilde{f}(x) = f(-x)$. One easily sees that $\tau$ is a homotopy equivalence. Let $D$ be a $C^*$-algebra over $X$. We get the following commutative diagram

$$
\begin{array}{c}
E_0(X; B, D) \xrightarrow{\pi^*} E_0(X; A, D) \xrightarrow{\iota^*} E_0(X; I, D) \\
\downarrow \Sigma \downarrow \Sigma \downarrow \Sigma \\
E_0(X; \Sigma B, \Sigma D) \xrightarrow{(\Sigma \pi)^*} E_0(X; \Sigma A, \Sigma D) \xrightarrow{(\Sigma \alpha)^*} E_0(X; \Sigma I, \Sigma D) \\
\downarrow \tau^* \downarrow \tau^* \downarrow \tau^* \\
E_0(X; C_\alpha, \Sigma D) \xrightarrow{\beta^*} E_0(X; \Sigma A, \Sigma D) \xrightarrow{(\Sigma \alpha)^*} E_0(X; \Sigma C_\pi, \Sigma D),
\end{array}
$$

and thus it suffices to show that the bottom row is exact. Note that $\beta(\Sigma \alpha)$ is null-homotopic. This is the case since it factors through $CC_\pi$ by the $X$-equivariant $\ast$-homomorphism

$$CC_\pi \ni f \otimes (a, g) \mapsto (f(0)(a, g), f \otimes a) \in C_\alpha.$$
Hence $(\Sigma \alpha)^* \beta^* = 0$. Let $(\phi_t) : \Sigma^3 A \otimes K \to \Sigma^3 D \otimes K$ such that $[\phi]$ is in the kernel of $(\Sigma \alpha)^*$. By Proposition 3.6.9 we may pick $(\phi'_t) : \Sigma^2 A \otimes K \to \Sigma^2 D \otimes K$ such that $[\Sigma \phi'] = [\phi]$. By naturality of $\Sigma$, $[\phi']$ is in the kernel of $\alpha^*$. Hence we can pick a homotopy $(\Phi_t) : \Sigma^2 C \pi \otimes K \to C_{-1} \Sigma^3 D \otimes K$ from 0 to $(\phi'_t)(\Sigma^2 \alpha \otimes 1)$. This induces an approximately $X$-equivariant asymptotic morphism of pull-backs

$$\Sigma^2 C \pi \otimes K \oplus \Sigma^2 A \otimes K \xrightarrow{\Phi_t \oplus C \phi'} C_{-1} \Sigma^2 D \otimes K \oplus \Sigma^2 D \otimes K \Sigma^2 D \otimes K,$$

where we note that the first pull-back is $\Sigma^2 C \pi \otimes K$, by the universal property of pull-backs, and that the second is $\Sigma^2 D \otimes K$. We finish the proof by observing that

$$\beta^* [\Phi_t \oplus C \phi'] = [\Sigma \phi'] = [\phi].$$

We may now apply Joachim Cuntz’ proof of Bott periodicity to get the following corollary. See Section A for the proof.

**Corollary 3.6.12.** The bifunctor $E_0(X; -, -)$ from $C^* \text{sep}(X)$ to $\text{Ab}$ satisfies Bott periodicity. Hence there are natural isomorphisms

$$E_0(X; A, B) \cong E_0(X; \Sigma^2 A, B) \cong E_0(X; A, \Sigma B) \cong E_0(X; A, \Sigma^2 B).$$

Moreover, for any extension $0 \to J \to A \to B \to 0$ in $C^* \text{sep}(X)$ and any separable $C^*$-algebra over $X, D$, we get six-term exact sequences

$$E_0(X; D, J) \to E_0(X; D, A) \to E_0(X; D, B) \quad \text{and} \quad E_0(X; J, D) \leftarrow E_0(X; A, D) \leftarrow E_0(X; B, D)$$

$$E_1(X; D, B) \leftarrow E_1(X; D, A) \leftarrow E_1(X; D, J) \quad \text{and} \quad E_1(X; B, D) \to E_1(X; A, D) \to E_1(X; J, D).$$

### 3.7 The Universal Property

In this section we prove the universal property of $E$-theory. Remarkably enough, the universal property of $E$-theory predates any form of $E$-theory in terms of asymptotic morphisms. To elaborate, Nigel Higson proved in [11] the existence of a bivariant theory satisfying excision and the universal properties, without constructing it. After having proved the existence of $E$-theory, Higson and Connes constructed it in terms of asymptotic morphisms in [4].

After proving the universal property, we apply it to show that certain functors of $C^*$-algebra categories descend to functors of $E$-theory categories. Then
we prove that if such functors are adjoint on the level of \( C^\ast \)-algebras, then they are adjoint in \( E \)-theory.

We start by stating the universal property.

**Theorem 3.7.1 (The Universal Property of \( E \)-theory).** The functor
\[
E^\ast \text{sep}(X) \to \mathcal{E}(X)
\]
is the universal half-exact, stable, homotopy functor, i.e. if \( F : E^\ast \text{sep}(X) \to \mathfrak{A} \) is a covariant (resp. contravariant), half-exact, stable, homotopy functor with \( \mathfrak{A} \) an abelian category, then \( F \) factors uniquely through \( \mathcal{E}(X) \).

In order to prove the universal property we will need a lemma. The corresponding results in ([1], Proposition 25.6.2) and ([7], Lemma 2.26) have mistakes in them. They represent an element in \( E_0(X; A, B) \) by an asymptotic morphism \( A \to B \) which one cannot necessarily do. We need to suspend and stabilize our \( C^\ast \)-algebras first. In doing this, the construction becomes a bit less pretty, but it still allows us to prove the universal property.

**Lemma 3.7.2.** Let \( \phi : \Sigma A \otimes K \to C_0(T, \Sigma B \otimes K) \) be an approximately \( X \)-equivariant asymptotic morphism. Then there exist \( X \)-equivariant \( \ast \)-homomorphisms \( \alpha \) and \( \beta \) such that
\[
[[\alpha][\beta]^{-1}] = [[\Sigma^2 \phi \otimes 1_K]].
\]

**Proof.** For simplicity let \( A' = \Sigma A \otimes K \) and \( B' = \Sigma B \otimes K \). We have the following pull-back diagram
\[
\begin{array}{cccccc}
0 & \to & C_0(T, \Sigma B') & \to & E & \to & \Sigma A' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \alpha' & & \downarrow \Sigma \phi & & \downarrow & \\
0 & \to & C_0(T, \Sigma B') & \to & C_0(T, \Sigma B') & \to & (\Sigma B')_\infty & \to & 0,
\end{array}
\]
and by Lemma 2.3.7 \( E \) is a \( C^\ast \)-algebra over \( X \), even though \( C_0(T, \Sigma B') \) and \( (\Sigma B')_\infty \) are only quasi \( C^\ast \)-algebras over \( X \). Since \( C_0(T, B') \) is contractible, we get by six-term exactness that
\[
\beta_* : E_0(X; \Sigma A', E) \to E_0(X; \Sigma A', \Sigma A')
\]
and
\[
\beta^* : E_0(X; \Sigma A', E) \to E_0(X; E, E)
\]
are isomorphisms. Since \( \beta_* \) and \( \beta^* \) are given by respectively left and right multiplication by \( [\beta] \), we get that \( [\beta] \) is invertible since we can map to both \([1_{\Sigma A'}]\) and \([1_E]\). Note that \( \alpha' \) is a (non-approximately) \( X \)-equivariant asymptotic morphism from \( E \) to \( \Sigma B' \) and that by construction \([[[\Sigma^2 \phi \otimes 1_K]]][[\beta]] = [[[\Sigma \alpha' \otimes 1_K]]]\).

Let \( \alpha : E \to \Sigma B' \) be the \( X \)-equivariant \( \ast \)-homomorphism given by \( \alpha(x) = \alpha'(x)(0) \). Clearly \([\alpha] = [[[\Sigma \alpha' \otimes 1_K]]]\) and thus
\[
[[[\Sigma^2 \phi \otimes 1_K]]] = [\alpha][\beta]^{-1}.
\]

\( \square \)
Proof of Theorem 3.7.1. We will prove the covariant case. The contravariant case is similar. Note that for any \( C^* \)-algebra over \( X \) there is a natural isomorphism \( F(A) \cong F(\Sigma^2 A \otimes \mathbb{K}) \) by Bott periodicity and stability. Using the notation from Lemma 3.7.2 we define \( F([\phi]) \) to be the composite

\[
F(A) \cong F(\Sigma^2 A \otimes \mathbb{K}) \xrightarrow{F(\beta)^{-1}} F(E) \xrightarrow{F(\alpha)} F(\Sigma^2 B \otimes \mathbb{K}) \cong F(B).
\]

We wish to show that this gives a well-defined functor. Note that \( F(\beta) \) is an isomorphism by six-term exactness.

Let \( \phi_0 \) and \( \phi_1 \) be homotopic approximately \( X \)-equivariant asymptotic morphism and \( \Phi \) be a corresponding homotopy. Construct the obvious associated pull-backs \( E_0, E_1 \) and \( E \) as in Lemma 3.7.2. For \( i = 0, 1 \) we get the following commutative diagram with exact rows

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & C_0(T, I\Sigma B') & \longrightarrow & E & \xrightarrow{\beta} & \Sigma A' & \longrightarrow & 0 \\
& & \downarrow{\text{ev}_i} & & & & \downarrow{\rho_i} & & \\
0 & \longrightarrow & C_0(T, \Sigma B') & \longrightarrow & E_i & \xrightarrow{\beta_i} & \Sigma A' & \longrightarrow & 0.
\end{array}
\]

Since \( F(\beta) \) and \( F(\beta_i) \) are isomorphisms, so is \( F(\rho_i) \). Observe that \( \alpha_i \rho_i = \text{ev}_i \alpha \). Hence we get that

\[
F(\alpha_i)F(\beta_i)^{-1} = F(\alpha_i)F(\rho_i)F(\beta)^{-1} = F(\text{ev}_i)F(\alpha)F(\beta)^{-1}
\]

and since \( \text{ev}_i \) are clearly homotopic we obtain

\[
F(\alpha_0)F(\beta_0)^{-1} = F(\alpha_1)F(\beta_1)^{-1}.
\]

If \( \phi : A \to B \) is an \( X \)-equivariant \( * \)-homomorphism then \( \beta \) is a homotopy equivalence with inverse given by

\[
\Sigma^2 A \otimes \mathbb{K} \ni a \mapsto (a, t \mapsto (\Sigma^2 \phi \otimes 1_\mathbb{K})(a)) \in E.
\]

The composite of this \( * \)-homomorphism with \( \alpha \) is \( \Sigma^2 \phi \otimes 1_\mathbb{K} \). Hence

\[
F([\phi]) = F(\phi).
\]

To see that \( F \) respects composition use that Bott periodicity and stability yield natural isomorphisms in \( E \)-theory. Then we may use Lemma 3.7.2 to write our composite as a product of classes of \( * \)-homomorphisms and inverses of such. By naturality of Bott periodicity and stability of \( F \) we see that \( F \) respects composition in \( E \)-theory. Hence \( F \) factors through \( \mathcal{E}(X) \). Uniqueness of this factorization follows from Lemma 3.7.2 an the fact that Bott periodicity and stability yield natural isomorphisms.

Proposition 3.7.3. Let \( X \) and \( Y \) be second countable sober spaces and \( F : \mathcal{C}^{* \text{sep}}(X) \to \mathcal{C}^{* \text{sep}}(Y) \) be an exact (co- or contravariant) functor which preserves homotopy and stability, i.e. \( F(\phi) \simeq F(\psi) \) if \( \phi \simeq \psi \) and \( F(A \otimes \mathbb{K}) \cong F(A) \otimes \mathbb{K} \) naturally. Then \( F \) descends uniquely to an additive functor \( F_E : \mathcal{E}(X) \to \mathcal{E}(Y) \).
Proposition 3.7.4. Let \( F \) that since \( F \) is exact and preserves homotopy and stability, the functors

\[
E_0(Y; A, F(-)) : \mathcal{C}^* \text{sep}(X) \to \mathcal{A}b, \quad E_0(Y, F(-), A) : \mathcal{C}^* \text{sep}(X)^{\text{op}} \to \mathcal{A}b
\]

are half-exact, stable, homotopy functors for every object \( A \) in \( \mathcal{C}^* \text{sep}(Y) \).

Proof. We do the covariant case. The contravariant is similar. One should note that since \( F \) is exact and preserves homotopy and stability, the functors

\[
E_0(Y; A, F(-)) : \mathcal{C}^* \text{sep}(X) \to \mathcal{A}b, \quad E_0(Y, F(-), A) : \mathcal{C}^* \text{sep}(X)^{\text{op}} \to \mathcal{A}b
\]

We may define \( F_{\mathcal{E}} \) exactly as in the proof of the universal property of \( E \)-theory. The proof that \( F_{\mathcal{E}} \) is well-defined and unique is identical to that in the proof of the universal property with exception of why \( F(\beta) \) is an isomorphism in \( \mathcal{E}(Y) \). To see this, we note that \( E_0(Y; F(\Sigma A'), F(-)) : \mathcal{C}^* \text{sep}(X) \to \mathcal{A}b \) is a half-exact, stable, homotopy functor and thus by six-term exactness

\[
E_0(Y; F(\Sigma A'), F(E)) \cong E_0(Y; F(\Sigma A'), F(\Sigma A')),
\]

where the isomorphism is given by left multiplication by \([F(\beta)]\). The element in \( E_0(Y; F(\Sigma A'), F(E)) \) which is mapped to \([1_{F(\Sigma A')}\]) is a right inverse of \([F(\beta)]\). A left inverse can be found similarly using the functor \( E_0(Y; F(-), F(E)) \) and thus \([F(\beta)]\) is an isomorphism.

To see that \( F_{\mathcal{E}} \) is additive recall that \( A \oplus B \) is the (categorical) product of the objects \( A \) and \( B \) in \( \mathcal{E}(Y) \). Hence it suffices to prove that \( F(A \oplus B) \cong F(A) \oplus F(B) \) in \( \mathcal{C}^* \text{sep}(Y) \). This follows from the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & F(A) & \longrightarrow & F(A \oplus B) & \longrightarrow & F(B) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & F(A) & \longrightarrow & F(A) \oplus F(B) & \longrightarrow & F(B) & \longrightarrow & 0
\end{array}
\]

which has exact rows. It follows from the five lemma that the \( X \)-equivariant \( * \)-homomorphism \( F(A \oplus B) \to F(A) \oplus F(B) \) is an isomorphism and thus \( F_{\mathcal{E}} \) is additive.

\[\square\]

Proposition 3.7.4. Let \( X \) and \( Y \) be second countable sober spaces and

\[
F : \mathcal{C}^* \text{sep}(X) \to \mathcal{C}^* \text{sep}(Y), \quad G : \mathcal{C}^* \text{sep}(Y) \to \mathcal{C}^* \text{sep}(X)
\]

be exact (co- or contravariant) which preserve homotopy and stability. If \( F \) is a left adjoint of \( G \) then \( F_{\mathcal{E}} \) is a left adjoint of \( G_{\mathcal{E}} \) such that

\[
E_0(Y; F_{\mathcal{E}}(A), B) \cong E_0(X; A, G_{\mathcal{E}}(B))
\]

is a natural group isomorphism.

Proof. Let \( \varepsilon : 1_X \to GF \) and \( \delta : FG \to 1_Y \) be the unit and counit respectively of the adjunction. Consider the two maps

\[
E_0(Y; F_{\mathcal{E}}(A), B) \ni [[\psi]] \mapsto G_{\mathcal{E}}([[\psi]])[\varepsilon_A] \in E_0(X; A, G_{\mathcal{E}}(B))
\]

\[
E_0(X; A, G_{\mathcal{E}}(B)) \ni [[\phi]] \mapsto [\delta_B]F_{\mathcal{E}}([[\phi]]) \in E_0(Y; F_{\mathcal{E}}(A), B)
\]

40
which are group homomorphisms since \( F_\xi \) and \( G_\xi \) are additive and since composition is bilinear. We claim that these two group homomorphisms are each others inverses. For \( ([\phi]) \in E_0(X; A, G_\xi(B)) \) we use Lemma 3.7.2 to construct the diagram

\[
\begin{array}{cccccc}
GF(A) & \xrightarrow{\cong} & GF(\Sigma^2 A \otimes \mathbb{K}) & \xrightarrow{[GF(\beta)]^{-1}} & GF(E) & \xrightarrow{[GF(\alpha)]} & GFG(\Sigma^2 B \otimes \mathbb{K}) & \xrightarrow{\cong} & GFG(B) \\
[\varepsilon_A] & & [\varepsilon_{\Sigma^2 A \otimes \mathbb{K}}] & & [\varepsilon_E] & & [G(\delta_{\Sigma^2 B \otimes \mathbb{K}})] & & [G(\delta_B)] \\
A \xrightarrow{\cong} \Sigma^2 A \otimes \mathbb{K} & \xrightarrow{[\beta]^{-1}} & E & \xrightarrow{[\alpha]} & G(\Sigma^2 B \otimes \mathbb{K}) & \xrightarrow{\cong} & G(B)
\end{array}
\]

for which the first, second and fourth square commute by naturality of \( \varepsilon \) and \( \delta \). The third square commutes by naturality and since

\[ [G(\delta_{\Sigma^2 B \otimes \mathbb{K}})][\varepsilon_{G(\Sigma^2 B \otimes \mathbb{K})}] = [1_{G(\Sigma^2 B \otimes \mathbb{K})}]. \]

The composite of the top row is exactly \( G_\xi F_\xi ([\phi]) \), since \( G_\xi F_\xi = (GF)_\xi \) by uniqueness of the descending functor, and the bottom row is exactly \( ([\phi]) \). Hence

\[ G_\xi ([\delta]) G_\xi F_\xi ([\phi]) [\varepsilon] = ([\phi]). \]

Similarly we get that

\[ [\delta] F_\xi G_\xi ([\psi]) F_\xi [\varepsilon] = ([\psi]) \]

and thus the group homomorphisms are each others inverses. Naturality of the group isomorphisms follows from naturality of \( \varepsilon \) and \( \delta \).

We will end this section by combining the above proposition with the results found in Section 2.4. Note that all the functors we worked with in that section preserve homotopy and stability. Hence we get the following corollary.

**Corollary 3.7.5.** Let \( X \) be a second countable sober space.

- If \( Y \subseteq X \) is a subspace, then
  \[
  E_0(X; i_Y^X(A), B) \cong E_0(Y; A, r_Y^X(B)), \text{ if } Y \text{ is open} \\
  E_0(X; B, i_Y^X(A)) \cong E_0(Y; r_Y^X(B), A), \text{ if } Y \text{ is closed},
  \]
  are natural isomorphisms, for \( A \in \text{Ob}(\mathcal{C}^*\text{sep}(Y)), B \in \text{Ob}(\mathcal{C}^*\text{sep}(X)) \).

- If \( x \in X \) then
  \[
  E_0(X; A, i_x(B)) \cong E_0(A(\overline{x}), B) \\
  E_0(X; i_x(B), A) \cong E_0(B, F_x(A))
  \]
  are natural isomorphisms, for \( A \in \text{Ob}(\mathcal{C}^*\text{sep}(X)), B \in \text{Ob}(\mathcal{C}^*\text{sep}) \).
  
  In particular, if \( x \) has a minimal open neighborhood \( U_x \), then
  \[
  E_0(X; i_x(B), A) \cong E_0(B, A(U_x))
  \]
  is a natural isomorphism, for \( A \in \text{Ob}(\mathcal{C}^*\text{sep}(X)), B \in \text{Ob}(\mathcal{C}^*\text{sep}) \).
• If $X$ is compact Hausdorff, then

$$E_0(X; C(X, A), B) \cong E_0(A, B(X))$$

is a natural isomorphism, for $A \in \text{Ob}(\mathcal{C}^{\text{sep}})$, $B \in \text{Ob}(\mathcal{C}^{\text{sep}}(X))$. 
4 \textit{E}-Theory over Finite Spaces

An important part of our study is \textit{E}-theory for $C^*$-algebras over finite sober spaces. In fact, most of our remaining research is only for finite spaces or, more precisely, only finite \textit{sober} spaces which is the same as finite $T_0$ spaces (also known as Kolmogorov spaces). Many mathematicians are fairly sceptical as to the amount of interesting results one can get by studying \textit{finite} topological spaces. To these people, I would like to show the following table.\footnote{The numbers in the table are from the Wikipedia page on finite topological spaces \url{http://en.wikipedia.org/wiki/Finite_topological_space}. The authenticity of these numbers are, for me, unknown.}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Number of elements in $X$ & Non-homeomorphic topologies on $X$ & Non-homeomorphic $T_0$-topologies on $X$ \\
\hline
1 & 1 & 1 \\
2 & 3 & 2 \\
3 & 9 & 5 \\
4 & 33 & 16 \\
5 & 139 & 63 \\
6 & 718 & 318 \\
7 & 4,535 & 2,045 \\
8 & 35,979 & 16,999 \\
9 & 363,083 & 183,231 \\
10 & 4,717,687 & 2,567,284 \\
\hline
\end{tabular}
\end{table}

Hopefully this indicates that there is a lot of theory to investigate.

There are two main results of this section. One says, informally, that the \textit{E}-theory over a second countable sober space may be approximated by \textit{E}-theory over finite sober spaces. We use this to prove our other main result which tells us, that for any second countable sober space $X$, a morphism $A \rightarrow B$ in $\mathcal{E}(X)$ is an isomorphism if and only if the induced morphisms $A(U) \rightarrow B(U)$ are isomorphisms for every $U \in \mathcal{O}(X)$. We will use this result in the Section 5.8, when giving our classification result.

4.1 The Canonical Filtration

In this section we let $X$ denote a \textit{finite} sober space. The canonical filtration in this section is due to [19].

Recursively we can construct a canonical filtration of $X$,

$$
\emptyset = \mathcal{F}_0X \subseteq \mathcal{F}_1X \subseteq \cdots \subseteq \mathcal{F}_lX = X,
$$

where each $\mathcal{F}_iX$ is open such that

$$
X_i = \mathcal{F}_iX \setminus \mathcal{F}_{i-1}X
$$

is discrete for each $i = 1, \ldots, l$. We do this by, in each step, letting $X_i$ be the set of all open points in the subspace $X \setminus \mathcal{F}_{i-1}X$ and defining $\mathcal{F}_iX = \mathcal{F}_{i-1}X \cup X_i$.\footnote{The numbers in the table are from the Wikipedia page on finite topological spaces \url{http://en.wikipedia.org/wiki/Finite_topological_space}. The authenticity of these numbers are, for me, unknown.}
In this construction, $X_i$ is the empty set only if $\mathcal{F}_{i-1}X = X$, since subspaces of finite sober spaces are again sober and every finite sober space contains at least one open point. Hence this filtration exists and is chosen in a unique way.

Let $P_Y := i_Y^* \circ r_Y^* : \mathcal{C}^{*}\text{sep}(X) \rightarrow \mathcal{C}^{*}\text{sep}(Y)$ for every $Y \in \mathbb{L}C(X)$, i.e.

$P_Y(A)(Z) = A(Y \cap Z)$ for every $Z \in \mathbb{L}C(X)$. We will use the notation $\mathcal{F}_iA := P_{\mathcal{F}_iA}$ for $i = 0, \ldots, l$. By Lemma 2.3.2 every separable $C^*$-algebra over $X$, $A$, and every $i = 1, \ldots, l$ gives a natural extension in $\mathcal{C}^{*}\text{sep}(X)$

$$0 \rightarrow \mathcal{F}_{i-1}A \rightarrow \mathcal{F}_iA \rightarrow P_{\mathcal{F}_i}(A) \rightarrow 0. \quad (3)$$

Furthermore, we observe, that since $X_i$ is discrete, we get natural isomorphisms

$$P_{\mathcal{F}_i}(A) \cong \bigoplus_{x \in X_i} i_{X_i}^*(i_x^{X_i}) \circ r_x^*(A) \cong \bigoplus_{x \in X_i} i_x(A(x)),$$

where $A(x) = A(\{x\})$ and $i_x = i_x^{X_i}$.

By using the canonical filtration and the universal property of $E$-theory, we get the following important proposition. Note that this proposition will be generalised in Theorem 4.4.5.

**Proposition 4.1.1.** Let $X$ be a finite sober space. A morphism $A \rightarrow B$ in $\mathcal{C}(X)$ is invertible if and only if the induced morphism $A(U) \rightarrow B(U)$ is invertible in $\mathcal{C}$ for each open subset $U$ of $X$.

When we say that a morphism $A \rightarrow B$ in $\mathcal{C}(X)$ induces a morphism $A(U) \rightarrow B(U)$ in $\mathcal{C}$, we are actually using that the evaluation functor $ev_U$ descends to $E$-theory.

**Proof.** Necessity is obvious. Let $A \rightarrow B$ be a morphism in $\mathcal{C}(X)$ such that $A(U) \rightarrow B(U)$ is invertible in $\mathcal{C}$. Let $U_x$ denote the minimal open set containing $x$. For any $C^*$-algebra $D$ we get a natural group isomorphism $E_0(A, U_x)) \rightarrow E_0(B, B(U_x))$. By Corollary 3.7.5 this induces the natural isomorphism

$$E_0(X; i_x(D), A) \rightarrow E_0(X; i_x(D), B).$$

Now, without fear of confusion, let $D$ be a $C^*$-algebra over $X$, let $E^C_0(D) = E_0(X; D, C)$ for $C = A, B$ and let $D_i = P_{\mathcal{F}_i}(D)$. Since the functors $P_Y$ commute with suspension, the extensions (3) and six-term exactness of $E$-theory give us the commutative diagram with exact rows

$$\begin{array}{ccccccc}
E^A_1(\mathcal{F}_{i-1}D) & \longrightarrow & E^A_0(D_i) & \longrightarrow & E^A_0(\mathcal{F}_iD) & \longrightarrow & E^A_0(\mathcal{F}_{i-1}D) & \longrightarrow & E^A_1(D_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
E^B_1(\mathcal{F}_{i-1}D) & \longrightarrow & E^B_0(D_i) & \longrightarrow & E^B_0(\mathcal{F}_iD) & \longrightarrow & E^B_0(\mathcal{F}_{i-1}D) & \longrightarrow & E^B_1(D_i)
\end{array}$$

for $i = 1, \ldots, l$. By construction $P_{\mathcal{F}_i}(D)$ and $\mathcal{F}_iD$ are (finite) direct sums of $C^*$-algebras over $X$ of the form $i_x(C)$. An induction argument invoking the five lemma gives us that all verticle maps in the above diagram are isomorphisms. Since $\mathcal{F}_iD = D$ we get that $E_0(X; D, A) \rightarrow E_0(X; D, B)$ is invertible for every $C^*$-algebra over $X$, $D$, and thus $A \rightarrow B$ is an isomorphism. □
4.2 Relations to $K$-Theory - Part II

We will in this section follow up on the the work done in Section 3.4. In this section $X$ will be a finite sober space. The goal of the section is, for every locally closed subset $Y$ of $X$, to construct a $C^*$-algebra over $X$, $R_Y$, such that $E_*(X; R_Y, B)$ is naturally isomorphic to $K_*(B(Y))$ for every object $B$ in $\mathcal{E}^{sep}(X)$. The constructions done in this section are as the constructions in [18].

We will equip $X$ with the specialisation preorder (see [19], §2.7), i.e. $x \leq y$ if and only if $\{x\} \subseteq \{y\}$. Since $X$ is a finite $T_0$ space, it carries the Alexandrov topology with respect to $\leq$, i.e. for $Y \subseteq X$ it holds that

- $Y$ is open if and only if $x \geq y \in Y$ implies $x \in Y$,
- $Y$ is closed if and only if $x \leq y \in Y$ implies $x \in Y$,
- $Y$ is locally closed if and only if $x \leq y \leq z$ and $x, z \in Y$ implies $y \in Y$.

We wish to construct a topological space $\text{Ch}(X)$ by using this partial order. Let $S_X$ the set of all strict chains $(x_0 < \cdots < x_n)$ in $X$. For each $I = (x_0 < \cdots < x_n) \in S_X$ of length $n + 1$ define a standard $n$-simplex

$$\Delta_I := \{(t_0, \ldots, t_n) \mid 0 \leq t_k \leq 1, \sum_{k=0}^{n} t_k = 1\} \subseteq \mathbb{R}^{n+1}$$

equipped with the subspace topology. The space $\text{Ch}(X)$ is obtained by taking the disjoint union $\bigsqcup_{I \in S_X} \Delta_I$ and identifying $\Delta_J$ with the corresponding face in $\Delta_I$ whenever $J$ is a subchain of $I$, i.e. $J$ contains only elements from $I$. If $J$ is a subchain of $I$, the face corresponding to $\Delta_J$ in $\Delta_I$ is the subspace

$$\{(t_0, \ldots, t_n) \mid t_k = 0 \text{ if } x_k \notin J\} \subseteq \Delta_I.$$ 

For every $I \in S_X$ let $\min I$ (resp. $\max I$) denote the unique minimal (resp. maximal) element of $I$. Define the interior $\Delta^\circ_I$ of $\Delta_I$ to be the interior in $\mathbb{R}^{n+1}$, unless $I$ is of length 1 for which we define $\Delta^\circ_I = \Delta_I$. Since $\text{Ch}(X) = \bigsqcup_{I \in S_X} \Delta^\circ_I$ as sets, we may define two maps $m, M : \text{Ch}(X) \to X$ by mapping a point in $\Delta^\circ_I$ to $\min I$ and $\max I$ respectively. Let $X^{\text{op}}$ be the set $X$ with the reversed partial order $\geq$, i.e. a subset of $X^{\text{op}}$ is open if and only if it is closed in $X$.

**Proposition 4.2.1.** The map $(m, M) : \text{Ch}(X) \to X^{\text{op}} \times X$ is continuous.

**Proof.** We will show that $m$ and $M$ are continuous. Let $I \subseteq X$ be open, i.e. closed in $X^{\text{op}}$. Let $I \in S_X$ such that $\min I \in Y$. For every subchain $J$ of $I$, $\min J \geq \min I \in Y$ and thus $\min J \in Y$ Hence $\Delta_I \subseteq m^{-1}(Y)$ and, moreover, $m^{-1}(Y) = \bigcup_{\min I \in Y} \Delta_I$, which is closed in $\text{Ch}(X)$ since $S_X$ is finite and $\Delta_I \subseteq \text{Ch}(X)$ is closed. Hence $m : \text{Ch}(X) \to X^{\text{op}}$ is continuous. That $M : \text{Ch}(X) \to X$ is continuous is very similar. $\square$

45
Now define the \( C^\ast \)-algebra over \( X^{\text{op}} \times X \),

\[
R := C(\text{Ch}(X)), \quad \text{Prim}(R) \cong \text{Ch}(X) \xrightarrow{(m,M)} X^{\text{op}} \times X.
\]

Let \( Y \in \mathcal{L}(X) \), and define \( R_Y \) to be the \( C^\ast \)-algebra over \( X \) given by

\[
\bigodot(X) \ni U \mapsto R_Y(U) := R(Y \times U) \in \mathbb{I}(R(Y \times X)).
\]

Note that \( R_Y(Y) = C(\text{Ch}(Y)) \) which is thus a unital \( C^\ast \)-algebra.

Let \( Y \in \mathcal{L}(X) \) and \( U \in \mathcal{O}(Y) \). By Lemma 2.3.2 we get an extension of \( C^\ast \)-algebras over \( X \),

\[
0 \rightarrow R_{Y \setminus U} \rightarrow R_Y \rightarrow R_U \rightarrow 0. \tag{4}
\]

For \( Y \in \mathcal{L}(X) \) and \( j \in \mathbb{Z}/2 \), define the functor \( \text{FK}_Y^j : \mathfrak{C}^{\ast \text{sep}}(X) \rightarrow \mathfrak{Ab} \) by \( \text{FK}_Y^j(A) = K_j(A(Y)) \). There is a canonical natural transformation \( E_0(X; R_Y, -) \Rightarrow \text{FK}_Y^1 \) given as follows: let \( A \) be a \( C^\ast \)-algebra over \( X \) and \(|[\phi]| \in E_0(X; R_Y, A)\). The evaluation functor \( ev_Y \), which descends to a functor \( \mathfrak{C}(X) \rightarrow \mathfrak{C} \), induces a class \(|[\tilde{\phi}]| \in E_0(C(\text{Ch}(Y)), A(Y))\). The unital inclusion \( i : \mathfrak{C} \rightarrow C(\text{Ch}(Y)) \) induces a class \(|[\tilde{\phi}][\iota]| \in E_0(C, A(Y)) = F_{Y^1}(A)\). This is easily seen to give a natural transformation. Composing with the suspension functor gives a natural transformation \( E_1(X; R_Y, -) \Rightarrow \text{FK}_Y^1 \).

Let \( \text{FK}_Y : \mathfrak{C}^{\ast \text{sep}}(X) \rightarrow \mathfrak{Ab}^{2/2} \) be the functor given by \( \text{FK}_Y(A) = K_+(A(Y)) \). The natural transformations above induce a natural transformation

\[
E_*(X; R_Y, -) \Rightarrow \text{FK}_Y.
\]

The main theorem of this section is that this natural transformation is a natural isomorphism.

**Theorem 4.2.2.** The functors

\[
E_*(X; R_Y, -), \text{FK}_Y : \mathfrak{C}^{\ast \text{sep}}(X) \rightarrow \mathfrak{Ab}^{2/2}
\]

are naturally isomorphic.

**Proof.** We will prove that \( E_0(X; R_Y, A) \cong K_0(A(Y)) \) naturally. Let \( U_x \) be the minimal open set containing \( x \). By Theorem 3.4.3 and Corollary 3.7.5 there are natural isomorphisms

\[
E_0(X; i_x(\mathfrak{C}), A) \cong E_0(\mathfrak{C}, A(U_x)) \cong K_0(A(U_x)),
\]

where the first isomorphism is induced by the evaluation functor \( \text{ev}_{U_x} \).

Note that \( U_x = \{ y \in X \mid y \geq x \} \). Let \( I \) be a strict chain in \( X \) such that \( \text{min } I \in U_x \). Then every element in \( \Delta I \) can be canonically contracted to \( \Delta x \), since \( \Delta I \subseteq \Delta_{U \{ x \}} \). This induces a homotopy of the \( * \)-homomorphisms

\[
1_{C(\text{Ch}(U_x))}, ev_{\Delta x} : C(\text{Ch}(U_x)) \rightarrow C(\text{Ch}(U_x))
\]

46
on the underlying $C^*$-algebra. Observing that $\max I = \max I \cup \{x\}$ it follows that the induced homotopy respects the ideal structure $\mathcal{R}_{U_x}(V)$ for $V \in \mathcal{O}(X)$ and thus this homotopy lifts to an $X$-equivariant homotopy. This implies that the unital inclusion $\iota : i_x(C) \to \mathcal{R}_{U_x}$ is a homotopy equivalence. Hence we get a commutative diagram of natural transformations

\[
\begin{array}{ccc}
E_0(X; \mathcal{R}_{U_x}, -) & \xrightarrow{\iota^*} & E_0(X; i_x(C), -) \\
ev_{U_x} & & \nev_{U_x} \\
E_0(C(Ch(U_x)), ev_{U_x}(-)) & \xrightarrow{\iota^*} & E_0(C, ev_{U_x}(-))
\end{array}
\]

and the lower composite is the canonical natural transformation, which is thus an isomorphism.

Let $\text{Good} \subseteq \mathcal{L}(X)$ denote the set of all locally closed subsets of $X$ for which the canonical natural transformation $E_0(X; \mathcal{R}_Y, -) \Rightarrow F K^0_Y$ is a natural isomorphism. We have just proven that $U_x \in \text{Good}$ for every $x \in X$, and we wish to prove that $\text{Good} = \mathcal{L}(X)$.

Let $Y \in \mathcal{L}(X)$ and $U \in \mathcal{O}(Y)$. The extension (4), six-term exactness of $E$-theory and the five lemma implies, just as in the proof of Proposition 4.1.1, that if two of $U, Y$ and $Y \setminus U$ are in Good, then so is the third. In particular, if $U, V \in \mathcal{O}(X)$ and $U, V, U \cap V \in \text{Good}$, then $U \cup V \in \text{Good}$ since $(U \cup V) \setminus U = V \setminus (U \cap V)$. An induction argument on the number of elements in the open subsets of $X$, gives that every open set is in Good. Since a locally closed set is the difference of two open sets, it follows that $\text{Good} = \mathcal{L}(X)$.

4.3 $E$-Theory as Homotopy Theory

In this section we take a step away from the theory for finite spaces, and try a new approach to $E$-theory. In this section we assume that $X$ is a (not necessarily second countable) sober space, and we do not assume that our $C^*$-algebras over $X$ are separable. Let $\text{Asym}(A, B)_X$ denote the set of all approximately $X$-equivariant asymptotic morphisms from $A$ to $B$. Note that $\text{Asym}(A, B)_X$ has the zero asymptotic morphism as a canonical basepoint. We would want to equip $\text{Asym}(A, B)_X$ with a topology such that we could approach the theory of asymptotic morphisms with homotopy theory. Unfortunately, as is written in [7], there has at this time not been found any natural topology to equip to $\text{Asym}(A, B)_X$. This is the motivation for introducing the theory of quasi-topological spaces which was studied in more detail by Spanier in [25].

The goal of this section is to develop homotopy theory for (pointed) quasi-topological spaces, and use this to give a new description of $E$-theory. This will be an important construction when approximating $E$-theory in Section 4.4.

**Definition 4.3.1.** A pointed quasi-topological space is a set $W$ with a basepoint together with sets of basepoint preserving maps $C(Y, W)$ from $Y$ to $W$, for each compact Hausdorff space $Y$ with a basepoint. The elements of $C(Y, W)$ are
called the \textit{quasi-continuous maps} from $Y$ to $W$. The quasi-continuous maps should satisfy the following conditions:

(i) The canonical map $\star \to W$ is quasi-continuous.

(ii) A map from a wedge sum $Y_1 \lor Y_2$ to $W$ is quasi-continuous if and only if the restrictions to $Y_1$ and $Y_2$ are quasi-continuous.

(iii) If $f : Y_1 \to Y_2$ is continuous and $g : Y_2 \to W$ is quasi-continuous then $g \circ f$ is quasi-continuous.

(iv) If $f : Y_1 \to Y_2$ is a continuous surjection (and thus a quotient map) and $g : Y_2 \to W$ satisfies that $g \circ f$ is quasi-continuous, then $g$ is quasi-continuous.

Moreover, we say that a map between pointed quasi-topological spaces $f : W_1 \to W_2$ is quasi-continuous if and only if for any quasi-continuous map $g : Y \to W_1$, the composite $f \circ g$ is quasi-continuous. We denote by $C(W_1, W_2)$ the set of all such quasi-continuous maps.

At times, it might be convenient to think of a quasi-topological space as a functor $\mathcal{LCH} \to \text{Set}^*$ from the category of locally compact Hausdorff spaces to the category of pointed sets, satisfying suitable conditions according to the items (i) – (iv) above. When doing this the quasi-continuous maps of two such functors is just a natural transformation.

\textbf{Example 4.3.2.} Any pointed topological space $W$ canonically has the structure of a pointed quasi-topological space with $C(Y, W)$ being the set of basepoint preserving continuous maps $Y \to W$ for each compact Hausdorff space $Y$ with a basepoint.

One could worry that this gives a notational problem of the sets $C(Y, W)$ by wether we consider the compact Hausdorff space $Y$ as a topological space or as a quasi-topological space. However, this is nothing to worry about. If $Y$ is a pointed compact Hausdorff space and $Y'$ is the corresponding pointed quasi-topological space then $C(Y, W) = C(Y', W)$ for any quasi-topological space $W$. To see this note first that $C(Y, W) \subseteq C(Y', W)$ follows from condition (iii). To get the other inclusion let $f \in C(Y', W)$. Since $id \in C(Y', Y)$ we get that $f \circ id = f \in C(Y, W)$.

The next example explains how we plan to use the theory of quasi-topological spaces.

\textbf{Example 4.3.3.} Let $X$ be a (not necessarily second countable) sober space. For (not necessarily separable) $C^*$-algebras over $X$, $A$ and $B$, we denote by $\text{Asym}(A, B)_X$ the set of approximately $X$-equivariant asymptotic morphisms from $A$ to $B$. The set $\text{Asym}(A, B)_X$ can be made into a pointed quasi-topological space by letting the zero asymptotic morphism be the basepoint and by defining $C(Y, \text{Asym}(A, B)_X) = \text{Asym}(A, C_0(Y, B))_X$. 

48
where we identify approximately $X$-equivariant asymptotic morphisms from $A$ to $C_0(Y,B)$ with maps $Y \rightarrow \text{Asym}(A,B)_X$ in the obvious way. We will sketch why the sets $C(Y,\text{Asym}(A,B)_X)$ satisfy conditions (i)-(iv).

Condition (i) follows since $C_0(\ast, B) = 0$ and condition (ii) follows since

\[
C_b(T, C_0(Y_1 \cup Y_2, B)) \cong C_b(T, C_0(Y_1, B) \oplus C_0(Y_2, B)) \\
\cong C_b(T, C_0(Y_1, B)) \oplus C_b(T, C_0(Y_2, B)).
\]

If $f : Y_1 \rightarrow Y_2$ is continuous let $f^* : C_0(Y_2, B) \rightarrow C_0(Y_1, B)$ be the induced $X$-equivariant *-homomorphism. If $\phi$ is in $\text{Asym}(A, C_0(Y_2, B))_X$ then

\[
(f \circ \phi)_I(a) = f^*(\phi_0(a)).
\]

Condition (iii) follows since $f^*$ is contractive and condition (iv) follows since $f^*$ is isometric if $f$ is surjective.

Our goal is to do homotopy theory on these pointed quasi-topological spaces. For a locally compact space $Y$ we let $Y^+$ denote the one point compactification of $Y$, where the additional point is the basepoint. Note that if $Y$ is compact Hausdorff then $Y^+$ is the disjoint union of $Y$ with an additional point. Recall that any compact Hausdorff space with basepoint is the one point compactification of a locally compact Hausdorff space and that the smash product $Y_1^+ \wedge Y_2^+ = (Y_1 \times Y_2)^+$.

Let $W$ be a pointed quasi-topological space and $Y$ a pointed compact Hausdorff space. The set $C(Y,W)$ has a canonical structure as a pointed quasi-topological space by defining

\[
C(Y_0, C(Y, W)) = C(Y_0 \wedge Y, W)
\]

for each pointed compact Hausdorff space $Y_0$. Let $f, g \in C(Y, W)$ and $I = [0,1]$ be the unit interval. We will say that $f$ and $g$ are homotopic if there is a quasi-continuous map $H : I^+ \rightarrow C(Y, W)$ such that $ev_j : Y \rightarrow Y \wedge I^+$ for $j = 0,1$ is the continuous map $ev_j(y) = (y,j)$ for $y$ not being the basepoint, then $f = H \circ ev_0$ and $g = H \circ ev_1$. Observe that as in the topological case, homotopy is an equivalence relation.

**Remark 4.3.4.** Note that homotopy is only defined for quasi-continuous maps from a compact Hausdorff space. We could extend this definition as in [25] but this would require quasi-topologizing $C(W_1, W_2)$ which we do not need in our case.

**Definition 4.3.5.** Let $W$ be a pointed quasi-topological space. For $n \in \mathbb{N}_0$ we define the $n$‘th homotopy group $\pi_n(W)$ to be the set of homotopy classes of $C((\mathbb{R}^n)^+, W)$ with the following group composition ‘$+$’ for $n \geq 1$: For $f, g \in C((\mathbb{R}^n)^+, W)$ we define the quasi-continuous map $f + g \in C((\mathbb{R}^n)^+, W)$ to be the composite

\[
(\mathbb{R}^n)^+ \rightarrow ((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1})^+ \cong (\mathbb{R}^n)^+ \vee (\mathbb{R}^n)^+ \overset{J}{\longrightarrow} W.
\]

We define the group composition by $[f] + [g] := [f + g]$.

$^1$Where $\mathbb{R}^0 = \ast$ is the one-point space.
Exactly as in the topological case (e.g. [10], §4) these groups are well defined
and abelian for \( n \geq 2 \). Clearly the ordinary homotopy groups of a pointed topo-
logical space agree with the homotopy groups of the induced quasi-topological
space. Note that \( \pi_0 \), as in the ordinary case, is just the set of path-components.

**Proposition 4.3.6.** Let \( A \) and \( B \) be \( C^* \)-algebras over \( X \). Then there are natural
group isomorphisms

\[
\pi_n(\mathrm{Asym}(A, B \otimes K)_X) \cong [[A, \Sigma^n B \otimes K]]_X.
\]

for every \( n \in \mathbb{N} \).

**Proof.** We should start by noting that

\[
C((\mathbb{R}^n)^+, \mathrm{Asym}(A, B \otimes K)_X) = \mathrm{Asym}(A, C_0(\mathbb{R}^n, B \otimes K)) \cong \mathrm{Asym}(A, \Sigma B' \otimes K),
\]

where \( B' = \Sigma^{n-1}B \). If \( \phi, \psi \in \mathrm{Asym}(A, \Sigma B' \otimes K) \) the homotopy group construction \( \phi + \psi \) corresponds to the composite

\[
A \xrightarrow{(\phi, \psi)} C_b(T, \Sigma B' \otimes K) \oplus C_b(T, \Sigma B' \otimes K) \hookrightarrow C_b(T, \Sigma B' \otimes K)
\]

which is exactly the same construction as the last construction in Remark 3.3.2.
Hence

\[
\pi_n(\mathrm{Asym}(A, B \otimes K)_X) \cong [[A, \Sigma^n B \otimes K]]_X = [[A, \Sigma^n B \otimes K]]_X.
\]

\( \square \)

Immediately we get the following corollary which relates \( E \)-theory and ho-
motopy theory.

**Corollary 4.3.7.** Let \( X \) be a second countable sober space and let \( A \) and \( B \) be separable \( C^* \)-algebras over \( X \). For \( n \in \mathbb{N} \) there are natural group isomorphisms

\[
\pi_n(\mathrm{Asym}(\Sigma A \otimes K, \Sigma B \otimes K)_X) \cong E_{n \mod 2}(X; A, B).
\]

**Proof.** This follows from Proposition 4.3.6 and Bott periodicity. \( \square \)

### 4.4 Approximation by Finite Spaces

In this section we will prove two of the main results of this paper. Informally,
we prove that the \( E \)-theory over a second countable sober space may be approx-
imated by \( E \)-theory over finite spaces. We prove this by using our description of
\( E \)-theory as homotopy theory, and known results from homotopy theory which
may be converted to the quasi-topological case. As a corollary we get our second
main theorem, which is a very good cricereon for checking that a morphism in
\( \mathcal{E}(X) \) is invertible.

The main theorems of this section are due to Marius Dadarlat and Ralf
Meyer in [7].
Let $X$ be a second countable sober space with basis $(U_n)_{n \in \mathbb{N}}$. We define for each $n$ a topological space $X_n$ as follows: let $\tau_n$ be the topology generated by $U_1, \ldots, U_n$, i.e. $\tau_n$ has $U_1, \ldots, U_n$ as subbasis or equivalently $\tau_n$ has the following basis

$$\big\{ \bigcap_{k \in F} U_k \mid F \subseteq \{1, \ldots, n\} \text{ is a finite subset} \big\} \cup \{\emptyset, X\}.$$ 

Define the set $X_n$ to be equivalence classes in $X$ under the relation

$$x \sim_n y \iff \{1 \leq k \leq n \mid x \in U_k\} = \{1 \leq k \leq n \mid y \in U_k\}$$

for $x, y \in X$. Clearly $\tau_n$ induces a topology on $X_n$ such that the space is $T_0$ which is equivalent to being sober on finite spaces. In fact, $X_n$ is the sobrification of $(X, \tau_n)$.

Given a $C^*$-algebra over $X$ we may regard it as a $C^*$-algebra over $X_n$ by forgetting most of the structure. In other words, there exists a canonical forgetful functor $\mathcal{C}^*_{\text{sep}}(X) \to \mathcal{C}^*_{\text{sep}}(X_n)$ for every $n$. Similarly there are forgetful functors $\mathcal{C}^*_{\text{sep}}(X_{n+1}) \to \mathcal{C}^*_{\text{sep}}(X_n)$ for every $n$. The main focus of this section is proving the following theorem.

**Theorem 4.4.1.** Let $A$ and $B$ be separable $C^*$-algebras over $X$ which we also view as $C^*$-algebras over $X_n$ for each $n \in \mathbb{N}$. Then there is a natural short exact sequence of $\mathbb{Z}/2$-graded abelian groups

$$0 \to \lim_{\leftarrow} E^1_{\ast+1}(X_n; A, B) \to E^\ast(X; A, B) \to \lim_{\leftarrow} E^\ast(X_n; A, B) \to 0.$$

The proof of this theorem uses results from topology which we may generalize to quasi-topological spaces. We will then obtain our result by applying Corollary 4.3.7.

Let $f, g : V \to W$ be quasi-continuous maps of pointed quasi-topological spaces. We define the homotopy equalizer $D(f, g)$ of $f$ and $g$ to be the pointed quasi-topological space with underlying set

$$D(f, g) = \{(v, w) \in V \times C(I^+, W) \mid f(v) = w(0), g(v) = w(1)\}$$

together with the sets of quasi-continuous maps

$$C(Y, D(f, g)) = \{(v, w) \in C(Y, V) \times C(Y \setminus I^+, W) \mid f \circ v = w(-, 0), g \circ v = w(-, 1)\}$$

for each pointed compact Hausdorff space $Y$.

Observe that if $(W_\lambda)_{\lambda \in \Lambda}$ is a collection of pointed quasi-topological spaces with basepoints $w_\lambda$, indexed over some set $\Lambda$, then we may define the product $\prod W_\lambda$ to be the pointed quasi-topological space with basepoint $(w_\lambda)$, given with the sets of quasi-continuous maps

$$C(Y, \prod W_\lambda) = \prod C(Y, W_\lambda)$$
for each pointed compact Hausdorff space $Y$. It is easy to see that $\prod W_\lambda$ is a pointed quasi-topological space and that it is the (categorical) product of $(W_\lambda)$ in the category of pointed quasi-topological spaces. Moreover, it is straightforward to check that

$$
\begin{array}{ccc}
D(f, g) & \longrightarrow & C(I^+, W) \\
\downarrow & & \downarrow (ev_0, ev_1) \\
V & \longrightarrow & W \times W
\end{array}
$$

is a pull-back in the category of pointed topological spaces.

These definitions and observations are all we need to do an analogous proof of Theorem C and the corollary which follows in [28] for quasi-topological spaces. The following lemma is a special case of the before mentioned corollary if we replace $Y$ with $(\mathbb{R}^k)_+$ for $k \geq 0$.

**Lemma 4.4.2** ([28], Corollary of Theorem C). Let $\cdots \rightarrow f_1 W_1 \rightarrow f_0 W_0$ be a projective system. If $id, f : \prod_{n=0}^\infty W_n \rightarrow \prod_{n=0}^\infty W_n$ where $f = (0, f_0, f_1, \ldots)$ and $k \geq 0$ is fixed, we get a short exact sequence of pointed sets

$$0 \rightarrow \varprojlim E_{1,k+1}(W_n) \rightarrow \pi_k(D(id, f)) \rightarrow \varprojlim E_{0,k}(W_n) \rightarrow 0.$$ 

Note that even though we consider the sequence as pointed sets we need the group structure on $\pi_{k+1}(W_n)$ in order to define $\varprojlim$. It is easily verified that if $k \geq 1$ then it is in fact a short exact sequence of groups.

We wish to apply this theory to our quasi-topological spaces $\text{Asym}(A,B)_X$. Let $X$ be a second countable sober space with basis $(U_n)_{n \in \mathbb{N}}$ and construct the finite sober spaces $X_n$. Let $A$ and $B$ be separable $C^*$-algebras over $X$ and define

$$\Gamma_n := \text{Asym}(\Sigma A \otimes K, \Sigma B \otimes K)_{X_n}.$$ 

We note that approximately $X_{n+1}$-equivariant asymptotic morphisms are also approximately $X_n$-equivariant. Thus we get a projective system of quasi-topological spaces

$$\cdots \subseteq \Gamma_{n+1} \subseteq \Gamma_n \subseteq \cdots \subseteq \Gamma_2 \subseteq \Gamma_1.$$ 

Note that the inclusions are quasi-continuous since $C(Y, \Gamma_{n+1}) \subseteq C(Y, \Gamma_n)$ for every pointed compact Hausdorff space $Y$. Define

$$id, f : \prod_{n=0}^\infty \Gamma_n \rightarrow \prod_{n=0}^\infty \Gamma_n$$

where $f((\phi_n)_{n \in \mathbb{N}}) = (\phi_{n-1})_{n \in \mathbb{N}}$. Combining Lemma 4.4.2 and Corollary 4.3.7 we get a short exact sequence of pointed sets

$$0 \rightarrow \varprojlim E_1(X_n; A, B) \rightarrow \pi_0(D(id, f)) \rightarrow \varprojlim E_0(X_n; A, B) \rightarrow 0.$$ 

We will denote by $\Gamma_X$ the quasi-topological space $\text{Asym}(\Sigma A \otimes K, \Sigma B \otimes K)_X$. The following lemma more or less finishes the proof of Theorem 4.4.1.

In the following we will denote asymptotic morphisms $(\phi_t)$ by $\phi_t$ as not to confuse these with tuples of asymptotic morphisms.
Lemma 4.4.3. With the same notation as above $\pi_k(\Gamma_X) \cong \pi_k(D(id, f))$.

Proof. Define $A' = \Sigma A \otimes K$ and $B' = \Sigma^{k+1} B \otimes K$ to ease notation. We start by observing that that

$$D(id, f) \cong \left\{ \left( \phi_n, \Phi_n \right) \in \prod_{n=1}^{\infty} (\Gamma_n \times C(I^+, \Gamma_n)) \mid \phi_n = \Phi_n(0), \phi_{n+1} = \Phi_n(1), \forall n \right\}$$

$$\cong \left\{ \Phi_n \in \prod_{n=1}^{\infty} C([n, n+1]^+, \Gamma_n) \mid \Phi_n(n+1) = \Phi_{n+1}(n+1) \text{ for all } n \right\}.$$

We will be identifying $D(id, f)$ with this latter quasi-topological space. Define the map $\Gamma_X \to D(id, f)$ by mapping $\phi$ to the constant sequence $(\phi)_{n \in \mathbb{N}}$, i.e. in the coordinate corresponding to $n$, $\phi_n(a) = \phi_t(a)$ for $s \in [n, n+1]$, $a \in A'$ and $t \in T$. This induces a map $\pi_0(\Gamma_X) \to \pi_0(D(id, f))$ which we wish to prove is a bijection. Note that if we replace $B'$ with $\Sigma B'$ we get a map $\pi_k(\Gamma_X) \to \pi_k(D(id, f))$ which is a group homomorphism. Thus it suffices to prove the case $k = 0$.

We will first prove that $\pi_0(\Gamma_X) \to \pi_0(D(f, id))$ is surjective. Let $(\Phi_n) \in D(id, f)$ and define $\phi_{s,t} : A' \to B'$ for $s, t \in T$ by

$$\phi_{s,t}(a) = \Phi_n(t)(a), \text{ for } s \in [n, n+1].$$

Let $A_0 = \{a_1, a_2, \ldots \}$ be a countable dense *-subring of $A'$, and $\{\lambda_1, \lambda_2, \ldots \}$ be a dense *-subring of $\mathbb{C}$. We may assume that $\lambda_i a_j \in A_0$ for each $i$ and $j$. We claim that there exists a suitable increasing continuous function $r_0 : T \to T$, such that for any continuous function $r \geq r_0$ the sequence of families of maps $H^t_{s,t} : A_0 \to B'$ for $x \in [0, 1], s \in [n, n+1], t \in T$ given by

$$H^t_{x,s,t}(a) = \phi_{(1-x)s+xt,t}(a)$$

extend to approximately $X_n$-equivariant asymptotic morphisms

$$A' \to C([0, 1] \times [n, n+1], B').$$

Hence $(H^n)_{n \in \mathbb{N}}$ will be a homotopy in $D(id, f)$ connecting the extension of $(\phi_{t,t}(t))_{n \in \mathbb{N}}$ to $(\Phi_n(t,t))_{n \in \mathbb{N}}$ which is canonically homotopic to $(\Phi_n,t)_{n \in \mathbb{N}}$. Since (the extension of) $\phi_{t,t}(t)$ is approximately $X_n$-equivariant for every $n$, it will be approximately $X$-equivariant. Hence (the extension of) $(\phi_{t,t}(t))_{n}$ is the image of (the extension of) $\phi_{t,t}(t) \in \Gamma_X$ which proves surjectivity. Moreover, if $\phi_0, \phi_1$ in $\Gamma_X$ are mapped to homotopic elements in $D(id, f)$ the exact same technique may be used to lift such a homotopy, thus proving injectivity.

We may assume that there exist subsequences $(a_{ij})_{i,j \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$ such that $(a_{ij})_{j \in \mathbb{N}}$ is dense in $A'(U_i)$ for each $i \in \mathbb{N}$. Choose an increasing sequence $(t_m)$ in $T$ such that for all $1 \leq i, j, k \leq m$ and all $t \geq t_m$ we have

$$\sup_{a \in [0, m+1]} \left\| \phi_{s,t}(a_i) \right\| \leq \left\| a_i \right\| + 1$$

$$\sup_{a \in [0, m+1]} \left\| \phi_{s,t}(a_i + \lambda_k a_j^*) - \phi_{s,t}(a_i) - \lambda_k \phi_{s,t}(a_j)^* \right\| < \frac{1}{m}$$

$$\sup_{a \in [0, m+1]} \left\| \phi_{s,t}(a_i a_j) - \phi_{s,t}(a_i) \phi_{s,t}(a_j) \right\| < \frac{1}{m}.$$
For each \( n \in \mathbb{N} \) construct a sequence \((r_{m,n})_{m \in \mathbb{N}}\) such that for all \( 1 \leq i \leq n, 1 \leq j \leq m \) and all \( t \geq r_{m,n} \) we have

\[
\sup_{s \in [n,m+1]} \|\phi_{s,t}(a_{ij})\|_{X \setminus U_i} < \frac{1}{m}.
\]

Moreover, these sequences should be chosen recursively such that \( r_{m,n+1} \geq r_{m,n} \) for every \( m \in \mathbb{N} \). Now let \( r_0 : T \to T \) be an increasing continuous function such that \( r_0(m) \geq \max\{t_m, r_{m,m}\} \) and \( \lim_{t \to \infty} r_0(t) = \infty \). We will prove that this \( r_0 \) satisfies the above claim. Let \( r \geq r_0 \) be continuous and construct \( H^m_{x,s,t} \) as above. The top of our inequalities above ensures that \( t \mapsto H^m_{x,s,t}(a_i) \) is (continuous and) bounded for every \( i \in \mathbb{N} \). Thus for the claim to be true we must show that for each \( n \) and all \( i, j, k \in \mathbb{N} \)

\[
\lim_{t \to \infty} \sup_{s \in [n,n+1], x \in [0,1]} \|H^m_{x,s,t}(a_i + \lambda_k a_j^* - H^m_{x,s,t}(a_i) - \lambda_k H^m_{x,s,t}(a_j)^*)\| = 0
\]

and for each \( 1 \leq i \leq n \) and \( j \in \mathbb{N} \)

\[
\lim_{t \to \infty} \sup_{s \in [n,n+1], x \in [0,1]} \|H^m_{x,s,t}(a_{ij})\|_{X \setminus U_i} = 0.
\]

Given \( n, i, j, k \in \mathbb{N} \) let \( m \geq n, t \in [m, m+1], s \in [n, n+1] \) and \( x \in [0,1] \). Since \( (1 - x)s + xt \leq m + 1 \) and \( r(t) \geq t_m \)

\[
\|H^m_{x,s,t}(a_i + \lambda_k a_j^* - H^m_{x,s,t}(a_i) - \lambda_k H^m_{x,s,t}(a_j)^*)\|
\]

\[
= \|\phi_{(1-x)s+xt,t(r(t))}(a_i + \lambda_k a_j^*) - \phi_{(1-x)s+xt,t(r(t))}(a_i) - \lambda_k \phi_{(1-x)s+xt,t(r(t))}(a_j)^*\|
\]

\[
< \frac{1}{m}.
\]

Hence the top of the above limits follows, since \( m \) was chosen arbitrarily. Similarly the two others follow, thus completing the proof.

With this lemma at our aid, we can now easily prove Theorem 4.4.1.

**Proof of Theorem 4.4.1.** By Lemma 4.4.2

\[
0 \rightarrow \lim_{k \to \infty} \pi_{k+1}(W_n) \rightarrow \pi_k(D(id, f)) \rightarrow \lim_{k \to \infty} \pi_k(W_n) \rightarrow 0
\]

is a short exact sequence of groups for \( k \geq 1 \). By Corollary 4.3.7 and Lemma 4.4.3 we get the desired short exact sequence.

**Remark 4.4.4.** In the proof of Theorem 4.4.1 we proved the existence of a short exact sequence, but never mention the corresponding maps. These require that one studies the construction in [28]. However, these are easily described in our case. The map \( \lim_{k \to \infty} E_1(X_n; A, B) \rightarrow \pi_0(D(id, f)) \) is given by

\[
[([\phi_0], [\phi_1], \ldots)] \mapsto [([\phi_0, \phi_1, \ldots])],
\]

54
where $\Phi_n : A' \to C_b(T, C_0((n, n + 1), B'))$ are approximately $X_n$-equivariant asymptotic morphisms. Moreover, the map $E_0(X; A, B) \to \lim E_0(X_n, A, B)$ is given by

$$[[\phi]] \mapsto ([[\phi]], [[\phi]], \ldots).$$

as one would hope.

Now, as promised, we prove a generalisation of Proposition 4.1.1.

**Theorem 4.4.5.** Let $X$ be a second countable sober space. A morphism $A \to B$ in $\mathcal{E}(X)$ is invertible if and only if the induced morphism $A(U) \to B(U)$ is invertible in $\mathcal{E}$ for each open subset $U$ of $X$.

**Proof.** Necessity is obvious. With the notation from Theorem 4.4.1, Theorem 4.1.1 implies that $A \to B$ is an isomorphism in $\mathcal{E}(X_n)$ for every $n$. Hence we get isomorphisms

$$\lim E_0(X_n; D, A) \xrightarrow{\approx} \lim E_0(X_n; D, B),$$

$$\lim E_1(X_n; D, A) \xrightarrow{\approx} \lim E_1(X_n; D, B)$$

for any $C^*$-algebra over $X$, $D$. In particular, by naturality of the short exact sequence of Theorem 4.4.1 and the five lemma, we get that $A \to B$ is an isomorphism in $\mathcal{E}(X)$.

$\square$
5 Filtrated $K$-Theory

In this final section we will be doing a lot of homological algebra in triangulated categories. The reason for doing this, is that the $E$-theory category $E(X)$ is (equivalent to) a triangulated category. When doing this we define derived functors and prove a general universal coefficient theorem (UCT). We prove, for $X$ being a finite sober space, that the filtrated $K$-theory functor $FK$ is the universal $K$-exact stable homological functor and prove simple UCT results for the $E$-theoretic bootstrap class $B_E(X)$. We also get an important theorem by applying our second main theorem, Theorem 4.4.5, and show that if $X$ is second countable and $A$ and $B$ are in the $E$-theoretic bootstrap class, then a morphism $A \to B$ in $E(X)$ is an isomorphism if and only it induces an isomorphism $K_*(A(U)) \to K_*(B(U))$ for every $U \in \mathcal{O}(X)$.

In the final section we give the reader different tools for applying $E$-theory to classification of $C^*$-algebras, and in particular use this to classify separable, nuclear $C^*$-algebras with Hausdorff primitive ideal space for which every ideal is $E$-contractible. In fact, if $A$ is such a $C^*$-algebra then $A \otimes O_{\infty} \otimes \mathbb{K} \cong C_0(\text{Prim}(A))$.

5.1 $E(X)$ as a Triangulated Category

In this section we prove that $E(X)$ is equivalent to a triangulated category. In our further applications, we will act as if $E(X)$ were triangulated, even though this is only true up to equivalence of categories.

We have written a small section on triangulated categories in Section B. The reader should note that the version of the octahedral axiom which we use is different, although equivalent, to the usual versions as (TR4) or (TR4') in [21].

If $E(X)$ should be triangulated we would want the functor $\Sigma : E(X) \to E(X)$ to be an automorphism of categories. Unfortunately $\Sigma$ is 'just' an equivalence of categories. To fix this 'tiny' problem we do the following trick from [17].

Let $\tilde{E}(X)$ be the category with objects being pairs $(A,n)$ with $A$ a separable $C^*$-algebra over $X$ and $n$ an integer, and the morphism sets being $\text{Hom}_{\tilde{E}(X)}((A,n),(B,m)) := \lim_{\rightarrow} E_0(X; \Sigma^{n+k}A, \Sigma^{m+k}B)$.

Since $\Sigma : E_0(X;A,B) \to E_0(X;\Sigma A,\Sigma B)$ is an isomorphism by Lemma 3.6.9 we would normally omit the direct limit. Why we do not omit it will be clarified in a moment. Define the translation (or suspension) automorphism $\Sigma : \mathcal{E}(X) \to \mathcal{E}(X)$ by $\Sigma(A,n) := (A,n+1)$. The reason for the direct limit above, is that if we omitted it, $\Sigma$ would again 'just' be an equivalence of categories, which is not entirely good enough. The canonical functor $\mathcal{E}(X) \to \tilde{E}(X)$ which maps $A$ to $(A,0)$ implies that $\mathcal{E}(X)$ is a full subcategory of $\tilde{E}(X)$. By Bott periodicity $(A,n)$ is naturally isomorphic to $(\Sigma^{n \mod 2}A,0)$ for any $n \in \mathbb{Z}$ and thus the functor $\mathcal{E}(X) \to \tilde{E}(X)$ is essentially surjective which implies that it is an equivalence of categories. Most of the time we will ignore the difference.
between $\mathcal{E}(X)$ and $\tilde{\mathcal{E}}(X)$, and we will almost always omit the integers for the objects of $\tilde{\mathcal{E}}(X)$.

Our goal is to prove that $\tilde{\mathcal{E}}(X)$ is a triangulated category. We say that a sequence in $\tilde{\mathcal{E}}(X)$ is a mapping cone triangle if it is of the form

$$\Sigma B \xrightarrow{[\iota]} C \xrightarrow{[\alpha]} A \xrightarrow{[\pi]} B$$

for an $X$-equivariant $*$-homomorphism $\pi : A \to B$. We define the exact triangles of $\tilde{\mathcal{E}}(X)$ to be sequences $\Sigma B' \to C' \to A' \to B'$ which are isomorphic to a mapping cone triangle. We will prove that the opposite category $\tilde{\mathcal{E}}(X)^{\text{op}}$ is triangulated,\(^{18}\) which will imply that $\tilde{\mathcal{E}}(X)$ with suspension automorphism $\Sigma^{-1}$ is also a triangulated category.

**Lemma 5.1.1.** The category $\tilde{\mathcal{E}}(X)$ is pre-triangulated, i.e. it satisfies the axioms (TR0)-(TR3).

**Proof.** We will show that $\tilde{\mathcal{E}}(X)^{\text{op}}$ is pre-triangulated. By replacing $\Sigma A$ with the isomorphic object $C_0(\mathbb{R}, A)$ for any object $A$ in $\tilde{\mathcal{E}}(X)$, we may consider everything as if it were happening in $\mathcal{E}(X)$, and then generalize to $\tilde{\mathcal{E}}(X)$. Hence when we write $\Sigma A$ we should think of it as $C_0(\mathbb{R}, A)$.

By definition $\Sigma$ is an automorphism and the class of exact triangles is closed under isomorphisms. The axiom (TR0) requires that $\Sigma A \to 0 \to A$ is an exact triangle which follows from $C_{1A} = CA$ being contractible.

The axiom (TR1) requires that every morphism $[[\phi]] : A \to B$ fits into an exact triangle $\Sigma B \to C \to A \xrightarrow{[[\phi]]} B$. By Lemma 3.7.2, $[[\Sigma^2 \phi \otimes K]] = [\alpha][\beta]^{-1}$ for $X$-equivariant $*$-homomorphisms $\alpha$ and $\beta$. Using the notation from this lemma we get the commutative diagram

\[
\begin{array}{c}
\Sigma^3 B \otimes K \xrightarrow{} C \xrightarrow{\alpha} E \xrightarrow{\beta} \Sigma^2 B \otimes K \\
\Sigma^3 B \otimes K \xrightarrow{} C \xrightarrow{\alpha} \Sigma^2 A \otimes K \xrightarrow{[[\Sigma^2 \phi \otimes 1_K]]} \Sigma^2 B \otimes K \\
\Sigma B \xrightarrow{} C \xrightarrow{\alpha} A \xrightarrow{\beta} B
\end{array}
\]

where the isomorphisms from the bottom to the middle row, are the ones induced by Bott periodicity and stability. Since the top row is a mapping cone triangle, the bottom row is an exact triangle.

The axiom (TR2) requires that $\Sigma B \to C \to A \to B$ is an exact triangle if and only if $\Sigma A \to \Sigma B \to C \to A$, where the morphisms have opposite signs, is an exact triangle. It is enough to prove the necessity since Bott periodicity is

\(^{18}\)I.e. we prove that $\mathcal{E}(X)$ satisfies the axioms (TR0)-(TR4) with all the arrows reversed.
natural also in $\tilde{\mathcal{E}}(X)$. Consider the commutative diagram

\[
\begin{array}{cccccc}
\Sigma A & \xrightarrow{-\Sigma[\pi]} & \Sigma B & \xrightarrow{[\beta]} & C_\pi & \xrightarrow{-[\alpha]} A \\
\downarrow \cong & & \downarrow \cong & \downarrow \cong & \downarrow &= \\
\Sigma A & \xrightarrow{[\pi]} & \Sigma B & \xrightarrow{[\beta]} & \Sigma C_\pi & \xrightarrow{[\alpha]} A \\
\downarrow \cong & & \downarrow \cong & \downarrow \cong & \downarrow &= \\
\Sigma A & \xrightarrow{C_\alpha} & C_\pi & \xrightarrow{[\alpha]} & A \\
\end{array}
\]

where the $X$-equivariant $*$-homomorphism $\tau : \Sigma B \to C_\alpha$ is given by $\tau(f \otimes b) = ((0, f \otimes b), 0)$, and the rest are the obvious ones. As usual we consider $\Sigma B$ as $C_0(\mathbb{R}, B)$, and $f(x) = f(-x)$. Note that $\tau$ is a homotopy equivalence even if $\pi$ is not surjective. The bottom row is an exact triangle by definition and thus so is the top row.

The axiom (TR3) says, that given a diagram as the solid diagram

\[
\begin{array}{cccccc}
\Sigma B & \xrightarrow{[\pi]} & C & \xrightarrow{[\alpha]} & A & \xrightarrow{B} B' \\
\downarrow \cong & & \downarrow \cong & \downarrow \cong & \downarrow \cong & \downarrow \cong \\
\Sigma B' & \xrightarrow{[\pi']} & C' & \xrightarrow{[\alpha']} & A' & \xrightarrow{B'} \\
\end{array}
\]

where the rows are exact triangles and the right square commutes, then there exists a morphism $C \to C'$ making the diagram commute. We may replace the exact triangles with mapping cone triangles induced by the $X$-equivariant $*$-homomorphisms $\pi : A \to B$ and $\pi' : A' \to B'$ and thus letting $C = C_\pi$ and $C' = C_\pi'$. Commutativity implies that

\[[([\Sigma \pi' \otimes 1_K] \circ \phi) = [\pi'][[\phi]] = [[\psi]][[\pi]] = [[\psi] (\Sigma \pi \otimes 1_K)]].

Let $\Phi_t : \Sigma A \otimes K \to \Sigma B' \otimes K$ be a corresponding homotopy. Then $[[([\phi_t, \Phi_t])]$ is a morphism from $A$ to $Z_{\pi' \otimes 1_K} \cong \Sigma Z_{\pi'} \otimes K$ by the universal property of pull-backs. We may, in a natural way, identify $\Sigma A \otimes K \oplus \Sigma B' \otimes K$ with the pull-back $\Sigma A \otimes K \oplus \Sigma B' \otimes K$ and $\Sigma C_{\pi'} \otimes K$ with the pull-back $\Sigma Z_{\pi'} \otimes K \oplus \Sigma B' \otimes K$. Under these identifications $[[([\phi, \Phi] \oplus C\psi)]$ is a morphism $C_\pi \to C_{\pi'}$ whic makes the above diagram commute.

We will say that a sequence $\Sigma B \to J \to A \to B$ in $\tilde{\mathcal{E}}(X)$ is an extension triangle if it is induced by an extension $0 \to J \to A \to B \to 0$ in $\mathcal{C}^*_\text{sep}(X)$, where the connecting morphism $\Sigma B \to J$ is induced by the Connes-Higson construction $\gamma$ of the extension $0 \to J \to A \to B \to 0$, i.e. $[[\Sigma \gamma \otimes 1_K]] : \Sigma B \to J$.

**Lemma 5.1.2.** The sequence $\Sigma B' \to C' \to A' \to B'$ in $\tilde{\mathcal{E}}(X)$ is an exact triangle if and only if it is isomorphic to an extension triangle.
Proof. Since the mapping cylinder $Z_\pi$ of the $X$-equivariant $*$-homomorphism $\pi : A \to B$ is homotopy equivalent to $A$, and $\pi$ is canonically isomorphic to an $X$-equivariant $*$-epimorphism $\tilde{\pi} : Z_\pi \to B$, with kernel $C_\pi$, any mapping cone triangle in $\mathcal{E}(X)$ is isomorphic to an extension triangle. We need to see that the class of the inclusion $\Sigma B \to C_\pi$ is homotopic to the Connes-Higson construction $\gamma$. When doing the Connes-Higson construction choose the lift $(0, x \mapsto xb) \in Z_\pi$ of $b \in B$. An argument similar to that in Lemma 3.6.2 gives that $\gamma$ is equivalent to $f \otimes b \mapsto (0, x \mapsto f(g_t(x))b)$ with $g_t$ defined as in the proof of Lemma 3.6.2. This is homotopic to the inclusion.

Recall that for an extension $0 \to J \to A \xrightarrow{\pi} B \to 0$ in $C^*\text{sep}(X)$, the canonical $X$-equivariant $*$-homomorphism $J \to C_\pi$ is an isomorphism in $\mathcal{E}(X)$. Commutativity of the left square in

$$
\begin{array}{ccc}
\Sigma B & \xrightarrow{\phi} & J \\
\downarrow & \cong & \downarrow \pi \\
\Sigma B & \xrightarrow{\varepsilon} & C_\pi
\end{array}
$$

follows almost immediately when replacing the class of the inclusion $J \to C_\pi$ with its inverse from Proposition 3.6.3. Hence every extension triangle is isomorphic an exact triangle. \qed

Theorem 5.1.3. The pre-triangulated category $\mathcal{E}(X)$ satisfies the octahedral axiom and is thus triangulated.

Proof. We will prove that the opposite category $\mathcal{E}(X)^{op}$ satisfies the octahedral axiom. Let $[[\phi]] : B \to D$ and $[[\psi]] : A \to B$ be morphisms in $\mathcal{E}(X)$ which we may assume up to isomorphism are represented by morphisms in $\mathcal{E}(X)$. Using Lemma 3.7.2 we decompose $[[\Sigma^2 \phi \otimes 1_K]] = [\alpha_\phi][\beta_\phi]^{-1}$ with $\alpha_\phi : E_\phi \to \Sigma^2 D \otimes K$ an $X$-equivariant $*$-homomorphism. Similarly we decompose $\Sigma^2(1 \otimes [\beta_\psi]^{-1}[[\Sigma^2 \psi \otimes 1_K]]) \otimes 1_K = [\alpha_\psi][\beta_\psi]^{-1}$.

Hence $[[\phi]]$ and $[[\psi]]$ are isomorphic in $\mathcal{E}(X)$ to

$$
\Sigma^2 \alpha_\phi \otimes 1_K : \Sigma^2 E_\phi \otimes K \to \Sigma^4 D \otimes K \otimes K,
\quad [\alpha_\psi] : E_\psi \to \Sigma^2 E_\psi \otimes K.
$$

In particular, we may, by renaming everything, assume that $[[\phi]] = [\pi]$ and $[[\psi]] = [\rho]$ for $X$-equivariant $*$-homomorphisms $\pi : B \to D$ and $\rho : A \to B$.

Define

$$
Z_{\pi,\rho} := \{(a, b, d) \in A \oplus IB \oplus ID \mid \rho(a) = b(0), \pi(b(1)) = d(0)\}
$$

which is a separable $C^*$-algebra over $X$ since it is the mapping cylinder of the $X$-equivariant $*$-homomorphism $\pi' : Z_\rho \to D$ given by $\pi'(a, b) = \pi(b(1))$. Consider the $X$-equivariant $*$-homomorphisms

$$
\begin{array}{c}
p_A : Z_{\pi,\rho} \to A, \\
j_A : A \to Z_{\pi,\rho}, \\
j_B : B \to Z_{\pi,\rho}, \\
\tilde{\pi} : Z_\pi \to D, \\
\tilde{\rho} : Z_{\pi,\rho} \to Z_\pi
\end{array}
\begin{array}{c}
(a, b, d) \mapsto a \\
(a) \mapsto (a, x \mapsto \rho(a), x \mapsto \pi(\rho(a))) \\
b \mapsto (b, x \mapsto \pi(b)) \\
(b, d) \mapsto d(1) \\
(a, b, d) \mapsto (b(1), d).
\end{array}
$$
Note that \( p_A j_A = 1_A \) and \( j_A p_A \) is homotopic to \( 1_{Z_{\pi,\rho}} \) in \( \mathcal{C}^{sep}(X) \). Hence \( p_A, j_A \) and also \( j_B \) are homotopy equivalences. Since \( \tilde{\pi} j_B = \pi \) and \( \tilde{\rho} j_A = j_B \rho \) we have that \( \pi \) (resp. \( \rho \)) is isomorphic to \( \tilde{\pi} \) (resp. \( \tilde{\rho} \)). Let \( C_{\pi,\rho} \) be the kernel of \( \tilde{\pi} \tilde{\rho} \). Since the kernel of \( \tilde{\rho} \) is naturally isomorphic to \( C_{\pi} \) we get the following commutative diagram

\[
\begin{array}{cccccc}
\Sigma C_{\pi} & \longrightarrow & \Sigma Z_{\pi} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma Z_{\pi} & \longrightarrow & C_{\rho} & \longrightarrow & C_{\rho} \\
\downarrow \Sigma[\tilde{\pi}] & & \downarrow & & \downarrow & & \downarrow \\
\Sigma D & \longrightarrow & C_{\pi,\rho} & \longrightarrow & Z_{\pi,\rho} & \longrightarrow & D \\
\downarrow & & \downarrow \tilde{\rho} & & \downarrow & & \downarrow \\
\Sigma D & \longrightarrow & C_{\pi} & \longrightarrow & Z_{\pi} & \longrightarrow & D \\
\end{array}
\]

in which the two lower rows and the two middle columns are extension triangles. The only squares which do not obviously commute are the top square and the two left squares. The top square commutes by Lemma 3.6.1. To see that the lower left square commutes let the approximate unit in \( C_{\pi} \) be the image of the approximate unit in \( C_{\pi,\rho} \), when doing the Connes-Higson construction. Commutativity follows almost immediately. It remains to show that the upper left square commutes.

Doing just as in the proof of Lemma 3.6.1 with the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C_{\pi} & \longrightarrow & Z_{\pi,\rho} & \longrightarrow & Z_{\pi} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \tilde{\rho} & & \downarrow \tilde{\pi} & & \downarrow & & \downarrow \\
0 & \longrightarrow & C_{\pi,\rho} & \longrightarrow & Z_{\pi,\rho} & \longrightarrow & D & \longrightarrow & 0 \\
\end{array}
\]

we get that the composites \( \Sigma Z_{\pi} \to \Sigma D \to C_{\pi,\rho} \) and \( \Sigma Z_{\pi} \to C_{\rho} \to C_{\pi,\rho} \) in \( \mathcal{E}(X) \) coincide.

\[ \square \]

So in conclusion we have proved that \( \mathcal{E}(X) \) is equivalent to a triangulated category. However, at no time in the remaining part of this thesis do we use that \( \Sigma \) is an an automorphism where the same would not hold for an equivalence. Hence we will abuse notation slightly and say that \( \mathcal{E}(X) \) is triangulated with suspension functor \( \Sigma \).\(^{19}\) Furthermore, we will, without even lifting an eyebrow, treat \( \mathcal{E}(X) \) as if it were a triangulated category by for example doing homological algebra.

\(^{19}\)Note that according to the construction, our suspension functor should be \( \Sigma^{-1} \) but Bott periodicity implies that this makes no significant difference.
5.2 Ideals in Triangulated Categories

In this section we define the basic theory for ideals in triangulated categories and give a few examples of such by the means of $E$-theory. We also discuss chain complexes in triangulated categories and what it means for them to be exact with respect to an ideal.

The ideas and definitions in section are from [18] and [20].

For a category $C$ we denote the set of morphisms from $A$ to $B$ by $C(A,B)$.

**Definition 5.2.1.** An ideal $\mathcal{I}$ in an additive category $C$, is a family of subgroups $\mathcal{I}(A,B) \subseteq C(A,B)$ for $A,B \in \text{Ob}(C)$, such that

$$C(C,D) \circ \mathcal{I}(B,C) \circ C(A,B) \subseteq \mathcal{I}(A,D)$$

for every $A,B,C,D \in \text{Ob}(C)$.

If $I_1$ and $I_2$ are ideals in $C$ we will write that $I_1 \subseteq I_2$ if $I_1(A,B) \subseteq I_2(A,B)$ for all objects $A$ and $B$ in $C$.

Let $C$ and $C'$ be additive categories and let $F: C \to C'$ be an additive functor. The kernel of $F$, $\ker F$, is the ideal in $C$ for which

$$\ker F(A,B) = \{ f \in C(A,B) \mid F(f) = 0 \}.$$

**Example 5.2.2.** An easy, and important, example is the additive functor $K_*: E \to \mathbb{Z}/2^0$, where $\mathbb{Z}/2^0$ is the category of $\mathbb{Z}/2$-graded abelian groups with evenly graded maps. Clearly $\mathbb{Z}/2^0$ is abelian, and thus additive, and $K_*$ is additive. This induces the ideal $K = \ker K_*$ where

$$K(A,B) = \{ [[\phi]] \mid [[\phi]]_*: K_*(A) \to K_*(B) \text{ vanishes} \} \subseteq E_0(A,B)$$

for every $C^*$-algebra $A$ and $B$.

In the following, $\mathcal{T}$ will be denoting a triangulated category.

Recall that a functor $F: \mathcal{T} \to C$, where $C$ is an abelian category, is called homological if it maps exact triangles to exact sequences. See Section B for more details.

**Definition 5.2.3.** A stable additive category is an additive category $C$ with an additive automorphism $\Sigma: C \to C$, called the suspension.

A stable homological functor is a homological functor $F: \mathcal{T} \to C$, where $C$ is a stable abelian category, such that $F$ intertwines with the suspension, i.e. there is a natural isomorphism $F(\Sigma_X A) \cong \Sigma C F(A)$ for every $A \in \text{Ob}(\mathcal{T})$.

An ideal $\mathcal{I}$ in a triangulated category $\mathcal{T}$ is said to be homological if it is the kernel of a stable homological functor.

**Example 5.2.4.** Let $X$ be a finite sober space and recall that for every $Y \in \text{LC}(X)$ we define the functor $FK_Y: \mathcal{C}(X) \to \mathbb{Z}/2^0$ by $FK_Y(A) = K_*(A(Y))$. We claim that the ideal $K$ given by

$$K(A,B) = \{ [[\phi]] \mid [[\phi]]_*: K_*(A(Y)) \to K_*(B(Y)) \text{ vanishes for all } Y \in \text{LC}(X) \}$$

for every $C^*$-algebra $A$ and $B$. 

61
for $A, B \in \text{Ob}(\mathcal{E}(X))$, is a homological ideal. Observe, that $\mathbb{A}^{Z/2}_0$ comes naturally equipped with a suspension functor $\Sigma : \mathbb{A}^{Z/2}_0 \to \mathbb{A}^{Z/2}_0$ given by $\Sigma(G_0, G_1) = (G_1, G_0)$. Define the functor

$$FK : \mathcal{E}(X) \to \prod_{Y \in \text{LC}(X)} \mathbb{A}^{Z/2}_0, \quad FK(A) = (K_*(A(Y)))_{Y \in \text{LC}(X)}$$

for $A \in \text{Ob}(\mathcal{E}(X))$. Clearly $FK$ is a stable homological functor and $\ker FK = K$.

**Definition 5.2.5.** Let $\mathcal{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$.

- Let $f : A \to B$ be a morphism in $\mathfrak{T}$ and embed $f$, by (TR1), into an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$. We say that $f$ is
  - $\mathcal{I}$-epic if $g \in \mathcal{I}$,
  - $\mathcal{I}$-monic if $h \in \mathcal{I}$,
  - an $\mathcal{I}$-equivalence if $f$ is both $\mathcal{I}$-epic and $\mathcal{I}$-monic, i.e. if $g, h \in \mathcal{I}$,
  - an $\mathcal{I}$-phantom map if $f \in \mathcal{I}$.

- An object $A$ in $\mathfrak{T}$ is called $\mathcal{I}$-contractible if $1_A \in \mathcal{I}(A, A)$.

- An exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is called $\mathcal{I}$-exact if $h \in \mathcal{I}$.

Note that the notions of $\mathcal{I}$-epic and $\mathcal{I}$-monic do not depend on the choices of $g$ and $h$. In fact let $A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A$ be another exact triangle. Then by (TR3) we may complete the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A.
\end{array}$$

By the five lemma for triangulated categories, the morphism $C \to C'$ is an isomorphism. It follows from the definition of an ideal that $g \in \mathcal{I}$ if and only if $g' \in \mathcal{I}$ and similarly for $h$ and $h'$.

The following lemma is just a reformulation of the above definition. Every assertion is obvious and thus the proof is omitted.

**Lemma 5.2.6.** Let $F : \mathfrak{T} \to \mathcal{C}$ be a stable homological functor.

- Let $f$ be a morphism in $\mathfrak{T}$. Then
  - $f$ is ker $F$-epic if and only if $F(f)$ is epic,
  - $f$ is ker $F$-monic if and only if $F(f)$ is monic,
  - $f$ is a ker $F$-equivalence if and only if $F(f)$ is invertible,
  - $f$ is a ker $F$-phantom map if and only if $F(f) = 0$.  

62
• An object \( A \) in \( \mathcal{T} \) is \( \mathcal{I} \)-contractible if and only if \( F(A) \cong 0 \).

• An exact triangle \( A \xrightarrow{f} B \xrightarrow{h} C \xrightarrow{\Sigma} \) is \( \mathcal{I} \)-exact if and only if
  \[ 0 \to F(A) \to F(B) \to F(C) \to 0 \]
  is a short exact sequence in \( \mathcal{C} \).

We end this section by defining exactness of chain complexes in triangulated categories with respect to some ideal. Our definition is different, yet equivalent, to the corresponding definition in [20]. Exact chain complexes will be crucial when defining derived functors.

**Definition 5.2.7.** Let

\[ C_\bullet := (\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots) \]

be a chain complex in \( \mathcal{T} \), indexed by \( \mathbb{Z} \), let \( F : \mathcal{T} \to \mathcal{C} \) be a stable homological functor and let \( \mathcal{I} := \ker F \). Then \( C_\bullet = (C_n, \partial_n) \) is called \( \mathcal{I} \)-exact in degree \( n \) if

\[ F(C_{n+1}) \xrightarrow{F(\partial_{n+1})} F(C_n) \xrightarrow{F(\partial_n)} F(C_{n-1}) \]

is exact.

We say that \( C_\bullet \) is \( \mathcal{I} \)-exact if it is \( \mathcal{I} \)-exact in every degree \( n \in \mathbb{Z} \).

### 5.3 Projective Objects and Derived Functors

In this section we define what we mean about projective objects in a triangulated category, we define projective resolutions and construct derived functors. Most of what is done in this section is done in more detail in [20].

In this section \( \mathcal{I} \) is always a homological ideal in the triangulated category \( \mathcal{T} \).

**Definition 5.3.1.** A homological functor \( F : \mathcal{T} \to \mathcal{C} \) is called \( \mathcal{I} \)-exact if \( \mathcal{I} \subseteq \ker F \), i.e. if \( F(f) = 0 \) for all \( \mathcal{I} \)-phantom maps \( f \).

An object \( P \) in \( \mathcal{T} \) is called \( \mathcal{I} \)-projective if the functor \( \mathcal{T}(P, -) : \mathcal{T} \to \text{Ab} \) is \( \mathcal{I} \)-exact. Dually, an object \( I \) in \( \mathcal{T} \) is called \( \mathcal{I} \)-injective if the functor \( \mathcal{T}(-, I) : \mathcal{T} \to \text{Ab}^{\text{op}} \) is \( \mathcal{I} \)-exact.

We let \( \mathcal{P}_\mathcal{I} \) denote the class of all \( \mathcal{I} \)-projective objects in \( \mathcal{T} \).

Note that the notions of \( \mathcal{I} \)-projective and \( \mathcal{I} \)-injective are each others duals. In fact, \( P \) is an \( \mathcal{I} \)-projective object in \( \mathcal{T} \) if and only if it is an \( \mathcal{I} \)-injective object in \( \mathcal{T}^{\text{op}} \). Hence we will only be working with \( \mathcal{I} \)-projective objects. The theory for injective objects is obtained by dualising everything.

Observe, as one would expect from classical homological algebra, that an \( \mathcal{I} \)-projective object \( P \) has the following lifting property: let \( f : A \to B \) be an \( \mathcal{I} \)-epic morphism in \( \mathcal{T} \) and \( g : P \to B \) be any morphism. Then there exists a morphism \( h : P \to A \) such that \( f \circ h = g \). To see this note that since \( \mathcal{T}(P, -) \) is \( \mathcal{I} \)-exact it maps \( \mathcal{I} \)-epimorphisms to epimorphisms. Hence \( \mathcal{T}(P, A) \xrightarrow{f} \mathcal{T}(P, B) \) is surjective and thus \( g \in \mathcal{T}(P, B) \) lifts to a morphism \( h \). The following lemma gives an other way to think of projectivity.
Lemma 5.3.2. An object $P$ in $\mathfrak{T}$ is $\mathfrak{I}$-projective if and only if $\mathfrak{I}(P, A) = 0$ for every object $A$ in $\mathfrak{T}$.

Proof. If $f \in \mathfrak{I}(P, A)$ then $f = f_\ast(1_P)$ vanishes if $P$ is $\mathfrak{I}$-projective. Conversely, if $\mathfrak{I}(P, A) = 0$ for every object $A$ in $\mathfrak{T}$, let $f \in \mathfrak{I}(A, B)$. Then $f_\ast : \mathfrak{I}(P, A) \to \mathfrak{I}(P, B)$ factors through $\mathfrak{I}(P, B) = 0$ and thus $f_\ast = 0$. Hence $P$ is $\mathfrak{I}$-projective. 

Lemma 5.3.3. The class $\mathfrak{P}_\mathfrak{I}$ of all $\mathfrak{I}$-projective objects in $\mathfrak{T}$, is closed under $(\mathrm{de})$suspensions, retracts, and arbitrary direct sums if they exist in $\mathfrak{T}$.

Proof. Let $F : \mathfrak{T} \to \mathfrak{C}$ be a stable homological functor such that $\mathfrak{I} = \ker F$. Since $F$ intertwines with suspension, a morphism $f$ is $\mathfrak{I}$-phantom if and only if $\Sigma f$ is $\mathfrak{I}$-phantom. Since $\mathfrak{T}(\Sigma P, -) \cong \mathfrak{T}(P, \Sigma^{-1}-)$, the class $\mathfrak{P}_\mathfrak{I}$ is closed under $(\mathrm{de})$suspension.

If $P_0$ is a retract of the $\mathfrak{I}$-projective object $P$, let $f \in \mathfrak{I}(P_0, A)$ for some object $A$. Since $f$ is equal to the composite $P_0 \to P \to P_0 \xrightarrow{f} A$, and $P \to P_0 \xrightarrow{f} A$ is zero by Lemma 5.3.2, it follows that $f = 0$ and $P_0$ is projective.

Let $(P_\alpha)$ be a class of objects which has a direct sum in $\mathfrak{T}$ and for which $P_\alpha$ is $\mathfrak{I}$-projective for every $\alpha$. Then $\mathfrak{T}(\bigoplus P_\alpha, -) \cong \prod \mathfrak{T}(P_\alpha, -)$ clearly is $\mathfrak{I}$-exact. Hence $\bigoplus P_\alpha$ is projective.

We can use the above lemma to obtain a lot of projective objects from only knowing very few. The following is an example of such.

Example 5.3.4. Let $K_* : \mathfrak{E} \to \mathbb{Z}/2\mathbb{Z}$ be the $K$-theory functor and $K := \ker K_*$. Since $E_0(\mathbb{C}, -)$ is naturally isomorphic to $K_0$, it follows that $\mathbb{C}$ is a $K$-projective object in $\mathfrak{E}$. By Lemma 5.3.3 it follows that $C_0(\mathbb{R})$ is also $K$-projective and, moreover, that $\bigoplus \mathbb{C} \oplus \bigoplus C_0(\mathbb{R})$ are $K$-projective objects for any countable index sets $I_0$ and $I_1$.

Definition 5.3.5. Let $A$ be an object of $\mathfrak{T}$. A one-step $\mathfrak{I}$-projective resolution of $A$ is an $\mathfrak{I}$-epimorphism $\pi : P \to A$ where $P$ is $\mathfrak{I}$-projective. An $\mathfrak{I}$-projective resolution of $A$ is an $\mathfrak{I}$-exact chain complex

$$\cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A$$

where $P_n$ is $\mathfrak{I}$-projective for every $n \geq 0$.

We say that $\mathfrak{I}$ has enough projective objects if every object $A$ in $\mathfrak{T}$ has a one-step $\mathfrak{I}$-projective resolution.
Just as in ordinary homological algebra we may define chain homotopies which is an equivalence relation. Furthermore, we may as usual define the homotopy category $\mathcal{H}(\mathcal{I})$ of chain complexes in the triangulated category $\mathcal{I}$. The following proposition is a well-known result from classical homological algebra and the proof is identical to the usual proof, using only the lifting property of $\mathcal{I}$-projective objects. The usual relevant proofs can be found in [12] sections IV.2-4.

**Proposition 5.3.6.** Assume $\mathcal{I}$ has enough projective objects. Then every object $A$ in $\mathcal{I}$ has an $\mathcal{I}$-projective resolution.

Let $P \to A$ and $P' \to A'$ be $\mathcal{I}$-projective resolutions and let $f : A \to A'$ be a morphism. Then $f$ lifts to a chain morphism $P \to P'$, and this lift is unique up to homotopy. In particular, two $\mathcal{I}$-projective resolutions of an object $A$ in $\mathcal{I}$ are homotopy equivalent, and thus there is an induced functor

$$P : \mathcal{I} \to \mathcal{H}(\mathcal{I}).$$

From this point on we assume that $\mathcal{I}$ has enough projective objects in order for us to know the existence of the functor $P : \mathcal{I} \to \mathcal{H}(\mathcal{I})$. Note that this implies that $\mathcal{I}^{\text{op}}$ has enough injective objects in $\mathcal{I}^{\text{op}}$ and that there exists a similar functor $I : \mathcal{I}^{\text{op}} \to \mathcal{H}(\mathcal{I}^{\text{op}})$, where $\mathcal{H}(\mathcal{I}^{\text{op}})$ is the homotopy category of cochain complexes in $\mathcal{I}^{\text{op}}$. Let $F : \mathcal{I} \to \mathcal{C}$ be an additive covariant functor, where $\mathcal{C}$ is an abelian category. Recall that an additive functor preserves chain homotopy when applying it on each entry of a chain complex, and thus it induces a functor $\mathcal{H}(F) : \mathcal{H}(\mathcal{I}) \to \mathcal{H}(\mathcal{C})$. Similarly, if $F : \mathcal{I}^{\text{op}} \to \mathcal{C}$ is an additive contravariant functor there is an induced functor $\mathcal{C}(F) : \mathcal{C}(\mathcal{I}^{\text{op}}) \to \mathcal{C}(\mathcal{C})$.

Let $H_n : \mathcal{H}(\mathcal{C}) \to \mathcal{C}$ be the $n$th homology functor and $H^n : \mathcal{C}(\mathcal{C}) \to \mathcal{C}$ be the $n$th cohomology functor.

**Definition 5.3.7.** Let $F : \mathcal{I} \to \mathcal{C}$ be an additive covariant functor. We define the $n$th left derived functor of $F$, $L_n F$, for $n \geq 0$ to be the composite

$$\mathcal{I} \xrightarrow{P} \mathcal{H}(\mathcal{I}) \xrightarrow{\mathcal{H}(F)} \mathcal{H}(\mathcal{C}) \xrightarrow{H_n} \mathcal{C}.$$

Similarly let $F : \mathcal{I}^{\text{op}} \to \mathcal{C}$ be an additive contravariant functor. We define the $n$th right derived functor of $F$, $R_n F$, for $n \geq 0$ to be the composite

$$\mathcal{I}^{\text{op}} \xleftarrow{I} \mathcal{C}(\mathcal{I}^{\text{op}}) \xrightarrow{\mathcal{C}(F)} \mathcal{C}(\mathcal{C}) \xrightarrow{H^n} \mathcal{C}.$$

In particular, for each object $B$ in $\mathcal{I}$ we define the Ext functors, $\text{Ext}^n_{\mathcal{I},\mathcal{C}}(-, B)$, to be the $n$th right derived functor of the additive contravariant functor

$$\mathcal{I}(-, B) : \mathcal{I}^{\text{op}} \to \mathfrak{A}.$$

We should note that there is a natural transformation $\mathcal{I}(-, B) \Rightarrow \text{Ext}^0_{\mathcal{I},\mathcal{C}}$, but that this is not in general a natural isomorphism as in the classical case.

We end this section with a remark on how to generalize the construction of derived functors, to the case where $F$ is not additive.

---

20If this were always an isomorphism then UCT would be remarkably uninteresting.
Remark 5.3.8. In the above construction of the derived functor of $F$, the only reason for $F$ to be additive, is so that it induces a functor $\mathcal{H}(\mathcal{T}) \rightarrow \mathcal{H}(\mathcal{C})$. We may, without altering the construction, redefine $\mathcal{H}(\mathcal{T})$ to be the homotopy category of non-negative chains. We can easily get an induced functor $\mathcal{H}(\mathcal{T}) \rightarrow \mathcal{H}(\mathcal{C})$ without the assumption that $F$ is additive. The functors $\Gamma : \text{Ch}_{\geq 0}(\mathcal{T}) \rightarrow \text{Ch}_{\geq 0}(\mathcal{C})$, defined as in the Dold-Kan correspondence, preserve homotopy, and thus induce functors on the corresponding homotopy categories. Note that $\Gamma$ can be constructed even though $\mathcal{T}$ is not abelian.

5.4 Getting Enough Projective Objects

In this section we wish to develop methods of ensuring that our triangulated category has enough projective objects. In particular, we show that $E(X)$ has enough $K$-projective objects. In doing this we define the categories $\mathcal{N}\mathcal{T}$ and $\text{Mod}(\mathcal{N}\mathcal{T})$ which are used to define the filtrated $K$-theory functor $FK$.

We let $\mathcal{T}$ denote a triangulated category, $\mathcal{C}$ a stable additive category, $F : \mathcal{T} \rightarrow \mathcal{C}$ a stable additive functor and $I := \ker F$.

Definition 5.4.1. A covariant functor $\mathcal{T} \rightarrow \text{Ab}$ is called representable if it is naturally isomorphic to the functor $\mathcal{T}(A, -)$ for some object $A$ in $\mathcal{T}$.

Let $\mathcal{C}$ be a stable additive category and $F : \mathcal{T} \rightarrow \mathcal{C}$ is stable additive functor. If the functor $\mathcal{C}(A, F(-)) : \mathcal{T} \rightarrow \text{Ab}$ is representable then we denote a representing object by $F^\dagger(A)$. Note that this object in $\mathcal{T}$ is unique up to isomorphism.

If $A$ is an object in $\mathcal{C}$ for which $F^\dagger$ is defined, and $B$ is an object in $\mathcal{T}$, then

$$\mathcal{T}(F^\dagger(A), B) \cong \mathcal{C}(A, F(B)).$$

Hence we want to think of $F^\dagger$ as a partially defined functor which is the left adjoint of $F$. More precisely, if $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$, of objects $A$ for which $\mathcal{C}(A, F(-))$ is representable, then we may fix a functor $F^\dagger : \mathcal{C}' \rightarrow \mathcal{T}$.

The following lemma tells us that $F^\dagger$ is the left adjoint of a restriction of $F$ under certain conditions.

Lemma 5.4.2. Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$ consisting of objects where $F^\dagger$ is defined, such that if $F^\dagger(A) \cong F^\dagger(B)$ and $A$ is an object of $\mathcal{C}'$ then so is $B$. Let $\mathcal{T}'$ be the full subcategory of $\mathcal{T}$ consisting of representing objects $F^\dagger(A)$ for objects $A$ in $\mathcal{C}'$. Then $F$ restricts to an additive fully faithful $F : \mathcal{T}' \rightarrow \mathcal{C}'$.

\[21\] This can be done by fixing a skeleton in $\mathcal{T}$ and a corresponding equivalence, and then mapping $F^\dagger(A)$ to the corresponding object in the skeleton.
In particular, $F^\dagger : \mathcal{C}' \to \mathcal{T}'$ is left adjoint to $F : \mathcal{T}' \to \mathcal{C}'$.

Moreover, if $F \circ F^\dagger(A) \cong A$ for every object $A$ in $\mathcal{C}'$ then $F : \mathcal{T}' \to \mathcal{C}'$ induces an equivalence of categories with left adjoint $F^\dagger : \mathcal{C}' \to \mathcal{T}'$.

**Proof.** Fix a functor $F^\dagger : \mathcal{C}' \to \mathcal{T}'$ and let $A$ be an object of $\mathcal{C}'$. Consider the composite of natural transformations

$$\mathcal{C}(A,F(-)) \cong \mathcal{T}(F^\dagger(A),- \overset{\delta}{\longrightarrow} \mathcal{C}(F \circ F^\dagger(A),F(-)) \Rightarrow \mathcal{C}(A,F(-))$$

where the last natural transformation is obtained by composing with the morphism in $\mathcal{C}(A,F \circ F^\dagger(A))$ which corresponds to $1_{F^\dagger(A)}$. This composite is the identity on $F^\dagger(A)$ and by naturality it is the identity on every object in $\mathcal{T}'$.

Similarly, the composite

$$\mathcal{C}(F \circ F^\dagger(A),F(-)) \Rightarrow \mathcal{C}(A,F(-)) \cong \mathcal{T}(F^\dagger(A),- \overset{\delta}{\longrightarrow} \mathcal{C}(F \circ F^\dagger(A),F(-))$$

is the identity. Hence $F$ restricted to $\mathcal{T}'$ is fully faithful and $F^\dagger \circ F \circ F^\dagger(A) \cong F^\dagger(A)$ which implies that $F \circ F^\dagger(A)$ is an object of $\mathcal{C}'$. Hence $F : \mathcal{T}' \to \mathcal{C}$ factors through $\mathcal{C}'$. \hfill $\square$

If $A$ is an object in $\mathcal{C}$ for which $F^\dagger$ is defined, then $\mathcal{T}(F^\dagger(A),-) \cong \mathcal{C}(A,F(-))$ vanishes on $\mathcal{I} := \ker F$. Hence $F^\dagger(A)$ is $\mathcal{I}$-projective. We will use this observation to ensure that $\mathcal{T}$ has enough $\mathcal{I}$-projective objects in certain cases. This is a special case, which suffices in our applications, of Proposition 3.37 of [20].

**Proposition 5.4.3.** Let $\mathcal{C}$ be a stable abelian category, let $\mathcal{PC}$ be a full subcategory of $\mathcal{C}$, and let $F : \mathcal{T} \to \mathcal{C}$ be a stable homological functor. Suppose that $F^\dagger$ is defined on $\mathcal{PC}$, and that for every object $A$ in $\mathcal{T}$ there is an epimorphism $P \to F(A)$ in $\mathcal{C}$, where $P$ is an object in $\mathcal{PC}$. Let $\mathcal{I} := \ker F$. Then there are enough $\mathcal{I}$-projective objects and $\mathcal{P}_{\mathcal{I}}$ is generated by the objects $F^\dagger(B)$ where $B$ are objects of $\mathcal{PC}$.

More precisely, an object of $\mathcal{I}$ is $\mathcal{I}$-projective if and only if it is a retract of $F^\dagger(B)$ for an object $B$ in $\mathcal{PC}$.

**Proof.** Let $A$ be an object of $\mathcal{T}$ and $\phi : P \to F(A)$ be an epimorphism for some object $P$ in $\mathcal{PC}$. By the adjointness relation $\mathcal{C}(P,F(A)) \cong \mathcal{T}(F^\dagger(P),A)$ there is an induced morphism $\hat{\phi} : F^\dagger(P) \to A$ which we wish to show is $\mathcal{I}$-epic. Let $\alpha : P \to F \circ F^\dagger(P)$ be the morphism corresponding to $1_{F^\dagger(P)}$ by the adjointness relation. Then $F(\hat{\phi}) \circ \alpha = \phi$ by naturality of the adjointness and since $\phi$ is epic, so is $F(\hat{\phi})$. Hence $\hat{\phi} : F^\dagger(P) \to A$ is a one-step $\mathcal{I}$-projective resolution of $A$, since $F^\dagger(P)$ is $\mathcal{I}$-projective by the remark prior to the proposition. This implies that $\mathcal{T}$ has enough $\mathcal{I}$-projective objects.

Any retract of an $\mathcal{I}$-projective object is again $\mathcal{I}$-projective by Lemma 5.3.3. Let $A$ be $\mathcal{I}$-projective and $\phi : F^\dagger(P) \to A$ be a one-step $\mathcal{I}$-projective resolution as above. Embed this into an exact triangle $N \to F^\dagger(P) \to \widehat{\phi} \to A \to \Sigma N$. The morphism $A \to \Sigma N$ is in $\mathcal{I}$ and thus vanishes, since $\phi$ is $\mathcal{I}$-epic and $A$ is $\mathcal{I}$-projective. This implies that $A$ is a retract of $F^\dagger(P)$. \hfill $\square$
For the rest of the section we let $X$ be a finite sober space. We will denote by $\mathcal{LC}(X)^*$ the subset of $\mathcal{LC}(X)$ which consists of all non-empty connected locally closed subsets of $X$.

**Definition 5.4.4.** Define $\mathcal{N}\mathcal{T}$ to be the small $\mathbb{Z}/2$-graded preadditive category with object set $\mathcal{LC}(X)^*$ and morphism sets being the $\mathbb{Z}/2$-graded abelian groups $\mathcal{N}\mathcal{T}_*(Y, Z) := E_*(X; R_Z, R_Y)$ with the composition inherited from composition of asymptotic morphisms.

**Remark 5.4.5 (Warning!).** The definition of $\mathcal{N}\mathcal{T}$ here is somewhat different than that of R. Meyer and R. Nest in [18]. They define the morphism sets $\mathcal{N}\mathcal{T}_*(Y, Z)$ to be the $\mathbb{Z}/2$-graded abelian groups $[FK_Y, FK_Z]$ of natural transformations $FK_Y \rightarrow FK_Z$ and then claim that these two definitions are equivalent by the Yoneda lemma. This is in fact incorrect! We will now explain why.

There is an obvious forgetful functor $F : \mathfrak{Ab}^{\mathbb{Z}/2} \rightarrow \mathfrak{Ab}$ given by $F(G_0, G_1) = G_0 \oplus G_1$. It is easily seen that $[FK_Y, FK_Z] \cong [F \circ FK_Y, F \circ FK_Z]$. The functor $F \circ FK_Y$ is representable as $E_0(X; R_Y \oplus \Sigma R_Y, -)$ and thus the Yoneda lemma yields $[FK_Y, FK_Z] \cong E_*(X; R_Z, R_Y) \oplus E_*(X; R_Z, R_Y)$.

The reason for this mistake is simple. We will quickly explain what was meant in the definition.

Let $\mathcal{E}_*(X)$ be the $\mathbb{Z}/2$-graded additive category with objects being separable $C^*$-algebras over $X$, and the morphism sets being the $\mathbb{Z}/2$-graded abelian groups $E_*(X; A, B)$. Clearly $\mathcal{E}(X)$ is a subcategory of $\mathcal{E}_*(X)$. Note that we normally do not want to work with $\mathcal{E}_*(X)$ since it is not triangulated. We may define $FK_Y^* : \mathcal{E}_*(X) \rightarrow \mathfrak{Ab}^{\mathbb{Z}/2}$ analogously to how we defined $FK_Y : \mathcal{E}(X) \rightarrow \mathfrak{Ab}^{\mathbb{Z}/2}$. The functors $FK_Y^*$ are representable as $E_*(X; R_Y, -)$. The object sets $\mathcal{N}\mathcal{T}_*(Y, Z)$ may now defined to be the $\mathbb{Z}/2$-graded abelian group $[FK_Y^*, FK_Z^*]$ which by the Yoneda lemma is isomorphic to $E_*(X; R_Z, R_Y)$.

This is in fact the intended definition, which can be seen in the proof of Theorem 4.7 in [18].

**Definition 5.4.6.** A module over $\mathcal{N}\mathcal{T}$ is a grading preserving additive functor $M : \mathcal{N}\mathcal{T} \rightarrow \mathfrak{Ab}^{\mathbb{Z}/2}$.

We let $\mathfrak{Mod}(\mathcal{N}\mathcal{T})$ denote the category with objects being all modules over $\mathcal{N}\mathcal{T}$ and morphisms being evenly graded natural transformations.$^{23}$ Moreover, we let $\mathfrak{Mod}(\mathcal{N}\mathcal{T})_c$ denote the full subcategory of countable $\mathcal{N}\mathcal{T}$-modules.

We define the filtrated $K$-theory functor $FK : \mathcal{E}(X) \rightarrow \mathfrak{Mod}(\mathcal{N}\mathcal{T})_c$ to be given by $FK_Y(A) = FK_Y(A)$ for $Y \in \mathcal{LC}(X)^*$ and

$$FK(A)(\phi_0, \phi_1) = \left( \begin{array}{cc} \phi_0 & \phi_1 \\ \phi_1 & \phi_0 \end{array} \right) : FK_Y(A) \rightarrow FK_Z(A),$$

$^{22}$The reader should be warned, that in this proof, R. Meyer and R. Nest do not make any distinction in the notation between the category $\mathfrak{T}$ with morphism sets $\mathfrak{T}(A, B)$ and the category with morphism sets $\mathfrak{T}_*(A, B)$. I would suggest to denote the latter category by $\mathfrak{T}_*$.

$^{23}$In [18] they do not mention that the natural transformations should be evenly graded, but they do in fact use it. This can easily be seen in the UCT Theorem 4.8, where the shifting $FK(A)[j]$ would not matter, unless the morphisms were evenly graded.
for \((\phi_0, \phi_1) \in E_*(X; \mathcal{R}_Z, \mathcal{R}_Y) = \mathcal{N}T_*(Y, Z)\), where we use that \(FK_Y(A) = E_*(X; \mathcal{R}_Y, A)\).

Note that the filtrated \(K\)-theory modules \(FK(A)\) are in fact countable, since the \(K\)-groups of separable \(C^*\)-algebras are countable. It is easily verified that \(\text{Mod}(\mathcal{N}T)c\) is a stable abelian category with countable coproducts and that \(FK\) is a stable homological functor. Moreover, it is easily seen that the ideal \(K\) in \(\mathcal{C}(X)\) defined in Example 5.2.4 is \(\text{ker} FK\), since if \(Y = Y_1 \sqcup Y_2\) then \(FK_Y = FK_{Y_1} \oplus FK_{Y_2}\).

Remark 5.4.7. One should note that when given the filtrated \(K\)-theory \(FK(A)\) of a separable \(C^*\)-algebra over \(X, A\), we are not only given all the \(K\)-groups of \(A(Y)\) for \(Y \in \mathcal{LC}(X)\). For every \(Y \in \mathcal{LC}(X)\) and \(U \in \mathcal{O}(Y)\) we are also given the six-term exact sequence

\[
\begin{align*}
K_0(A(U)) & \longrightarrow K_0(A(Y)) \longrightarrow K_0(A(Y)/A(U)) \\
\uparrow & \quad \uparrow \\
K_1(A(Y)/A(U)) & \leftarrow K_1(A(Y)) \leftarrow K_1(A(U))
\end{align*}
\]

due to the extension \(0 \to \mathcal{R}_Y \setminus U \to \mathcal{R}_Y \to \mathcal{R}_U \to 0\). Hence one should not think of filtrated \(K\)-theory as all the \(K\)-groups of \(A(Y)\) for \(Y \in \mathcal{LC}(X)\), but rather of all six-term exact sequences of such.

Example 5.4.8. Let \(X = \ast\) be the one-point space. Then \(\mathcal{LC}(X)^* = \{X\}\). Hence \(\mathcal{N}T\) has only one object and since \(\mathcal{R}_X = \mathbb{C}\) the \(\mathbb{Z}/2\)-graded group of morphisms is \(E_*(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[0]\), i.e. \(\mathbb{Z}\) in degree 0. Let \(M : \mathcal{N}T \to \mathfrak{Ab}^{\mathbb{Z}/2}\) be an \(\mathcal{N}T\)-module. Since \(M\) is additive, the morphisms are of the form

\[M(n, 0) = nM(1_X) = n \cdot : M(X) \to M(X).\]

Hence \(M\) is uniquely determined by \(M(X)\) and every \(\mathbb{Z}/2\)-graded abelian group determines an \(\mathcal{N}T\)-module. Since the morphisms of \(\mathcal{N}T\)-modules are evenly graded natural transformations, it follows that \(\text{Mod}(\mathcal{N}T)\) is isomorphic to the category \(\mathfrak{Ab}^{\mathbb{Z}/2}\).

Define for \(Y \in \mathcal{LC}(X)^*\) the free \(\mathcal{N}T\)-module on \(Y\) by

\[P_Y = \mathcal{N}T_*(Y, -) : \mathcal{N}T \to \mathfrak{Ab}^{\mathbb{Z}/2}.\]

An \(\mathcal{N}T\)-module is called free if it is isomorphic to a direct sum of degree-shifted free \(\mathcal{N}T\)-modules \(P_Y[j]\) for \(j \in \mathbb{Z}/2\).

Proposition 5.4.9. Let \(M\) be an \(\mathcal{N}T\)-module. Then

\[\text{Hom}_{\mathcal{N}T}(P_Y[j], M) \cong M(Y)_j\]

naturally for \(j \in \mathbb{Z}/2\) and \(Y \in \mathcal{LC}(X)^*\). Here \(M(Y) = (M(Y)_0, M(Y)_1)\).
Proof. The proof is a Yoneda lemma type argument. Assume that \( j = 0 \). The proof is similar for \( j = 1 \). Let \( \theta : P_Y \Rightarrow M \) be an evenly graded natural transformation and \( \phi_* \in \mathcal{N}T_*(Y,Z) \). By naturality

\[
\theta_Z(\phi_*) = \theta_Z \circ \phi_*(1_Y) = M(\phi_*)(\theta_Y(1_Y)),
\]

and thus \( \theta \) is determined by where it maps \( 1_Y \). Since \( 1_Y = ([1_R_Y],0) \in E_*(X;\mathcal{R}_Y,\mathcal{R}_Y) \) and since \( \theta \) is evenly graded, \( \theta_Y(1_Y) \) is an element of \( M(Y)_0 \). If \( Z = Y \) and \( \phi_* = 1_Y \) then \( M(\phi_*)(\theta_Y(1_Y)) = \theta_Y(1_Y) \) and thus no two different values in \( M(Y)_0 \) define the same natural transformation. It is easily verified that any element in \( M(Y)_0 \) defines such a natural transformation and thus \( \text{Hom}_{\mathcal{N}T}(P_Y,M) \cong M(Y)_0 \). Naturality is obvious.

\[
\square
\]

The following theorem will be important once we have the Universal Coefficient Theorem in the next section.

**Proposition 5.4.10.** The category \( \mathcal{E}(X) \) has enough \( K \)-projective objects.

**Proof.** Let \( \mathfrak{M}\mathfrak{d}(\mathcal{N}\mathcal{T})_f \) be the full subcategory of \( \mathfrak{M}\mathfrak{d}(\mathcal{N}\mathcal{T})_c \) of free countable \( \mathcal{N}\mathcal{T} \)-modules. We wish to show that \( \mathfrak{M}\mathfrak{d}(\mathcal{N}\mathcal{T})_f \) satisfies the conditions of Proposition 5.4.3, thus proving that \( \mathcal{E}(X) \) has enough \( K \)-projective objects.

Let \( M \) be a countable \( \mathcal{N}\mathcal{T} \)-module and define the free countable \( \mathcal{N}\mathcal{T} \)-module

\[
P := \bigoplus_{Y \in \mathcal{L}\mathcal{C}(X),j \in \mathbb{Z}/2} \bigoplus_{m \in M(Y)_j} P_Y[j].
\]

To the direct summand corresponding to \( m \in M(Y)_j \) there is a morphism \( P_Y[j] \rightarrow M \) which hits \( m \) by Proposition 5.4.9. The direct sum of these defines an epimorphism \( P \twoheadrightarrow M \).

By Proposition 5.4.9

\[
\text{Hom}_{\mathcal{N}T}(P_Y[j],\text{FK}(-)) \cong \text{FK}^\dagger_P \cong E_0(X;\mathcal{R}_Y[j],-)
\]

i.e. \( \text{FK}^\dagger(P_Y[j]) \cong \mathcal{R}_Y[j] \) for every \( Y \in \mathcal{L}\mathcal{C}(X) \) and \( j \in \mathbb{Z}/2 \). Here \( \mathcal{R}_Y[0] = \mathcal{R}_Y \) and \( \mathcal{R}_Y[1] = \Sigma\mathcal{R}_Y \). Hence \( \text{FK}^\dagger \) is defined on \( \mathfrak{M}\mathfrak{d}(\mathcal{N}\mathcal{T})_f \) and thus \( \mathcal{E}(X) \) has enough \( K \)-projective objects.

\[
\square
\]

# 5.5 The Universal Property of Filtrated \( K \)-Theory

In this section we will prove the universal property of the filtrated \( K \)-theory functor. Roughly speaking this will imply that doing homological algebra in \( \mathcal{E}(X) \) is the same as doing homological algebra in \( \mathfrak{M}\mathfrak{d}(\mathcal{N}\mathcal{T})_c \).

**Theorem 5.5.1.** Let \( G : \mathcal{E}(X) \rightarrow \mathcal{C} \) be a stable homological functor. Then there is a unique right-exact stable functor \( \overline{G} : \mathfrak{M}\mathfrak{d}(\mathcal{N}\mathcal{T})_c \rightarrow \mathcal{C} \) such that \( L_nG \cong L_n\overline{G} \circ \text{FK} \) for every \( n \). A similar statement holds for stable cohomological functors.

70
Proof. We only prove the covariant case. Let \( \text{Mod}(\mathcal{N}\mathcal{T})_f \) denote the subcategory of \( \text{Mod}(\mathcal{N}\mathcal{T})_c \) consisting of free countable \( \mathcal{N}\mathcal{T} \)-modules, and \( \text{FK}^! : \text{Mod}(\mathcal{N}\mathcal{T})_f \to \mathcal{E}(X) \) denote the induced functor as in Lemma 5.4.2. Define the functor \( \mathcal{G} : \text{Mod}(\mathcal{N}\mathcal{T})_f \to \mathcal{E}(X) \) to be the composite \( \text{FK}^! \). Define \( \mathcal{G}_n \) to be the composite

\[
\text{Mod}(\mathcal{N}\mathcal{T})_c \xrightarrow{P} \text{Ho}(\text{Mod}(\mathcal{N}\mathcal{T})_f) \xrightarrow{\text{Ho}(\mathcal{G})} \text{Ho}(\mathcal{C}) \xrightarrow{H_n} \mathcal{C}.
\]

It follows that \( \mathcal{G}_0 \cong \mathcal{G} \) on \( \text{Mod}(\mathcal{N}\mathcal{T})_f \) and hence \( \mathcal{G}_n \cong L_n \mathcal{G}_0 \) for every \( n \).

Redefine \( \mathcal{G} := \mathcal{G}_0 \). By classical homological algebra \( \mathcal{G} \) is right-exact since \( L_0 \mathcal{G} \cong \mathcal{G} \), and clearly \( \mathcal{G} \) is stable. Now \( L_n \mathcal{G} \cong L_n (\mathcal{G} \circ \text{FK}^!) \circ \text{FK} \cong L_n \mathcal{G} \circ \text{FK} \), where the first natural isomorphism follows, since the following diagram commutes up to natural isomorphism

\[
\begin{array}{ccc}
\mathcal{E}(X) & \xrightarrow{P} & \text{Ho}(\mathcal{P} \mathcal{E}(X)) \\
\downarrow \text{FK} & & \downarrow \text{Ho}(\text{FK}^!)
\end{array}
\]

\[
\begin{array}{ccc}
\text{Mod}(\mathcal{N}\mathcal{T})_c & \xrightarrow{P} & \text{Ho}(\text{Mod}(\mathcal{N}\mathcal{T})_f) \\
\end{array}
\]

If \( \mathcal{H} : \text{Mod}(\mathcal{N}\mathcal{T})_c \to \mathcal{C} \) is any other right-exact stable functor such that \( L_0 \mathcal{G} \cong L_0 \mathcal{H} \circ \text{FK} \) then \( \mathcal{H} \) and \( \mathcal{G} \) agree on \( \text{Mod}(\mathcal{N}\mathcal{T})_f \) up to natural isomorphism. But this implies that \( \mathcal{H} \cong \mathcal{G} \) since they are both right-exact. Hence \( \mathcal{G} \) is unique.

Corollary 5.5.2. For any objects \( A \) and \( B \) in \( \mathcal{E}(X) \),

\[
\text{Ext}^n_{\mathcal{E}(X), K}(A, B) \cong \text{Ext}^n_{\mathcal{N}\mathcal{T}}(\text{FK}(A), \text{FK}(B))
\]

naturally.

Proof. Let \( \mathcal{G} = E_0(X; -, B) \) and construct \( \mathcal{G} \) as in Theorem 5.5.1. Let \( P \) be a free countable \( \mathcal{N}\mathcal{T} \)-module. Then

\[
\mathcal{G} \circ \text{FK}(P) \cong E_0(X; \text{FK}^!(\text{FK}(P)), B) \cong \text{Hom}_{\mathcal{N}\mathcal{T}}(\text{FK}(P), \text{FK}(B)).
\]

The functor \( \text{Hom}_{\mathcal{N}\mathcal{T}}(-, \text{FK}(B)) \) is right-exact and agrees with \( \mathcal{G} \) on \( \text{Mod}(\mathcal{N}\mathcal{T})_f \), and thus it follows that \( \text{Hom}_{\mathcal{N}\mathcal{T}}(-, \text{FK}) \cong \mathcal{G} \). The natural isomorphisms of Ext functors follows immediately.

Corollary 5.5.3. The functor \( \mathcal{G} : \mathcal{E}(X) \to \text{Mod}(\mathcal{N}\mathcal{T})_c \) is the universal \( K \)-exact stable homological functor, i.e. if \( \mathcal{G} : \mathcal{E}(X) \to \mathcal{C} \) is a \( K \)-exact stable homological functor, then there exists a unique exact stable functor \( \mathcal{G} : \text{Mod}(\mathcal{N}\mathcal{T})_c \to \mathcal{C} \) such that \( \mathcal{G} = \mathcal{G} \circ \text{FK} \).

Proof. By Theorem 5.5.1 we may construct a unique right-exact stable homological functor \( \mathcal{G} \) such that \( L_n \mathcal{G} \cong L_n \mathcal{G} \circ \text{FK} \). Since \( L_0 \mathcal{G} \cong \mathcal{G} \) it suffices to show that \( L_0 \mathcal{G} \cong \mathcal{G} \). But this follows from exactness and classical homological algebra.

71
5.6 A Universal Coefficient Theorem

In certain cases we get a Universal Coefficient Theorem (UCT) for derived functors in triangulated categories. We will start this section with the theorem and then apply it to get the classical Künneth theorem for tensor products in $\mathcal{E}$.

**Theorem 5.6.1** (Universal Coefficient Theorem). Let $I$ be a homological ideal in the triangulated category $\mathcal{T}$. Let $A$ be an object of $\mathcal{T}$ such that $A$ has an $I$-projective resolution of length 1 and suppose that $\mathcal{T}(A,B) = 0$ for every $I$-contractible object $B$. Let $F : \mathcal{T} \to \mathcal{C}$ be a homological functor and $\tilde{F} : \mathcal{T}^{op} \to \mathcal{C}$ be a contravariant homological functor. Then there are natural short exact sequences

$$
0 \to L_0 F_\ast(A) \to F_\ast(A) \to L_1 F_{\ast-1}(A) \to 0,
$$

$$
0 \to R^1 \tilde{F}^{\ast-1}(A) \to \tilde{F}^\ast(A) \to R^0 F^\ast(A) \to 0.
$$

In particular, if $B$ is any object of $\mathcal{T}$ we get a natural short exact sequence

$$
0 \to \text{Ext}^1_{\mathcal{T},I}(\Sigma A, B) \to \mathcal{T}(A,B) \to \text{Ext}^0_{\mathcal{T},I}(A, B) \to 0.
$$

(5)

Whenever an object $A$ in $\mathcal{T}$ fits naturally in the exact sequence (5) for any object $B$, we say that $A$ satisfies UCT. In these terms, the 'in particular' part of the above theorem, could be stated as the following corollary.

**Corollary 5.6.2.** If $A$ has an $I$-projective resolution of length 1 and $\mathcal{T}(A,B) = 0$ for every $I$-contractible object $B$, then $A$ satisfies UCT.

Before proving the UCT above we will need a lemma about projective resolutions of length 1.

**Lemma 5.6.3.** Let $I$ be a homological ideal in $\mathcal{T}$. Let $0 \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0$ be $I$-exact and construct an exact triangle $0 \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \tilde{A} \to \Sigma P_1$. Then this exact triangle is $I$-exact and there exists an $I$-equivalence $\alpha : \tilde{A} \to A$ such that the following diagram commutes

$$
\begin{array}{ccc}
P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & \tilde{A} \\
\downarrow & & \downarrow & & \downarrow \alpha \\
P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & A.
\end{array}
$$

**Proof.** Let $F$ be a stable homological functor such that $I = \ker F$. Note that since the given sequence is $I$-exact, $F(\partial_1)$ is a monomorphism and thus $\partial_1$ is $I$-monic. Since the functor $\mathcal{T}(-,A)$ is cohomological we have an exact sequence

$$
\mathcal{T}(\tilde{A},A) \xrightarrow{\partial_0^*} \mathcal{T}(P_0,A) \xrightarrow{\partial_1^*} \mathcal{T}(P_1,A).
$$

Since $\partial_0 \in \ker \partial_1^*$ there exists a morphism $\alpha : \tilde{A} \to A$ such that $\alpha \circ \tilde{\partial}_0 = \partial_0$. Since $\partial_1$ is $I$-monic the exact triangle is $I$-exact. If we apply $F$ to the above diagram, the five lemma implies that $F(\alpha)$ is invertible and thus $\alpha$ is an $I$-equivalence. □
This lemma is a special case of Lemma 3.12 in [20]. For our application there is no reason to make the lemma more general than above.

**Proof of the Universal Coefficient Theorem.** We will only prove the theorem in the covariant case. The contravariant is similar.

Construct the same diagram as in Lemma 5.6.3. We claim that \( \alpha \) is invertible in \( T \). Embed \( \alpha \) into the exact triangle \( \Sigma^{-1}B \to \hat{A} \xrightarrow{\alpha} A \xrightarrow{\beta} B \). By Lemma 5.2.6 \( B \) is \( I \)-contractible and thus \( \mathcal{T}(A, B) = 0 \) by assumption. In particular \( \beta \) vanishes, and thus \( A \cong \hat{A} \oplus B \). Hence

\[
\mathcal{T}(B, B) \subseteq \mathcal{T}(\hat{A}, B) \oplus \mathcal{T}(B, B) \cong \mathcal{T}(A, B) = 0
\]

and thus \( B \cong 0 \) which implies that \( \alpha \) is invertible.

Since \( F \) is a homological functor we get a long exact sequence

\[
\cdots \to F_*(P_1) \xrightarrow{F_*(\partial_1)} F_*(P_0) \to F_*(A) \to F_{*-1}(P_1) \xrightarrow{F_{*-1}(\partial_1)} F_{*-1}(P_0) \to \cdots
\]

where we used that \( A \cong \hat{A} \). This may be cut into short exact sequences

\[
0 \to \text{coker}(F_*(\partial_1)) \to F_*(A) \to \text{ker}(F_{*-1}(\partial_1)) \to 0
\]

and since \( \mathbb{L}_0F_*(A) = \text{coker}(F_*(\partial_1)) \) and \( \mathbb{L}_1F_{*-1}(A) = \text{ker}(F_{*-1}(\partial_1)) \), we get our short exact sequence. The sequence is natural for all objects with projective resolutions of length 1 by an argument identical to that of the construction of the functor \( P : \mathcal{T} \to \mathcal{H}_0(\mathcal{T}) \).

We will use this general UCT to prove the following Künneth theorem for tensor products.

We should note that if \( G = (G_0, G_1), H = (H_0, H_1) \) are \( \mathbb{Z}/2 \) graded abelian groups, we define the tensor product in \( \mathbb{A}^{\mathbb{Z}/2} \) to be

\[
G \otimes H = ((G_0 \otimes H_0) \oplus (G_1 \otimes H_1), (G_0 \otimes H_1) \oplus (G_1 \otimes H_0)).
\]

**Theorem 5.6.4** (The Künneth Theorem). Let \( A \) and \( B \) be a separable \( C^* \)-algebra such that \( A \) satisfies UCT. Then we have a short exact sequence

\[
K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to \text{Tor}_{\mathbb{A}^{\mathbb{Z}/2}}(K_{*-1}(A), K_*(B)).
\]

where both maps have degree 0.

**Proof.** Let \( B \) be a separable \( C^* \)-algebra and let \( F : \mathcal{E} \to \mathbb{A}^{\mathbb{Z}/2} \) be the functor given by \( F(A) = K_*(A \otimes B) \). By Example 5.4.8 we note that \( \mathcal{M}\mathcal{D}(\mathcal{N} \mathcal{T})_c = \mathbb{A}^{\mathbb{Z}/2} \). Moreover, one can easily see that the free \( \mathcal{N} \mathcal{T} \)-modules are exactly the free countable \( \mathbb{Z}/2 \)-graded abelian groups.

Let \( \mathcal{F} : \mathbb{A}^{\mathbb{Z}/2} \to \mathbb{A}^{\mathbb{Z}/2} \) be given by \( \mathcal{F}(G) = G \otimes K_*(B) \). Observe that \( F \) is stable and homologicol. We claim that this functor is the one induced by \( F \) in Theorem 5.5.1. Since \( \mathcal{F} \) is right-exact and stable it suffices to prove that
$F(K^1_*(G)) \cong \overline{F}(G)$ for every free countable \( \mathbb{Z}/2 \)-graded abelian group \( G \). We may write

$$G = \bigoplus_{i \in I_0} \mathbb{Z}[0] \oplus \bigoplus_{i \in I_1} \mathbb{Z}[1], \quad K^1_*(G) \cong \bigoplus_{i \in I_0} C \oplus \bigoplus_{i \in I_1} C_0(\mathbb{R}),$$

where \( I_0 \) and \( I_1 \) are countable sets. This yields

$$F(K^1_*(G)) \cong K_*(\bigoplus_{i \in I_0} B \oplus \bigoplus_{i \in I_1} \Sigma B) \cong \bigoplus_{i \in I_0} K_*(B) \oplus \bigoplus_{i \in I_1} K_{*+1}(B) \cong G \otimes K_*(B).$$

The short exact sequence now follows from general UCT, if \( A \) has a projective resolution of length 1 and if \( E_0(A, C) = 0 \) for every \( K \)-contractible object \( C \) in \( \mathcal{E} \). It is explained in greater detail in the next section why this is equivalent to \( A \) satisfying UCT.

\[ \Delta \]

5.7 The \( E \)-Theoretic Bootstrap Class

In this section we define the \( E \)-theoretic bootstrap class and use this to prove some important theorems. The bootstrap class plays an important role in classification of \( C^* \)-algebras.

Denote by \( \mathcal{B}_E \) the localizing subcategory of \( \mathcal{E} \) generated by \( C \), i.e. the smallest triangulated subcategory of \( \mathcal{E} \) containing \( C \), and which is closed under countable direct sums. This subcategory is called the \( E \)-theoretic bootstrap class. By Proposition 5.4.3 \( \mathcal{B}_E \) contains all \( K \)-projective objects of \( \mathcal{E} \). We will start by studying the category \( \mathcal{E} \) and then extend our knowledge to \( \mathcal{E}(X) \).

By Example 5.4.8, if \( X = \star \) then the categories \( \mathcal{M}od(NT) \) and \( \mathbb{A}_0^{\mathbb{Z}/2} \) are isomorphic. Hence by Corollary 5.5.2 a \( C^* \)-algebra \( A \) satisfies UCT if and only if

$$\text{Ext}_{\mathbb{A}_0^{\mathbb{Z}/2}}(K_{*+1}(A), K_*(B)) \hookrightarrow E_0(A, B) \twoheadrightarrow \text{Hom}_{\mathbb{A}_0^{\mathbb{Z}/2}}(K_*(A), K_*(B))$$

is a short exact sequence for every \( C^* \)-algebra \( B \).

The following theorem classifies objects of \( \mathcal{B}_E \) in terms of UCT.

**Proposition 5.7.1.** Let \( A \) be a separable \( C^* \)-algebra. Then \( A \) is in the bootstrap class \( \mathcal{B}_E \) if and only if \( A \) satisfies UCT.

**Proof.** Note that since every object in \( \mathbb{A}_0^{\mathbb{Z}/2} \) has a projective resolution of length 1, the same is true in \( \mathcal{E} \).

If \( A \) satisfies UCT, then \( E_0(A, B) \) vanishes for any \( K \)-contractible \( C^* \)-algebra \( B \), since \( \text{Ext} \) and \( \text{Hom} \) vanish. By the proof of UCT, Theorem 5.6.1, there is an exact triangle in \( \mathcal{E} \)

$$P_1 \rightarrow P_0 \rightarrow A \rightarrow \Sigma P_1$$
with \( P_0 \) and \( P_1 \) \( K \)-projective. Hence \( A \) is an object of any triangulated subcategory of \( \mathcal{E} \) containing the \( K \)-projective objects by the two-out-of-three property, and in particular \( A \) is an object of \( \mathcal{B}_E \).

We will make use of the following general fact for any triangulated category \( T \). If \( B \) is an object of \( T \), then the full subcategory of \( T \) of objects \( D \) for which \( T(D,B) = 0 \), is triangulated and closed under arbitrary coproducts (as far as they exist). Moreover, intersections of such triangulated categories are again triangulated and closed under coproducts. This follows easily since \( T(-,B) \) is cohomological. Since \( \mathcal{B}_E \) is the smallest triangulated category closed under countable coproducts which contains \( \mathcal{C} \), and since \( E_0(\mathcal{C},B) \) vanishes for any \( K \)-contractible object \( B \), it follows that \( E_0(A,B) = 0 \) for any object \( A \) in \( \mathcal{B}_E \). Hence \( A \) satisfies UCT by Theorem 5.6.1.

Recall that two separable \( C^* \)-algebras are called \( E \)-equivalent if they are isomorphic in \( \mathcal{E} \). Similarly two \( C^* \)-algebras over \( X \) are \( E(X) \)-equivalent if they are isomorphic in \( \mathcal{E}(X) \). The following corollary of the above proposition shows that elements of \( \mathcal{B}_E \) are classified up to \( E \)-equivalence by their \( K \)-theory.

**Corollary 5.7.2.** Let \( A \) and \( B \) be separable \( C^* \)-algebras in \( \mathcal{B}_E \). Then \( A \) and \( B \) are \( E \)-equivalent if and only if \( K_* (A) \cong K_* (B) \).

**Proof.** Necessity is obvious. If \( K_* (A) \cong K_* (B) \) we may lift an isomorphism to a morphism \( A \to B \) in \( \mathcal{E} \) by UCT. By naturality of UCT we get a commutative diagram with exact rows

\[
\begin{array}{ccc}
\Ext_{\mathbb{H}_c,0}^{\mathbb{Z}/2} (K_{*+1} (B), K_*(D)) & \to & \Ext_{\mathbb{H}_c,0}^{\mathbb{Z}/2} (K_*(B), K_*(D)) \\
\downarrow \cong & & \downarrow \cong \\
\Ext_{\mathbb{H}_c,0}^{\mathbb{Z}/2} (K_{*+1} (A), K_*(D)) & \to & \Ext_{\mathbb{H}_c,0}^{\mathbb{Z}/2} (K_*(A), K_*(D))
\end{array}
\]

for any separable \( C^* \)-algebra \( D \). By the five lemma the middle vertical map is an isomorphism which implies that the morphism \( A \to B \) in \( \mathcal{E} \) is invertible.

We extend our notion of the \( E \)-theoretic bootstrap class.

**Definition 5.7.3.** Let \( X \) be a second countable sober space. We define the \( E \)-theoretic bootstrap class \( \mathcal{B}_E(X) \) to be the full subcategory of \( \mathcal{E}(X) \) consisting of objects \( A \) for which \( A(U) \in \Ob(\mathcal{B}_E) \) for every \( U \in \mathcal{O}(X) \).

**Theorem 5.7.4.** Let \( X \) be a finite sober space and \( A \) be a separable \( C^* \)-algebra over \( X \). Suppose that \( A \) has a projective resolution of length 1. Then \( A \) is in the bootstrap class \( \mathcal{B}_E(X) \) if and only if \( A \) satisfies UCT.

**Proof.** We will show that \( \mathcal{B}_E(X) \) contains \( \mathcal{R}_Y \) for every \( Y \in \mathcal{L}(X) \). This will imply that \( \mathcal{B}_E(X) \) contains every \( K \)-projective object. When this is proven the proof is identical to that of Proposition 5.7.1.
Let $U_x$ denote the smallest open subset of $X$ containing $x$. It was proven in Theorem 4.2.2 that $i_x(C)$ and $R_{U_x}$ are $E(X)$-equivalent, and since $i_x(C)$ is in the bootstrap class, so is $R_{U_x}$. A two-out-of-three argument similar to that in the proof of Theorem 4.2.2 yields that $R_Y$ is in the bootstrap class $B_E(X)$ for every $Y \in \mathbb{L}C(X)$. 

Using this theorem we immediately get the following corollary. The proof is exactly as in Corollary 5.7.2. This shows that elements of $B_E(X)$ are classified up to $E(X)$-equivalence by their filtrated $K$-theory.

**Corollary 5.7.5.** Let $X$ be a finite sober space and let $A$ and $B$ be separable $C^*$-algebras over $X$ in $B_E(X)$. Suppose that $A$ and $B$ have projective resolutions of length 1. Then $A$ and $B$ are $E(X)$-equivalent if and only if $FK(A) \cong FK(B)$.

The following last theorem is a very important theorem. This tells us that a morphism in the bootstrap class of $E(X)$ is an isomorphism if it induces isomorphisms of the $K$-theory of the ideals induced by open subsets of $X$.

**Theorem 5.7.6.** Let $X$ be a second countable sober space and let $A$ and $B$ be $C^*$-algebras over $X$ in $B_E(X)$. Then a morphism $A \to B$ in $E(X)$ is invertible if and only if the induced maps $K_*(A(U)) \to K_*(B(U))$ are invertible in $\mathbb{A}b_{0}^{\mathbb{Z}/2}$ for every $U \in \mathcal{O}(X)$.

**Proof.** Assume that we have proven the theorem for $X = \star$ being the one-point space. Then we have $K_*(A(U)) \to K_*(B(U))$ is invertible for every $U \in \mathcal{O}(X)$ if and only if $A(U) \to B(U)$ is invertible in $\mathcal{E}$ for every $U \in \mathcal{O}(X)$ if and only if $A \to B$ is invertible in $\mathcal{E}(X)$. The last 'if and only if' is Theorem 4.4.5. Hence it suffices to prove the theorem for $X = \star$, which was done in Theorem 5.7.2.

We will end this section with a remark on why, unfortunately, the finite approximation of Section 4.4 and filtrated $K$-theory do not work as well together as one might hope.

**Remark 5.7.7.** Let $X$ be a second countable sober space, with finite approximation $X_1, X_2, \ldots$ as in Section 4.4, and let $A$ and $B$ be separable $C^*$-algebras over $X$. Suppose that these induce a UCT short exact sequence, with respect to the finite filtration, i.e.

$$\text{Ext}_{\mathcal{N}T_n}(FK(A)[1], FK(B)) \to E_0(X_n; A, B) \to \text{Hom}_{\mathcal{N}T_n}(FK(A), FK(B))$$

is a short exact sequence for each $n$. Then we get a short exact sequence of projective systems, and thus $\lim$ induces a six-term exact sequence by the snake lemma. What is interesting for us, in this sequence, is the part

$$\lim E_0(X_n; A, B) \longrightarrow \lim \text{Hom}_{\mathcal{N}T_n}(FK(A), FK(B))$$

$$\longrightarrow \lim^{1} \text{Ext}_{\mathcal{N}T_n}(FK(A)[1], FK(B)).$$
This tells us is, that in favorable cases we may lift elements of
\[ \lim Hom_{\mathcal{N}}(FK(A), FK(B)) \]

to elements of \( \lim E_0(X_n; A, B) \) and by Theorem 4.4.1 lift these to \( E_0(X; A, B) \). One could hope that this would help us lift isomorphisms of the filtrated \( K \)-theory to isomorphisms in \( \mathcal{E}(X) \) by Theorem 5.7.6. Unfortunately, by the construction of the connecting morphism, which is the vertical line above, one checks that an element is in the kernel by checking that it is in the image of the previous map. Hence there is no obvious way to check that an element of \( \lim Hom_{\mathcal{N}}(FK(A), FK(B)) \) lifts to \( \mathcal{E}_0(X) \).

5.8 \( KK \)-Theory and Applications

In this section we will discuss \( KK \)-theory and how it intertwines with \( E \)-theory. We then give a classification result of separable nuclear \( C^* \)-algebras with Hausdorff primitive ideal space, for which every ideal is \( E \)-contractible. This classification result is due to [7].

One thing we deliberately have avoided talking about throughout this thesis is \( KK \)-theory. In Ralf Meyer’s and Ryszard Nest’s articles they define \( KK \)-theory for \( C^* \)-algebras over topological spaces, which coincides with the definitions already made by Kirchberg and Bonkat. We refer to [19] for more details. We may define \( KK \)-groups and the category \( KK(X) \) in a similar way as to how we defined the \( E \)-groups and the category \( E(X) \). Most classification results uses Kirchberg’s famous and very deep result from [13] which we will state a simple version of.

\textbf{Theorem 5.8.1 (Kirchberg).} Let \( A \) and \( B \) be separable nuclear \( C^* \)-algebras, let \( X = \text{Prim} A \) and suppose that \( \text{Prim} B \cong X \). Then \( A \) and \( B \) are \( KK(X) \)-equivalent if and only if there is an \( X \)-equivariant isomorphism
\[ A \otimes O_\infty \otimes K \cong B \otimes O_\infty \otimes K. \]

In the theorem we consider \( A \) and \( B \) as \( C^* \)-algebras over \( X = \text{Prim} A \) in the canonical way. When given a \( C^* \)-algebra over \( X \), \((B, \psi)\) we say that it is tight if \( \psi : \text{Prim} B \to X \) is a homeomorphism. Hence in the theorem we could have said that \( B \) should be a tight separable nuclear \( C^* \)-algebra over \( X = \text{Prim} A \).

The functor \( \mathcal{C} \text{sep}(X) \to \mathfrak{R}(X) \) is the universal split-exact, stable, homotopy functor. By the universal property this induces a functor \( \mathfrak{R}(X) \to \mathcal{E}(X) \) which, it turns out, often induces an isomorphism between \( KK \)-groups and \( E \)-groups. This is how we usually apply \( E \)-theory when doing classification. We prove some statement for \( E \)-theory, show that the \( E \)-theory and \( KK \)-theory coincide and then apply Kirchberg’s theorem.

We say that a \( C^* \)-algebra over \( X \), \( A \), is continuous if the map \( \mathcal{O}(X) \to \mathcal{I}(A) \) respects arbitrary infima. It turns out, that if \( X \) is a locally compact Hausdorff

\[ \text{Yet very different.} \]

\[ \text{Within Elliott’s program to classify} \ C^* \text{-algebras by methods of} \ K \text{-theory.} \]
space, then a continuous $C^*$-algebra over $X$ is the same as a continuous $C_0(X)$-algebra. Hence we get the following theorem from [23]. An alternative proof is given in [7].

**Theorem 5.8.2** ([23], Theorem 4.7). Let $X$ be a second countable locally compact Hausdorff space, let $A$ be a separable, nuclear, continuous $C^*$-algebra over $X$ and let $B$ be any separable $C^*$-algebra over $X$. Then the canonical map $KK_0(X; A, B) \rightarrow E_0(X; A, B)$, induced by the functor $\mathfrak{R}(X) \rightarrow \mathfrak{E}(X)$, is an isomorphism.

An other way of obtaining an isomorphism of $KK$-theory and $E$-theory is by extending our homological algebra from earlier. Most results of homological algebra given in this thesis, was originally constructed for the category $\mathfrak{R}(X)$. We should note that the theory simplifies for $E$-theory, since extension triangles are not in general exact in $\mathfrak{R}(X)$. In [16], Ralf Meyer constructs a spectral sequence which, in some cases, converge to the $KK$-groups and $E$-groups respectively. In some cases the induced spectral sequences in $KK$-theory and $E$-theory are identical and both converge to the respectable $KK$-groups or $E$-groups. This is how the following theorem is proved in [7].

**Theorem 5.8.3** ([7], Theorem 5.5). Let $X$ be a finite sober space and let $A$ and $B$ be separable $C^*$-algebras over $X$. Suppose that $A$ is in the bootstrap class of $KK(X)$, which is defined in [19]. Then the canonical map $KK_0(X; A, B) \rightarrow E_0(X; A, B)$ is an isomorphism.

An important part of Elliott’s classification program is to get UCT results. An important application of UCT is, that it allows to lift isomorphisms of $K$-theoretic data to morphisms in $\mathfrak{E}(X)$ or $\mathfrak{R}(X)$. A deeper study of filtrated $K$-theory, as in [18], give us more UCT results. In fact, in the before mentioned article they prove UCT for certain topological spaces with four points.

An other recent UCT result is due to Marius Dadarlat and Ralf Meyer in [7]. They prove a universal multicoefficient theorem over totally disconnected, compact, metrisable spaces, such as the cantor set. The theorem is as follows.

**Theorem 5.8.4** ([7], Theorem 6.11). Let $X$ be a second countable, totally disconnected, compact, metrisable space and let $A$ and $B$ be separable $C^*$-algebras over $X$. If $A$ is in the bootstrap class $B_E(X)$, then there is a short exact sequence

$$\text{Ext}_{C(X, A)}(K(A), K(\Sigma B)) \rightarrow E_0(X; A, B) \rightarrow \text{Hom}_{C(X, A)}(K(A), K(B)).$$

We refer the reader to ([7], §6) for the relevant definitions. It should be noted that since the constructions of $K$-theory with coefficients only use $C^*$-algebras in the bootstrap class of $\mathfrak{R}$, it could just as well be defined using $E$-theory.

Very recently, $E$-theory for $C^*$-algebras over topological spaces has been used to give classification results. In [26], Mitsuharu Takeori proves that every full, continuous, separable, nuclear $C^*$-algebra over a sober space $X$ is $KK(X)$-equivalent to a stable Kirchberg algebra over $X$, by using $E$-theory. We refer the reader to [26] for the relevant definitions.
We will end this thesis with a classification result due to Marius Dadarlat and Ralf Meyer in [7]. The result uses our main theorems, Theorem 4.4.5 and 5.7.6.

**Theorem 5.8.5.** Let $A$ be a separable, nuclear $C^*$-algebra with Hausdorff primitive ideal space $X$, and suppose that every ideal in $A$ is $E$-contractible. Then

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong C_0(X) \otimes \mathcal{O}_2 \otimes \mathbb{K}. $$

*Proof.* Since $A$ is tight it is continuous, and since every ideal is $E$-contractible it follows from Theorem 4.4.5 that $A$ is $E(X)$-contractible. Let $B := C_0(X) \otimes \mathcal{O}_2$. By Theorem 5.7.6 we get that $B$ is $E(X)$-contractible since $C_0(U) \otimes \mathcal{O}_2$ is in the bootstrap class and has trivial $K$-theory for every $U \in \mathcal{O}(X)$. Hence $0 \in E_0(X; A, B)$ is an $E(X)$-equivalence. By Theorem 5.8.2, $0 \in KK_0(X; A, B)$ is a $KK(X)$-equivalence and by the deep theorem of Kirchberg, $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$. The result follows since $\mathcal{O}_2$ is $\mathcal{O}_\infty$-absorbing.

$\square$
A Bott Periodicity

In this section we will reconstruct Joachim Cuntz’ proof of the Bott periodicity theorem from [5]. Although we assume that the reader is already familiar with Bott periodicity and six-term exact sequences, we have chosen to include the proof, since parts of Section 3.6 depend on details from this proof. We have extended the proof such that it holds in any “fitting” subcategory of $\mathcal{C}^*_{\text{alg}}(X)$ where $X$ is a sober space.

### A.1 The Toeplitz Algebra

Before proving the periodicity theorem, we will include some properties of the Toeplitz algebra.

Let $v$ denote the unilateral shift operator on $\ell^2(\mathbb{N})$, i.e. if $(\xi_n)$ is the canonical orthonormal basis in $\ell^2(\mathbb{N})$ then $v\xi_n = \xi_{n+1}$. Recall that the Toeplitz algebra $\mathcal{T}$ is the $C^*$-subalgebra of $B(\ell^2(\mathbb{N}))$ which is generated by $v$. By Coburn’s theorem $\mathcal{T}$ is the universal $C^*$-algebra generated by a (proper) isometry.

**Proposition A.1.1.** The Toeplitz algebra $\mathcal{T}$ is an extension of $\mathbb{K}$ by $\mathbb{K}$, i.e. there is a short exact sequence of $C^*$-algebras

$$0 \rightarrow \mathbb{K} \rightarrow \mathcal{T} \rightarrow C(\mathcal{T}) \rightarrow 0.$$

**Proof.** Observe that $e_{ij} := v^{i-1}(1 - vv^*)(v^*)^{j-1}$ is the $ij$'th matrix unit in $B(\ell^2(\mathbb{N}))$, i.e. $e_{ij}\xi_j = \xi_i$ and $e_{ij}$ is zero on $\{\xi_j\}^\perp$. It is well-known that $\mathbb{K}$ is generated by these matrix units, and thus $\mathbb{K}$ is a $C^*$-subalgebra of $\mathcal{T}$ which is clearly an ideal.

Let $\pi : \mathcal{T} \rightarrow \mathcal{T}/\mathbb{K}$ be the quotient map. Since $\pi(1 - v^*v) = \pi(1 - vv^*) = 0$, $\pi(v)$ is a unitary which generates $\mathcal{T}/\mathbb{K}$ and thus $\mathcal{T}/\mathbb{K} \cong C(\text{spec}(\pi(v)))$. For a fixed $\lambda \in \mathbb{T}$ the universal property of $\mathcal{T}$ gives a $*$-homomorphism $\phi : \mathcal{T} \rightarrow \mathbb{C}$ such that $\phi(v) = \lambda$. Since $\phi(e_{ij}) = 0$ for all $i, j$ it follows that $\phi$ induces a $*$-homomorphism $\mathcal{T}/\mathbb{K} \rightarrow \mathbb{C}$ such that $\pi(v) \mapsto \lambda$. Hence $\text{spec}(\pi(v)) = \mathbb{T}$. \qed

Throughout the rest of the section we let $\pi : \mathcal{T} \rightarrow C(\mathcal{T})$ be the quotient map. Let $C_0(\mathbb{R})$ be the kernel of the evaluation map $\text{ev}_1 : C(\mathcal{T}) \rightarrow \mathbb{C}$, with $\text{ev}_1(f) = f(1)$. We denote by $\mathcal{T}_0$ the reduced Toeplitz algebra which is the pull-back $C_0(\mathbb{R}) \bowtie_{C(\mathcal{T})} \mathcal{T}$. By the corresponding pull-back diagram this fits into the exact sequence

$$0 \rightarrow \mathbb{K} \rightarrow \mathcal{T}_0 \rightarrow C_0(\mathbb{R}) \rightarrow 0.$$

**Remark A.1.2.** Since $\mathbb{K}$ and commutative $C^*$-algebras are nuclear, and nuclearity is preserved by extensions, we get that $\mathcal{T}_0$ and $\mathcal{T}$ are nuclear.

Denote by $\hat{\mathcal{T}}$ the $C^*$-algebra $(\mathbb{K} \otimes \mathcal{T} + \mathcal{T} \otimes 1) \subseteq \mathcal{T} \otimes \mathcal{T}$ which fits into the exact sequence

$$0 \rightarrow \mathbb{K} \otimes \mathcal{T} \rightarrow \hat{\mathcal{T}} \rightarrow C(\mathcal{T}) \rightarrow 0

26 Another reason for including the proof, is the pure brilliance in this version of the proof, which every operator algebraist should see at least once in their lifetime.

27 Whatever that means.
where $\hat{\pi}(a \otimes b) = \pi(a)$.

Denote by $\mathcal{T}$ the pull-back $\hat{T} \oplus_{C(T)} T$. This gives the extension

$$0 \rightarrow \mathbb{K} \otimes T \xrightarrow{i} \hat{T} \xrightarrow{\pi} T \rightarrow 0$$

where $\pi(a \otimes b, x) = x$. Note that this extension has a splitting given by $x \mapsto (x \otimes 1, x)$.

**Lemma A.1.3.** Let $\alpha_i, \beta : T \rightarrow \mathcal{T}$ for $i = 0, 1$ be the $\ast$-homomorphisms given by

$$\alpha_0(v) = (e_{11} \otimes v, 0), \quad \alpha_1(v) = (e_{11} \otimes 1, 0), \quad \beta(v) = (v(1 - e_{11}) \otimes 1, v).$$

Then $\alpha_i + \beta$ are $\ast$-homomorphisms which are homotopic.

**Proof.** Note that $\alpha_0$ and $\alpha_1$ are well-defined since $(e_{11} \otimes v, 0)$ and $(e_{11} \otimes 1, 0)$ are isometries in $(e_{11} \otimes 1, 0)T(e_{11} \otimes 1, 0) \subseteq \mathcal{T}$. Similarly $\beta$ is well-defined. That $\alpha_i + \beta$ are $\ast$-homomorphisms follows from

$$\alpha_i(T) \subseteq (e_{11} \otimes 1, 0)\mathcal{T}(e_{11} \otimes 1, 0), \quad \beta(T) \subseteq ((1 - e_{11}) \otimes 1, 1)\mathcal{T}((1 - e_{11}) \otimes 1, 1),$$

which are orthogonal $C^\ast$-subalgebras of $\mathcal{T}$. It suffices to find a continuous path of isometries connecting $\alpha_0(v) + \beta(v)$ and $\alpha_1(v) + \beta(v)$. Then the universal property of $T$ induces the desired homotopy $T \rightarrow C([0,1], \mathcal{T})$. Let

$$u_0 = v(1 - e_{11})v^* \otimes 1 + e_{11}v^* \otimes v + ve_{11} \otimes v^* + e_{11} \otimes e_{11} \in \mathcal{T},$$

let $\tilde{u}_1 \in \mathcal{T}$ be the unitary which interchanges $\xi_1$ and $\xi_2$ and is the identity on $\xi_n$ for $n > 2$, and let $u_1 = \tilde{u}_1 \otimes 1 \in \mathcal{T}$. Clearly the $u_i$ are self-adjoint and straight forward computations give that both $u_i$ are unitary. Moreover, one easily gets that

$$\alpha_i(v) + \beta(v) = (u_i(v \otimes 1), v)$$

Since $\text{spec}(u_i) \subseteq \{\pm 1\}$ we can choose a branch of logarithm $\log$, holomorphic on $\{z \in \mathbb{C} \mid \text{Im}~z \leq 0\}$, and by functional calculus get two continuous paths of unitaries

$$U_i : [0, 1] \ni s \mapsto \exp(s \log(u_i))$$

connecting $u_i$ to 1. Moreover, $\hat{\pi}(U_i(s)) = \exp(s \log(\hat{\pi}(u_i))) = 1$ and thus

$$s \mapsto (U_i(s)(v \otimes 1), v)$$

for $i = 0, 1$ induce a continuous path of isometries in $\mathcal{T}$ from $\alpha_0(v) + \beta(v)$ to $\alpha_1(v) + \beta(v)$.

\[\square\]
A.2 The Periodicity Theorem in $\mathcal{C}^*\text{alg}(X)$

In this section we prove the periodicity theorem. The reader should have read sections 2.1, 2.2 and 2.3 in order to understand everything that is going on. We start by recalling some relevant definitions and properties.

Let $X$ be a sober space. We will make the very vague definition of saying that a subcategory $\mathcal{C}^*$ of $\mathcal{C}^*\text{alg}(X)$ is 'fitting', if all the constructions done in this section, can be done in $\mathcal{C}^*$. We note that $\mathcal{C}^*\text{alg}(X)$ and $\mathcal{C}^*\text{scp}(X)$ are 'fitting', the latter being our motivation for this definition. Another example of a 'fitting' category is the $E$-theoretic bootstrap class $\mathcal{B}_E(X)$ defined in Section 5.7.

For the rest of the section, $\mathcal{C}^*$ denotes a 'fitting' subcategory of $\mathcal{C}^*\text{alg}(X)$, and $\mathfrak{A}$ denotes an abelian category. We will recall some definitions and properties of functors. For details see ([1], §IX.21). Note that the translations from $\mathcal{C}^*$-algebras to $\mathcal{C}^*$-algebras over $X$ and from $\mathfrak{Ab}$ to any abelian category are trivial.

Recall that a covariant functor $F : \mathcal{C}^* \to \mathfrak{A}$ is called

(H) a homotopy functor if $F$ is homotopy invariant, i.e. $\phi_*=\psi_*$ when $\phi \simeq \psi$ in $\mathcal{C}^*$,

(S) a stable functor if the $X$-equivariant $\ast$-homomorphism $1_A \otimes e_{11} : A \to A \otimes \mathbb{K}$, i.e. $a \to a \otimes e_{11}$ where $e_{11}$ is any rank one projection in $\mathbb{K}$, induces a isomorphism $F(A) \cong F(A \otimes \mathbb{K})$,

(HX) a half-exact functor if for every extension $0 \to J \to A \to B \to 0$ in $\mathcal{C}^*$ the induced sequence $F(J) \to F(A) \to F(B)$ is exact (in the middle).

Similar definitions can be made if $F$ is contravariant.

Recall that if $F$ is a half-exact homotopy functor and $A$ and $B$ are in $\mathcal{C}^*$, then $F(A \oplus B) \cong F(A) \oplus F(B)$. Moreover, if $0 \to J \to A \xrightarrow{i} B \to 0$ is an extension in $\mathcal{C}^*$, then the canonical map $j : J \to C_\pi$ is invertible in $\mathfrak{A}$. Letting $i : \Sigma B \to C_\pi$ be the canonical map and $\partial$ be the composite $F(\Sigma B) \xrightarrow{i_*} F(C_\pi) \xrightarrow{j^{-1}} F(J)$ we get a natural long exact sequence

$$\cdots \to F_{n+1}(A) \to F_{n+1}(B) \xrightarrow{\partial} F_n(J) \to F_n(A) \to F_n(B) \to \cdots$$

where $F_n(A) := F(\Sigma^n A)$. Similarly if $F$ is contravariant we get a natural long exact sequence

$$\cdots \leftarrow F^{n+1}(A) \leftarrow F^{n+1}(B) \xleftarrow{\partial} F^n(J) \leftarrow F^n(A) \leftarrow F^n(B) \leftarrow \cdots$$

where $F^n(A) := F(\Sigma^n A)$.

It is easily seen that if $F$ is stable then the isomorphism $F(A) \cong F(A \otimes \mathbb{K})$ is natural.

For the rest of the section we let $F$ denote a (co- or contravariant) half-exact, stable, homotopy functor from $\mathcal{C}^*$ to $\mathfrak{A}$.

Lemma A.2.1. Let $\phi, \psi : A \to B$ be $X$-equivariant $\ast$-homomorphisms such that $\phi + \psi$ is a $X$-equivariant $\ast$-homomorphism. Then $F(\phi + \psi) = F(\phi) + F(\psi)$. 

82
Proof. Since $\phi(A) \cdot \psi(A) = 0$ we get that $\phi(A) \oplus \psi(A) \subseteq B$. Identifying $F(\phi(A) \oplus \psi(A)) = F(\phi(A)) \oplus F(\psi(A))$ we get that the diagram

\[ \begin{array}{c}
F(\phi(A)) \\
\downarrow \phi_*
\end{array} \xrightarrow{\phi} \begin{array}{c}
F(A) \\
\oplus F(\phi(A)) \oplus F(\psi(A)) \\
\downarrow \psi_*, \phi_*
\end{array} \xrightarrow{\sim} \begin{array}{c}
F(\psi(A))
\end{array} \]

commutes for exactly one morphism $F(A) \rightarrow F(\phi(A)) \oplus F(\psi(A))$ (similarly in the contravariant case). Now we note that both $F(\phi + \psi)$ and $F(\phi) + F(\psi)$ make the diagram commute. \qed

**Theorem A.2.2.** For any object $A$ in $\mathcal{C}$, $F(T_0 \otimes A) = 0$.

**Proof.** We prove this for $F$ being covariant. The contravariant case is very similar. By Lemmas A.1.3 and A.2.1

\[ F(\alpha_0 \otimes 1_A) + F(\beta \otimes 1_A) = F(\alpha_1 \otimes 1_A) + F(\beta \otimes 1_A) \]

where $(\alpha_i + \beta) \otimes 1_A : T \otimes A \rightarrow T \otimes A$. Hence $F(\alpha_0 \otimes 1_A) = F(\alpha_1 \otimes 1_A)$. Since the exact sequence (6) splits, so does the exact sequence

\[ 0 \rightarrow K \otimes T \otimes A \xrightarrow{\iota \otimes 1_A} T \otimes A \rightarrow T \otimes A \rightarrow 0 \]

and thus the morphism $F(\iota \otimes 1_A)$ is monic. Let $\varepsilon : T \rightarrow C_{1T} \subseteq T$ be the $*$-homomorphism given by $\varepsilon(x) = \pi(x)(1)_{1T}$ and note that $\ker \varepsilon = T_0$. Define $\alpha'_0 : T \rightarrow K \otimes T$ for $i = 0, 1$ by

\[ \alpha'_0(x) = e_{11} \otimes x, \quad \alpha'_1(x) = e_{11} \otimes \varepsilon(x). \]

Then $(\iota \otimes 1_A)(\alpha'_i \otimes 1_A) = \alpha_i \otimes 1_A$ and since $F(\iota \otimes 1_A)$ is monic, $F(\alpha'_0 \otimes 1_A) = F(\alpha'_1 \otimes 1_A)$. But $F(\alpha'_0 \otimes 1_A)$ is an isomorphism by stability and since the kernel of $\alpha'_1 \otimes 1_A$ is $T_0 \otimes A$, the inclusion $T_0 \otimes A \rightarrow T \otimes A$ vanishes when applying $F$. Moreover, since the extension of $C^*$-algebras over $X$

\[ 0 \rightarrow T_0 \otimes A \rightarrow T \otimes A \rightarrow C \otimes A \rightarrow 0 \]

is split, the morphism $F(T_0 \otimes A) \rightarrow F(T \otimes A)$ is monic. Hence $F(T_0 \otimes A) = 0$. \qed

**Theorem A.2.3** (Bott Periodicity). For any object $A$ in $\mathcal{C}$, $F(A) \cong F(\Sigma^2 A)$ naturally.

**Proof.** Again we show this for $F$ covariant. The contravariant case is similar. We know that

\[ 0 \rightarrow K \otimes A \rightarrow T_0 \otimes A \rightarrow C_0(\mathbb{R}) \otimes A \rightarrow 0 \]
is an extension of \( C^* \)-algebras over \( X \). Hence we get the induced long exact sequence

\[ \cdots \to F(\Sigma T_0 \otimes A) \to F(\Sigma C_0(\mathbb{R}) \otimes A) \to F(\mathbb{K} \otimes A) \to F(T_0 \otimes A) \to \cdots . \]

By Theorem A.2.2 both \( F(\Sigma T_0 \otimes A) \) and \( F(T_0 \otimes A) \) are zero and thus

\[ F(\Sigma^2 A) \cong F(\mathbb{K} \otimes A) \cong F(A). \]

This isomorphism is natural since the long exact sequence and stability is natural.

**Corollary A.2.4.** Given an extension \( 0 \to J \to A \to B \to 0 \) in \( \mathcal{C}^* \) we get the exact diagram

\[
\begin{array}{ccc}
F(J) & \longrightarrow & F(A) \longrightarrow F(B) \\
\uparrow & & \downarrow \\
F(\Sigma B) & \leftarrow & F(\Sigma A) \leftarrow F(\Sigma J)
\end{array}
\]

if \( F \) is covariant. If \( F \) is contravariant we get the exact diagram

\[
\begin{array}{ccc}
F(J) & \leftarrow & F(A) \leftarrow F(B) \\
\downarrow & & \uparrow \\
F(\Sigma B) & \longrightarrow & F(\Sigma A) \longrightarrow F(\Sigma J).
\end{array}
\]

**Proof.** This follows from Bott Periodicity and the fact that the induced long exact sequence is natural. \( \square \)
B Triangulated Categories

In this section we define triangulated categories and state a few basic theorems which we will apply. Note that our version of the octahedral axiom is different, yet equivalent, to those of [21].

Definition B.0.5. A pre-triangulated category is an additive category $\mathcal{T}$ with an additive automorphism $\Sigma : \mathcal{T} \to \mathcal{T}$, called the suspension (or translation) functor, and a class of diagrams of the form $A \to B \to C \to \Sigma A$ called the exact triangles, satisfying the following axioms:

(TR0) Any diagram isomorphic to an exact triangle is exact, i.e. if the diagram

\[
\begin{array}{c}
A & \to & B & \to & C & \to & \Sigma A \\
| & & \downarrow f & & \downarrow \Sigma f & & \\
A' & \to & B' & \to & C' & \to & \Sigma A'
\end{array}
\]

commutes, the vertical maps are isomorphisms and one row is an exact triangle, then so is the other row. Moreover, the diagram

\[
\begin{array}{c}
A & \to & A & \to & 0 & \to & \Sigma A
\end{array}
\]

is an exact triangle.

(TR1) For any morphism $f : A \to B$ in $\mathcal{T}$ there is an exact triangle of the form

\[
\begin{array}{c}
A & \overset{f}{\to} & B & \to & C & \to & \Sigma A
\end{array}
\]

(TR2) If one of the following diagrams

\[
\begin{array}{c}
A & \overset{u}{\to} & B & \overset{v}{\to} & C & \overset{w}{\to} & \Sigma A
\end{array}
\]

and

\[
\begin{array}{c}
B & \overset{v}{\to} & C & \overset{w}{\to} & \Sigma A & \overset{-\Sigma u}{\to} & \Sigma B
\end{array}
\]

is an exact triangle, then so is the other.

(TR3) For any commutative diagram as the following solid diagram

\[
\begin{array}{c}
A & \overset{f}{\to} & B & \overset{g}{\to} & C & \overset{h}{\to} & \Sigma A \\
| & & \downarrow & & \downarrow & & \downarrow \\
A' & \overset{f'}{\to} & B' & \overset{g'}{\to} & C' & \overset{h'}{\to} & \Sigma A'
\end{array}
\]

where the rows are exact triangles, there exists a (not necessarily unique) morphism $C \to C'$ making the above diagram commute.
Definition B.0.6. Let $\mathcal{T}$ be a pre-triangulated category and $\mathcal{C}$ be an abelian category. A covariant functor $F : \mathcal{T} \to \mathcal{C}$ is said to be homological if every exact triangle $A \to B \to C \to \Sigma A$ in $\mathcal{T}$ is mapped to an exact sequence $F(A) \to F(B) \to F(C)$. A contravariant functor $F : \mathcal{T}^{\text{op}} \to \mathcal{C}$ is said to be cohomological on $\mathcal{T}$ if it is homological on the triangulated category $\mathcal{T}^{\text{op}}$.

Note that by (TR2), a homological functor $F : \mathcal{T} \to \mathcal{C}$ actually induces a natural long exact sequence

$$\cdots \to F_{n+1}(C) \to F_n(A) \to F_n(B) \to F_n(C) \to F_{n-1}(A) \to \cdots$$

when applying it to an exact triangle $A \to B \to C \to \Sigma A$. Here $F_n(A) := F(\Sigma^{-n}A)$ for $n \in \mathbb{Z}$. Similarly, if $F : \mathcal{T}^{\text{op}} \to \mathcal{C}$ is cohomological we get a natural long exact sequence

$$\cdots \leftarrow F^{n+1}(C) \leftarrow F_n(A) \leftarrow F_n(B) \leftarrow F_n(C) \leftarrow F_{n-1}(A) \leftarrow \cdots$$

where $F^n(A) := F(\Sigma^{-n}A)$.

The following result is Lemma 1.1.10 and Remark 1.1.11 of [21].

Proposition B.0.7. Let $\mathcal{T}$ be a pre-triangulated category. The functors

$$\mathcal{T}(A, -) : \mathcal{T} \to \mathsf{Ab}, \quad \mathcal{T}(-, A) : \mathcal{T}^{\text{op}} \to \mathsf{Ab}$$

are respectively homological and cohomological for every $A \in \mathsf{Ob}(\mathcal{T})$.

We will be calling the following theorem the five lemma in (pre-)triangulated categories, due its resemblance of the usual five lemma of homological algebra. It is a special case of Proposition 1.1.20 of [21].

Theorem B.0.8 (Five Lemma in (Pre-)Triangulated Categories). Consider the diagram from (TR3). If the solid vertical arrows are isomorphisms, then so is the arrow $C \to C'$.

Lemma B.0.9. Let $A$ be a retract of $B$, i.e. there exist morphisms $A \to B \to A$ for which the composite is the identity, and construct an exact triangle $A \to B \to C \to \Sigma A$ by (TR1). Then $B \cong A \oplus C$.

Proof. By (TR0) $D \overset{1}{\to} D \to 0 \to \Sigma D$ is exact triangle for $D = A, C$. By (TR2) and Proposition 1.2.1 of [21], we get an exact triangle $A \to A \oplus C \to C \to \Sigma A$. Applying (TR3) and the five lemma for pre-triangulated categories yields the result. □

Definition B.0.10. A pre-triangulated category $\mathcal{T}$ is said to be triangulated if it satisfies the octahedral axiom:
(TR4) If \( f : A \to B \) and \( g : B \to B' \) are morphisms then there is a commutative diagram

\[
\begin{array}{cccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{\Sigma f} & \Sigma A \\
\downarrow{g} & & \downarrow{\Sigma f} & & \downarrow{} & & \downarrow{} \\
A & \xrightarrow{gf} & B' & \xrightarrow{h} & C' & \xrightarrow{\Sigma f} & \Sigma A \\
\downarrow{h} & & \downarrow{} & & \downarrow{} & & \downarrow{} \\
B'' & \xrightarrow{h} & B'' & \xrightarrow{\Sigma f} & \Sigma B & & \\
\downarrow{} & & \downarrow{} & & \downarrow{} & & \downarrow{} \\
\Sigma B & \to & \Sigma C & & & & \\
\end{array}
\]

where the two top rows and the two middle columns are exact triangles.

Remark B.0.11. The definition of the octahedral axiom given above is different than that of [21] but however equivalent. To see this use that (TR4) of [21] is equivalent to (TR4) of [14], which is proven in the appendix of [14]. By the five lemma in pre-triangulated categories, we easily get that (TR4) of [14] is equivalent to our definition of the octahedral axiom.
C Notation

In the thesis we use the following notation

\( X \) topological space, often assumed to be second countable (or finite) and sober.

\( \mathcal{O}(X) \) the complete lattice of all open subsets of \( X \), equipped with the partial order \( \subseteq \) of inclusion.

\( \mathbb{L}\mathcal{C}(X) \) the set of all locally closed subsets of \( X \).

\( \mathbb{L}\mathcal{C}(X)^* \) the set of all non-empty connected locally closed subsets of \( X \).

\( A \) often a (separable) \( C^* \)-algebra (over \( X \)). Sometimes this denotes some object in a category.

\( \text{Prim}(A) \) the primitive ideal space of \( A \) equipped with the Jacobsen topology.

\( \mathbb{I}(A) \) the complete lattice of all (closed two-sided) ideals, equipped with the partial order \( \subseteq \) of inclusion.

*All ideals are assumed to be closed and two-sided.*

\( \mathcal{C}^\ast\text{alg}(X) \) the category with objects being \( C^* \)-algebras over \( X \), and morphisms being all \( X \)-equivariant \( \ast \)-homomorphisms.

\( \mathcal{C}^\ast\text{sep}(X) \) the full subcategory of \( \mathcal{C}^\ast\text{alg}(X) \) consisting of all separable \( C^* \)-algebras over \( X \).

\( \mathcal{E}(X) \) the \( E \)-theory category with objects being all separable \( C^* \)-algebras over \( X \), and morphism sets being \( E_0(X;A,B) \).

\( \mathfrak{A}\text{b} \) the category of abelian groups.

\( \mathfrak{A}\text{b}_{Z/2} \) the category of \( Z/2 \)-graded abelian groups.

\( \mathfrak{A}\text{b}_{Z/2}^0 \) the category of \( Z/2 \)-graded abelian groups and evenly graded morphisms.

\( i_X^Y \) the extension functor \( \mathcal{C}^\ast\text{alg}(Y) \rightarrow \mathcal{C}^\ast\text{alg}(X) \) or \( \mathcal{E}(Y) \rightarrow \mathcal{E}(X) \) for a subspace \( Y \subseteq X \).

\( i_x \) abbreviation of \( i^X_{\{x\}} \) for \( x \in X \).

\( r_X^Y \) the restriction functor \( \mathcal{C}^\ast\text{alg}(Y) \rightarrow \mathcal{C}^\ast\text{alg}(X) \) or \( \mathcal{E}(Y) \rightarrow \mathcal{E}(X) \) for \( Y \in \mathbb{L}\mathcal{C}(X) \).

\( \text{ev}_Y \) the evaluation functor \( \mathcal{C}^\ast\text{alg}(X) \rightarrow \mathcal{C}^\ast\text{alg} \) or \( \mathcal{E}(X) \rightarrow \mathcal{E} \) for \( Y \in \mathbb{L}\mathcal{C}(X) \).

\( \mathcal{F}_S \) the filtration functor \( \mathcal{C}^\ast\text{alg}(X) \rightarrow \mathcal{C}^\ast\text{alg} \) or \( \mathcal{E}(X) \rightarrow \mathcal{E} \) for a subset \( S \) of \( X \).

\( \mathbb{K} \) the compact operators.

\( \Sigma \) suspension \( \Sigma B := C_0(\mathbb{R}, A) \). In Section 5 this may define the suspension functor in a triangulated category or in a stable additive category.

\( C^\ast \) cone \( CA := C_0([0,1), A) \).

\( I \) cylinder \( IA := C([0,1], A) \).
\[ C_\pi \text{ mapping cone, the pull-back } C_\pi := A \oplus_B CB \text{ where } \pi : A \to B \text{ is an } X\text{-equivariant } \ast\text{-homomorphism.} \]

\[ Z_\pi \text{ mapping cylinder, the pull-back } Z_\pi := A \oplus_B IB \text{ where } \pi : A \to B \text{ is an } X\text{-equivariant } \ast\text{-homomorphism.} \]

\[ T \text{ the interval } T := [0, \infty). \]

\[ \mathcal{A}_\infty \text{ the quotient } C_b(T, A)/C_0(T, A). \]

\[ [[A, B]]_X \text{ the set of homotopy classes of approximately } X\text{-equivariant asymptotic morphisms from } A \text{ to } B. \]

\[ [[\phi]] \text{ homotopy class of the asymptotic morphism } \phi. \]

\[ [\pi] \text{ homotopy class of the constant asymptotic morphism } \Sigma\pi \otimes 1_K, \text{ where } \pi \text{ is an } X\text{-equivariant } \ast\text{-homomorphism.} \]

\[ E_0(X; A, B) \text{ the } E\text{-group with underlying set equal to } [[\Sigma A \otimes K, \Sigma B \otimes K]]_X. \]

\[ \mathcal{R}_Y \text{ a } C^\ast\text{-algebra over } X \text{ such that } E_*(X; \mathcal{R}_Y, A) \cong K_*(A(Y)) \text{ naturally for } Y \in \mathcal{L}\mathcal{C}(X). \]

\[ \pi_k \text{ the } k\text{'th homotopy group.} \]

\[ \Sigma \text{ a triangulated category} \]

\[ \Sigma(A, B) \text{ morphism set in } \Sigma. \]

\[ \mathcal{J} \text{ an ideal in } \Sigma, \text{ often assumed to be homological.} \]

\[ \mathcal{C} \text{ most often denotes a stable abelian category.} \]

\[ \mathcal{H} \text{ homotopy category of chain complexes in a given additive category.} \]

\[ \mathcal{C} \text{ homotopy category of cochain complexes in a given additive category.} \]

\[ \mathbb{L}_nF \text{ the } n\text{'th left derived functor of } F, \text{ } F \text{ being covariant} \]

\[ \mathbb{R}^nF \text{ the } n\text{'th right derived functor of } F, \text{ } F \text{ being contravariant} \]

\[ \mathbb{E}xt_{\Sigma, \mathbb{C}}(-, B) \text{ the } n\text{'th right derived functor of } \Sigma(-, B). \]

\[ F^\dagger \text{ partially defined functor for which } \Sigma(F^\dagger(A), B) \cong \mathcal{C}(A, F(B)). \]

\[ \mathcal{N}\mathcal{T} \text{ small category with object set } \mathbb{L}\mathcal{C}(X)^* \text{ and } \mathbb{Z}/2\text{-graded morphism groups } \mathcal{N}\mathcal{T}_*(Y, Z) = E_*(X; \mathcal{R}_Z, \mathcal{R}_Y). \]

\[ \mathcal{M}\mathcal{O}\mathcal{D}(\mathcal{N}\mathcal{T})_c \text{ category of countable } \mathcal{N}\mathcal{T}\text{-modules.} \]

\[ \mathcal{F}\mathcal{K} \text{ the filtrated } K\text{-theory functor.} \]

\[ P_Y \text{ free } \mathcal{N}\mathcal{T}\text{-module given by } \mathcal{N}\mathcal{T}_*(Y, -) \text{ for } Y \in \mathcal{L}\mathcal{C}(X)^*. \]

\[ \mathcal{B}_E(X) \text{ the } E\text{-theoretic bootstrap class.} \]
References


