

A Digitalized Employee Option

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This paper introduces a new employee option plan for low- and mid-level employees. The option plan is a possibility for the employee to exercise an accrual account at any time before maturity contingent on the stock price being above a strike level. The option is American in nature and is a combination of an accrual account and a digital option with a strike price. The optimal stopping problem associated with the valuation of the option has a discontinuous payoff. The key argument to solve the stopping problem is the principle of smooth fit at a single point. The optimal exercise strategy is displayed and the valuation formula is derived.

1. Introduction

For the last decade, many corporate employee incentive programs in various shapes have been introduced. Among the most popular incentive programs are programs involving plain vanilla options written with the company's stock price as the underlying. The incentives structure of these programs is obvious – the more the stock increases, the more the value of the option increases. However, a general rising market will often translate into an increased reward for the option owners in the company, even when the company has under performed. This has motivated the development of other incentive programs. For instance, Johnson & Titan [5] developed an option plan with indexed strike levels, (see also Jørgensen [6]). Since low-level employees may have limited influence on the company's stock price, "normal" plain-vanilla options might not be the right incentive program for these employees. In this paper, a new employee option is introduced that addresses this problem. The employee option plan is well suited for low- and mid-level employees. A key idea is that the option plan is sufficiently simple that the individual employee feels that he can manage it in a much more efficient way compared to the plain vanilla options. The option plan consists of two parts. One part is an employee account – an accrual account with an employee benefit rate associated as the accrual rate – and the other is an option part measuring the performance of the company. The payoff is constructed as follows: In the time window from the date of grant until the date of maturity, the option can be exercised. Thus, the option is an American type option as most employee options are (see Huddart [4]). If the stock price is above a strike level and the employee exercises the option, the employee achieves the employee account at the exercise time. The cost of the option, which the firm account for, is derived using the framework of Black & Scholes [1] for option valuation. Other pricing models available (see Huddart [4] and Carpenter [3]) that take into account the impact of special features of employee options which differ from stock options (see Rubinstein [13]).

The employee option has the property that the valuation formulae are simple and analytically tractable in contrast to an American put option (see Kim [8]). Furthermore, the method to derive the valuation formulae is also different due to the discontinuous payoff and breakdown of smooth fit. Another property of the option is that it might not be optimal to exercise it the first time the stock price exceeds the strike level. If the difference between the employee benefit

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rate and the interest rate is sufficient large, it might be optimal for the employee to keep the option alive longer, even though the stock price has reached the strike level.

The remainder of this paper is constructed as follows. The next section establishes the solution to the optimal stopping problem associated with the valuation of the option. Section 3 provides facts and formulae used in the derivation of the solution of previous section. Finally, Section 4 concludes on the valuation formula of the employee option.

2. The optimal stopping problem

Pricing the employee option reduces to solve a finite-horizon optimal stopping problem. In this section, the solution is established to this stopping problem. Some results on the involved functions are postponed to the next section.

Let $T > 0$ be the terminal date and let $(W_t)_{0 \leq t \leq T}$ be a standard Brownian motion started at zero. For fixed $\lambda > 0$ and $b \in \mathbb{R}$, introduce for $0 \leq t \leq T$ and $x \in \mathbb{R}$ the optimal stopping problem

$$(2.1) \quad V_*(t, x) = \sup_{t+\tau \leq T} \mathbf{E}_{t,x}(e^{\lambda(t+\tau)}; X_{t+\tau} \geq b)$$

where τ is a stopping time of the Brownian motion with drift $(X_{t+s})_{t \leq t+s \leq T}$ given by $X_{t+s} = x + W_s + \mu s$ where $\mu \in \mathbb{R}$ and with $X_t = x$ under $\mathbf{P}_{t,x}$. The problem is to compute the value function $V_*(t, x)$ and to find an optimal stopping time τ_* , that is, a stopping time for which the supremum is attained.

First, the structure of the optimal strategy is studied. The form of the gain function $g(t, x) = e^{\lambda t} \mathbf{1}_{[b, \infty)}(x)$ implies that it is only optimal stop on the boundary b , that is, $X_{t+\tau_*} = b$ on $\{t + \tau_* < T\}$. The conclusion to draw is that the stopping region is given by

$$D = \{(t, x) \in [0, T] \times \mathbb{R} \mid V_*(t, x) = g(t, x)\} = D^{(T)} \times \{b\}$$

where $D^{(T)} = \{t \in [0, T] \mid V_*(t, b) = e^{\lambda t}\}$. This fact yields that the principle of smooth fit – the value function is differentiable at all points of the boundary of the continuation region (see Shiryaev [14]) – is not satisfied in the usual sense. This is due to the discontinuous gain function. The set $D^{(T)}$ does not depend on b since a Brownian motion with drift is linear in the state variable x . The general theory of optimal stopping provides the following representation of an optimal strategy

$$\tau_* = \inf\{t < t + s \leq T : X_{t+s} \in D\} = \inf\{t < t + s \leq T : t + s \in D^{(T)} \text{ and } X_{t+s} = b\}.$$

Next, the set $D^{(T)}$ is studied in greater details. Consider the stopping problem (2.1) for only deterministic times $\tau = s$ and $x = b$, that is,

$$v(t) = \sup_{t+s \leq T} \mathbf{E}_{t,b}(g(t+s, X_{t+s})) = \sup_{t+s \leq T} e^{\lambda(t+s)} \mathbf{P}(W_s + \mu s \geq 0).$$

Define the set $I^{(T)} = \{t \in [0, T] \mid v(t) = e^{\lambda t}\} = \{t \in [0, T] \mid \sup_{t+s \leq T} e^{\lambda s} \mathbf{P}(W_s + \mu s \geq 0) = 1\}$ and elementary calculations give that $I^{(T)} = [T - \Delta T_d, T] \cap [0, T]$ where $0 < \Delta T_d \leq \infty$ is the unique root to the equation $\Phi(\mu \sqrt{\Delta T_d}) = \exp(-\lambda \Delta T_d)$. Note that $\Delta T_d = \infty$ happens only if $\mu < 0$ satisfying $\mu^2 \geq 2\lambda$. First observation to be made is that $D^{(T)} \subseteq I^{(T)}$. The second observation to be made is that there exists $\varepsilon > 0$ such that $[T - \varepsilon, T] \subseteq D^{(T)}$. Else there is a small interval $[T - \varepsilon, T] \cap D^{(T)} = \emptyset$ where $\tau_* = T - t$ is optimal for $x = b$ and any $T - \varepsilon \leq t \leq T$. But this is a contradiction due to the form of $I^{(T)}$.

Heuristic, these observations prompt that $D^{(T)}$ might be an interval of the form $[T_*, T]$ where T_* is to be found. The principle of smooth fit at a single point – the value function should be differentiable at the point (T_*, b) (see Pedersen [10]) – provides a method to determine T_* .

To do this, determine T_* , define the function

$$V_b(t, x) = \mathbf{E}_{t,x}(g(t + \tau_b \wedge (T - t), X_{t+\tau_b \wedge (T-t)})) = \mathbf{E}_{t,x}(e^{\lambda(t+\tau_b \wedge (T-t))}; X_{t+\tau_b \wedge (T-t)} \geq b)$$

where $\tau_b = \inf\{t + s > t : X_{t+s} = b\}$. By the fact that the Brownian motion with drift is linear in the state variable x it follows that $V_b(t, x) = e^{\lambda t} f(b - x, T - t)$ where f is given in equation (3.2). By Proposition 3.2, $t \mapsto f(t, b)$ is not differentiable at $b = 0$ for any $t > 0$ if $\mu < 0$ satisfying $\mu^2 \geq \lambda$. In this case set $\Delta T_* = \infty$. Moreover, in the other cases, let $\Delta T_* > 0$ be the root of the equation

$$(2.2) \quad \mu \sqrt{\Delta T_*} \Phi(\mu \sqrt{\Delta T_*}) + \varphi(\mu \sqrt{\Delta T_*}) = \Theta(\Delta T_*)$$

from Proposition 3.2. Therefore, $b \mapsto f(\Delta T_*, b)$ is differentiable at $b = 0$. Set

$$(2.3) \quad T_* = (T - \Delta T_*)^+.$$

Then T_* is defined such that $x \mapsto V_b(T_*, x)$ is differentiable at $x = b$ if $T_* > 0$. The function $x \mapsto V_b(t, x)$ is only continuous (not differentiable) at $x = b$ for $t \neq T_*$. Let $t < T$ and x be given and define the stopping time

$$\tau_\bullet = \inf\{(T_* \vee t) < t + s \leq T : X_{t+s} = b\} \quad (\inf \emptyset = T - t)$$

and the associated function $V_\bullet(t, x) = \mathbf{E}_{t,x}(g(t + \tau_\bullet, X_{t+\tau_\bullet}))$. By construction $V_\bullet(t, x) = V_b(t, x)$ for $t \geq T_*$ and direct calculations give for $t < T_*$ that

$$V_\bullet(t, x) = \int_{T_*}^T e^{\lambda u} h_{b-x}^{(T_*-t)}(u-t) du + e^{\lambda T} \left(\Phi\left(\frac{x-b+\mu(T_*-t)}{\sqrt{T_*-t}}\right) - \int_{T_*}^T \bar{h}_{b-x}^{(T_*-t)}(u-t) du \right),$$

where the densities $h_{b-x}^{(T_*-t)}(\cdot)$ and $\bar{h}_{b-x}^{(T_*-t)}(\cdot)$ are given in Proposition 3.3. It is immediate to check that $V_\bullet(t, x)$ is differentiable (smooth) at the point $(t, x) = (T_*, b)$ and hence satisfies the principle of smooth fit at a single point. The proposed solution is stated in the following theorem.

Theorem 2.1. *Consider the optimal stopping problem (2.1). Let $t \leq T$ and $x \in \mathbb{R}$ be given and fixed. Let T_* be defined as in equation (2.3). Then the optimal stopping strategy is given by*

$$\tau_* = \inf\{(T_* \vee t) < t + s \leq T : X_{t+s} = b\} \quad (\inf \emptyset = T - t).$$

The associated value function $V_*(t, x) = \mathbf{E}_{t,x}(e^{\lambda(t+\tau_*)}; X_{\tau_*} \geq b)$ is given by

$$V_*(t, x) = \begin{cases} L_{b-x}^{(\lambda)}(T-t) + e^{\lambda(T-t)} H_{b-x}(T-t) \mathbf{1}_{(-\infty, 0)}(b) & \text{if } t \geq T_* \\ \int_{T_*}^T e^{\lambda u} h_{b-x}^{(T_*-t)}(u) du + e^{\lambda T} \left(\Phi\left(\frac{x-b+\mu(T_*-t)}{\sqrt{T_*-t}}\right) - \int_{T_*}^T \bar{h}_{b-x}^{(T_*-t)}(u-t) du \right) & \text{if } t < T_* \end{cases}$$

where $\Phi(\cdot)$ is the normal distribution function, $L_b^{(\lambda)}(\cdot)$ is given in Proposition 3.1, $H_b(\cdot)$ is given in equation (3.1) and, $h_b^{(T_*)}(\cdot)$ and $\bar{h}_b^{(T_*)}(\cdot)$ are given in Proposition 3.3.

Proof. The aim is to show that the value function $V_*(t, x)$ of stopping problem (2.1) is equal to the proposed solution $V_\bullet(t, x)$. By the theory of Markov processes, the two functions $V_b(t, x)$ and $V_\bullet(t, x)$ solve the following partial differential equation

$$\frac{\partial V}{\partial t}(t, x) + \mu \frac{\partial V}{\partial x}(t, x) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}(t, x) = 0$$

for $t < T$ and $x \neq b$. This fact will be used below in the proof.

First step is to verify that V_\bullet is a majorant of the gain function g . It is enough to verify that $V_\bullet(t, x) \geq V_b(t, x)$ since $V_b(t, x) \geq g(t, x)$. If $t \geq T_*$, then due to the definitions of the functions it follows that $V_\bullet(t, x) = V_b(t, x)$. If $t < T_*$ (and hence $T_* > 0$), then $V(t, x) = V_\bullet(t, x) - V_b(t, x)$ is a bounded and continuous function such that $V(t, x)$ solves the above partial equation for $(t, x) \notin [0, T_*] \times \{b\}$. For $t + s < T$, then by an extended Itô formula, see Peskir [11], yields that

$$\begin{aligned} V(t + s, X_{t+s}) &= V(t, x) + M_s + \int_0^s \left(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \right)(t + u, X_{t+u}) \mathbf{1}_{\{X_{t+u} \neq b\}} du \\ &\quad + \frac{1}{2} \int_0^s \left(\frac{\partial V}{\partial x}(t + u, b+) - \frac{\partial V}{\partial x}(t + u, b-) \right) d\ell_u^b \\ &= V(t, x) + M_s - \frac{1}{2} \int_0^{(T_*-t) \wedge s} \left(\frac{\partial V_b}{\partial x}(t + u, b+) - \frac{\partial V_b}{\partial x}(t + u, b-) \right) d\ell_u^b \end{aligned}$$

where ℓ_s^b is the local time at b of (X_{t+s}) and $M_s = \int_0^s (\partial V / \partial x)(t + u, X_{t+u}) \mathbf{1}_{\{X_{t+u} \neq b\}} dW_u$ is a local martingale. From the proof of Proposition 3.2 it follows that $(\partial V_b / \partial x)(t, b+) > (\partial V_b / \partial x)(t, b-)$ for $t < T_*$ and hence M_s is bounded from below and therefore is a supermartingale. Pick $s > 0$ such that $t + s > T_*$, then $0 = V(t + s, X_{t+s}) \leq V(t, x) + M_s$ which implies $V_\bullet(t, x) \geq V_b(t, x)$ by taking expectation.

Next step is to verify that $(V_\bullet(t + s, X_{t+s}))_{t \leq t+s \leq T}$ is a supermartingale. Using the extended Itô formula as above for $t + s < T$ gives that

$$\begin{aligned} V_\bullet(t + s, X_{t+s}) &= V_\bullet(t, x) + M_s + \int_0^s \left(\frac{\partial V_\bullet}{\partial t} + \mu \frac{\partial V_\bullet}{\partial x} + \frac{1}{2} \frac{\partial^2 V_\bullet}{\partial x^2} \right)(t + u, X_{t+u}) \mathbf{1}_{\{X_{t+u} \neq b\}} du \\ &\quad + \frac{1}{2} \int_0^s \left(\frac{\partial V_\bullet}{\partial x}(t + u, b+) - \frac{\partial V_\bullet}{\partial x}(t + u, b-) \right) d\ell_u^b \\ &= V_\bullet(t, x) + M_s + \frac{1}{2} \int_{(T_*-t) \vee 0}^{(T_*-t) \vee s} \left(\frac{\partial V_b}{\partial x}(t + u, b+) - \frac{\partial V_b}{\partial x}(t + u, b-) \right) d\ell_u^b \end{aligned}$$

where M_s is a supermartingale (by similiary arguments as above). From the proof of Proposition 3.2 it follows that $(\partial V_b / \partial x)(t, b+) < (\partial V_b / \partial x)(t, b-)$ for $t > T_*$ and hence

$$\int_{(T_*-t) \vee 0}^{(T_*-t) \vee s} \left(\frac{\partial V_b}{\partial x}(t + u, b+) - \frac{\partial V_b}{\partial x}(t + u, b-) \right) d\ell_u^b \geq \int_{(T_*-t) \vee 0}^{(T_*-t) \vee v} \left(\frac{\partial V_b}{\partial x}(t + u, b+) - \frac{\partial V_b}{\partial x}(t + u, b-) \right) d\ell_u^b$$

for $t + s < t + v < T$. The inequality implies that $(V_\bullet(t + s, X_{t+s}))_{t \leq t+s \leq T}$ is a supermartingale.

Finally, let $t < T$ and $x \in \mathbb{R}$ be given and fixed. The supermartingale property and that V_\bullet is a majorant of g yield that $V_\bullet(t, x) \geq \mathbf{E}_{t,x}(V_\bullet(t + \tau, X_{t+\tau})) \geq \mathbf{E}_{t,x}(g(t + \tau, X_{t+\tau}))$ for any stopping time τ satisfying $t + \tau \leq T$. Therefore, $V_\bullet(t, x) \geq \sup_{t+\tau \leq T} \mathbf{E}_{t,x}(g(t + \tau, X_{t+\tau})) = V_*(t, x)$. It is clear that $V_\bullet(T, x) = g(T, x) = V_*(T, x)$. Therefore, V_\bullet is the value function of the optimal stopping problem (2.1) and the strategy τ_\bullet is optimal. \square

3. Facts and formulae

The aim of this section is to derive formulae used to solve the optimal stopping problem in the previous section. For some notation, let

$$\varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2} \quad \text{and} \quad \Phi(y) = \int_{-\infty}^y \varphi(u) du \quad (y \in \mathbb{R})$$

denote the density and the distribution function of a standard normal variable, respectively. The function $\Phi(\cdot)$ is analytic and the complex function $z \mapsto \Phi(z)$ for $z \in \mathbb{C}$ is well defined. Set $z = y - ia$ for $y, a \in \mathbb{R}$ then

$$\operatorname{Re}(\Phi(z)) = e^{a^2/2}\Phi_{\cos}(a; y) \quad \text{and} \quad \operatorname{Im}(\Phi(z)) = e^{a^2/2}\Phi_{\sin}(a; y)$$

where

$$\Phi_{\cos}(a; y) = \int_{-\infty}^y \cos(au)\varphi(u) du \quad \text{and} \quad \Phi_{\sin}(a; y) = \int_{-\infty}^y \sin(au)\varphi(u) du.$$

Let $(X_t)_{t \geq 0}$ denote a Brownian motion with drift given by $X_t = W_t + \mu t$ where $\mu \in \mathbb{R}$ and $(W_t)_{t \geq 0}$ is a Brownian motion. Let

$$\tau_b = \inf\{s > 0 : X_s = b\}$$

be the first hitting time to the level $b \in \mathbb{R}$. The density $\mathbf{P}(\tau_b \in dt) = h_b(t) dt$ and the distribution function $\mathbf{P}(\tau_b \leq t) = 1 - H_b(t)$ are for $t > 0$ given by (see Borodin & Salminen [2]),

$$h_b(t) = \frac{|b|}{t^{3/2}}\varphi\left(\frac{b - \mu t}{\sqrt{t}}\right)$$

$$(3.1) \quad H_b(t) = \Phi\left(\frac{|b| - \operatorname{sgn}(b)\mu t}{\sqrt{t}}\right) - e^{2b\mu}\Phi\left(\frac{-|b| - \operatorname{sgn}(b)\mu t}{\sqrt{t}}\right).$$

If $\operatorname{sgn}(b) \neq \operatorname{sgn}(\mu)$ then $\mathbf{P}(\tau_b = \infty) = 1 - e^{2b\mu}$. For $t > 0$, define the function

$$L_b^{(\lambda)}(t) = \mathbf{E}(e^{\lambda\tau_b}; \tau_b < t)$$

where $\lambda \in \mathbb{R}$ is a constant. The function $L_b^{(\lambda)}(t)$ is key to defining the other primary functions.

Proposition 3.1. *Let $t > 0$ be given and fixed. Then for $\mu^2 \geq 2\lambda$,*

$$L_b^{(\lambda)}(t) = e^{b\mu} \left(e^{b\sqrt{\mu^2 - 2\lambda}} \Phi\left(\frac{-\operatorname{sgn}(b)\sqrt{\mu^2 - 2\lambda}t - |b|}{\sqrt{t}}\right) + e^{-b\sqrt{\mu^2 - 2\lambda}} \Phi\left(\frac{\operatorname{sgn}(b)\sqrt{\mu^2 - 2\lambda}t - |b|}{\sqrt{t}}\right) \right)$$

and for $\mu^2 < 2\lambda$,

$$L_b^{(\lambda)}(t) = 2e^{b\mu + (\lambda - \mu^2/2)t} \left(\cos(b\sqrt{2\lambda - \mu^2}) \Phi_{\cos}(\sqrt{(2\lambda - \mu^2)t}; -|b|/\sqrt{t}) \right. \\ \left. - \sin(|b|\sqrt{2\lambda - \mu^2}) \Phi_{\sin}(\sqrt{(2\lambda - \mu^2)t}; -|b|/\sqrt{t}) \right).$$

Proof. Consider the case of a standard Brownian motion, that is, $\mu = 0$. Denote by $\sigma_b = \inf\{s > 0 : W_s = b\}$, the aim is to compute

$$\mathbf{E}(e^{\lambda\sigma_b}; \sigma_b < t) = \int_0^t e^{\lambda u} \frac{|b|}{u^{3/2}} \varphi\left(\frac{b}{\sqrt{u}}\right) du.$$

It is well known, e.g. pricing digital option that

$$\mathbf{E}(e^{\lambda\sigma_b}; \sigma_b < t) = e^{b\sqrt{-2\lambda}}\Phi\left(\frac{-|b| - \operatorname{sgn}(b)\sqrt{-2\lambda}t}{\sqrt{t}}\right) + e^{-b\sqrt{-2\lambda}}\Phi\left(\frac{-|b| + \operatorname{sgn}(b)\sqrt{-2\lambda}t}{\sqrt{t}}\right)$$

for $\lambda \leq 0$. An analytic continuation argument gives for $\lambda > 0$ that

$$\mathbf{E}(e^{\lambda\sigma_b}; \sigma_b < t) = e^{ib\sqrt{2\lambda}}\Phi\left(\frac{-|b| - \operatorname{sgn}(b)i\sqrt{2\lambda}t}{\sqrt{t}}\right) + e^{-ib\sqrt{2\lambda}}\Phi\left(\frac{-|b| + \operatorname{sgn}(b)i\sqrt{2\lambda}t}{\sqrt{t}}\right)$$

and without complex numbers the expression reads as

$$\mathbf{E}(e^{\lambda\sigma_b}; \sigma_b < t) = 2e^{\lambda t} \left(\cos(b\sqrt{2\lambda})\Phi_{\cos}(\sqrt{2\lambda}t; -|b|/\sqrt{t}) - \sin(|b|\sqrt{2\lambda})\Phi_{\sin}(\sqrt{2\lambda}t; -|b|/\sqrt{t}) \right).$$

Define the probability measure

$$d\mathbf{P}_{|\mathcal{F}_t}^{(\mu)} = \exp(-\mu W_t - (\mu^2/2)t) d\mathbf{P}_{|\mathcal{F}_t} = \exp(-\mu X_t + (\mu^2/2)t) d\mathbf{P}_{|\mathcal{F}_t}.$$

By Girsanov's theorem (see Revuz & Yor [12]), (X_t) is a Brownian motion under $\mathbf{P}^{(\mu)}$. Then

$$L_b^{(\lambda)}(t) = \mathbf{E}(e^{\lambda\tau_b}; \tau_b < t) = \mathbf{E}^{(\mu)}(e^{\mu X_{\tau_b} - (\mu^2/2)\tau_b} e^{\lambda\tau_b}; \tau_b < t) = e^{b\mu} \mathbf{E}(e^{(\lambda - \mu^2/2)\sigma_b}; \sigma_b < t)$$

and the result follows. \square

Define the function

$$(3.2) \quad f(t, b) = L_b^{(\lambda)}(t) + e^{\lambda t} H_b(t) \mathbf{1}_{(-\infty, 0)}(b) = \mathbf{E}(e^{\lambda\tau_b}; \tau_b < t) + e^{\lambda t} \mathbf{P}(\tau_b \geq t) \mathbf{1}_{(-\infty, 0)}(b).$$

For $t > 0$ fixed, the function $b \mapsto f(t, b)$ is in general only continuous at $b = 0$ and not differentiable.

Proposition 3.2. *There is at most one point t_0 where $f(t_0, b)$ is differentiable at $b = 0$. If $\mu < 0$ satisfying $\mu^2 \geq 2\lambda$, then $b \mapsto f(t, b)$ is not differentiable at $b = 0$ for any $t > 0$. Otherwise, there is exactly one point $0 < t_0 < \infty$ where $f(t_0, b)$ is differentiable at $b = 0$ which is the unique root of the equation*

$$\mu\sqrt{t}\Phi(\mu\sqrt{t}) + \varphi(\mu\sqrt{t}) = \Theta(t)$$

where

$$\Theta(t) = \begin{cases} \sqrt{(\mu^2 - 2\lambda)t} e^{-\lambda t} \left(2\Phi(\sqrt{(\mu^2 - 2\lambda)t}) - 1 \right) + 2\varphi(\mu\sqrt{t}) & \text{for } \mu^2 \geq 2\lambda \\ 2e^{-(\mu^2/2)t} \left(\sqrt{(2\lambda - \mu^2)t} \Phi_{\sin}(\sqrt{(2\lambda - \mu^2)t}; 0) + \frac{1}{\sqrt{2\pi}} \right) & \text{for } \mu^2 < 2\lambda. \end{cases}$$

Proof. Elementary calculations based on Proposition 3.1 show that

$$\frac{\partial f}{\partial b}(t, 0+) = \begin{cases} \mu - \sqrt{\mu^2 - 2\lambda} \left(2\Phi(\sqrt{(\mu^2 - 2\lambda)t}) - 1 \right) - \frac{2}{\sqrt{t}} \varphi(\sqrt{(\mu^2 - 2\lambda)t}) & \text{for } \mu^2 \geq 2\lambda \\ \mu - 2e^{(\lambda - \mu^2/2)t} \left(\Phi_{\sin}(\sqrt{(2\lambda - \mu^2)t}; 0) + \frac{1}{\sqrt{2\pi t}} \right) & \text{for } \mu^2 < 2\lambda \end{cases}$$

$$\frac{\partial f}{\partial b}(t, 0-) = 2\mu - \frac{\partial f}{\partial b}(t, 0+) - 2e^{\lambda t} \left(\mu\Phi(\mu\sqrt{t}) + \frac{1}{\sqrt{t}} \varphi(\mu\sqrt{t}) \right).$$

More calculations show that

$$\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial b}(t, 0+) \right) = \frac{e^{(\lambda - \mu^2/2)t}}{\sqrt{2\pi t^3}} > 0 \text{ and } \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial b}(t, 0-) \right) = -2\lambda e^{\lambda t} (e^{-(\mu^2/2)t} + \mu\Phi(\mu\sqrt{t})) < 0.$$

Therefore $t \mapsto (\partial f/\partial b)(t, 0+)$ is increasing and $t \mapsto (\partial f/\partial b)(t, 0-)$ is decreasing. It is straight forward that there are the following limits: $(\partial f/\partial b)(t, 0+)|_{t=0} = -\infty$, $(\partial f/\partial b)(t, 0-)|_{t=0} = 0$ and

$$\begin{aligned} \frac{\partial f}{\partial b}(t, 0+)|_{t=\infty} &= \begin{cases} \mu - \sqrt{\mu^2 - 2\lambda} & \text{for } \mu^2 \geq 2\lambda \\ \infty & \text{for } \mu^2 < 2\lambda \end{cases} \\ \frac{\partial f}{\partial b}(t, 0-)|_{t=\infty} &= \begin{cases} \mu + \sqrt{\mu^2 - 2\lambda} & \text{for } \mu < 0 \text{ satisfying } \mu^2 \geq 2\lambda \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

$f(t_0, b)$ is differentiable at $b = 0$ if $(\partial f/\partial b)(t_0, 0+) = (\partial f/\partial b)(t_0, 0-)$. Thus there is at most point where $b \mapsto f(t, b)$ is differentiable at $b = 0$. More precise, there is no point if $\mu < 0$ satisfying $\mu^2 \geq 2\lambda$. Otherwise there is a unique point t_0 which is the root in the equation given above. The equation is derived by rewriting the equation $(\partial f/\partial b)(t_0, 0+) = (\partial f/\partial b)(t_0, 0-)$. \square

The last part of this section is to study the law of the first hitting time to the level b after time $t_0 \geq 0$, that is,

$$\tau_b^{(t_0)} = t_0 + \tau_b \circ \theta_{t_0} = \inf\{s > t_0 : X_s = b\}$$

where θ_t is the shift operator (see Revuz & Yor [12]). The next proposition gives expressions for the densities of hitting the level b after time t_0 from above denoted by

$$\mathbf{P}(\tau_b^{(t_0)} \in dt; X_{t_0} > b) = \bar{h}_b^{(t_0)}(t) dt$$

and simply hitting the level b after time t_0 denoted by

$$\mathbf{P}(\tau_b^{(t_0)} \in dt) = h_b^{(t_0)}(t) dt.$$

Proposition 3.3. *The densities are for $t > t_0$ given by*

$$\bar{h}_b^{(t_0)}(t) = \frac{\sqrt{t_0}}{t\sqrt{t-t_0}} \varphi(\mu\sqrt{t-t_0}) \varphi\left(\frac{b-\mu t_0}{\sqrt{t_0}}\right) - \frac{b}{t^{3/2}} \varphi\left(\frac{b-\mu t}{\sqrt{t}}\right) \Phi\left(-b\sqrt{\frac{1}{t_0} - \frac{1}{t}}\right)$$

$$h_b^{(t_0)}(t) = \frac{2\sqrt{t_0}}{t\sqrt{t-t_0}} \varphi(\mu\sqrt{t-t_0}) \varphi\left(\frac{b-\mu t_0}{\sqrt{t_0}}\right) + \frac{|b|}{t^{3/2}} \varphi\left(\frac{b-\mu t}{\sqrt{t}}\right) \left(2\Phi\left(|b|\sqrt{\frac{1}{t_0} - \frac{1}{t}}\right) - 1\right).$$

Proof. For simplicity, consider the Brownian motion case and introduce the hitting time

$$\sigma_b^{(t_0)} = t_0 + \sigma_b \circ \theta_{t_0} = \inf\{s > t_0 : W_s = b\}.$$

Let $t > t_0$ be given and then an application of the Markov property (see Revuz & Yor [12]) shows that

$$\begin{aligned} \mathbf{P}(\sigma_b^{(t_0)} \in dt; W_{t_0} > b) &= \mathbf{E}\left(\mathbf{P}(\sigma_{b-y} \in d(t-t_0))\Big|_{y=W_{t_0}}; W_{t_0} > b\right) \\ &= \mathbf{E}\left(\frac{|b-W_{t_0}|}{(t-t_0)^{3/2}} \varphi\left(\frac{b-W_{t_0}}{\sqrt{t-t_0}}\right); W_{t_0} > b\right) dt. \end{aligned}$$

The integration by parts yield that

$$\begin{aligned} \mathbf{P}(\sigma_b^{(t_0)} \in dt; W_{t_0} > b) &= \left(\int_0^\infty \frac{|y|}{\sqrt{t_0}(t-t_0)^{3/2}} \varphi\left(\frac{|y|}{\sqrt{t-t_0}}\right) \varphi\left(\frac{y+b}{\sqrt{t_0}}\right) dy \right) dt \\ &= \left(\frac{\sqrt{t_0}}{t\sqrt{t-t_0}} \varphi(0) \varphi\left(\frac{b}{\sqrt{t_0}}\right) - \frac{b}{t^{3/2}} \varphi\left(\frac{b}{\sqrt{t}}\right) \Phi\left(-b\sqrt{\frac{1}{t_0} - \frac{1}{t}}\right) \right) dt. \end{aligned}$$

The expression for $\bar{h}_b^{(t_0)}(\cdot)$ in the general cases is obtained by applying the Girsanov theorem as in Proposition 3.1. The expression for $h_b^{(t_0)}(\cdot)$ is computed by the same method. \square

Remark 3.4. For completeness, it follows from Proposition 3.3 that the density of hitting the level b after time t_0 from below is for $t > t_0$ given by

$$\begin{aligned} \mathbf{P}(\tau_b^{(t_0)} \in dt; X_{t_0} < b) \\ = \left(\frac{\sqrt{t_0}}{t\sqrt{t-t_0}} \varphi(\mu\sqrt{t-t_0}) \varphi\left(\frac{b-\mu t_0}{\sqrt{t_0}}\right) + \frac{b}{t^{3/2}} \varphi\left(\frac{b-\mu t}{\sqrt{t}}\right) \Phi\left(b\sqrt{\frac{1}{t_0} - \frac{1}{t}}\right) \right) dt. \end{aligned}$$

Remark 3.5. In the special case of a Brownian motion and $b = 0$ there is a simple expression of the density of $\sigma_0^{(t_0)}$. It is for $t > t_0$ given by

$$\mathbf{P}(\sigma_0^{(t_0)} \in dt) = \frac{1}{\pi} \frac{\sqrt{t_0}}{t\sqrt{t-t_0}} dt.$$

4. Pricing the employee option

This section presents an analytical framework for the valuation of the employee option. The optimal exercise policies and valuation formulae provided below use the Black-Scholes model. A numerical example concludes the section.

The employee account (the accrual account) is determined by $e^{\alpha t}$ where α is the employee benefit rate. Given the employee account, the payoff of the option at time t is

$$\gamma_t = e^{\alpha t} \mathbf{1}_{[K, \infty)}(S_t)$$

where S_t denotes the company's stock price and $K > 0$ is the strike price. The stock price S_t has the dynamics – under the risk neutral measure – given by

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \quad S_0 > 0$$

where (W_t) is a Brownian motion, $\sigma > 0$ is the volatility coefficient, r is the constant interest rate and δ is the dividend rate of the stock. To ensure that the option has the possibility to perform better than the bank account it is required that the benefit rate α is greater than the interest rate r , that is, $\alpha - r > 0$ (otherwise the employee option is trivially exercised instantaneous).

The arbitrage valuation of the option – by pricing theory of American options, Karatzas [7] and Musiela & Rutkowski [9] – is related to a finite-horizon optimal stopping problem. Indeed, the price $\pi(S_0)$ is given by

$$\pi(S_0) = \sup_{\tau \leq T} \mathbf{E}(e^{-r\tau} \gamma_\tau) = \sup_{\tau \leq T} \mathbf{E}(e^{(\alpha-r)\tau}; S_\tau \geq K).$$

The price process S_t is a geometric Brownian motion and has the representation

$$S_t = S_0 \exp(\sigma W_t + (r - \delta - \sigma^2/2)t) = S_0 \exp(\sigma X_t)$$

where X_t is a Brownian motion with drift $\mu = (r - \delta)/\sigma - \sigma/2 \in \mathbb{R}$, that is, $X_t = W_t + \mu t$. Then the above stopping problem can be rewritten as

$$\pi(S_0) = \sup_{\tau \leq T} \mathbf{E}(e^{(\alpha-r)\tau}; S_0 \exp(\sigma X_\tau) \geq K) = \sup_{\tau \leq T} \mathbf{E}(e^{(\alpha-r)\tau}; X_\tau \geq \sigma^{-1} \log(K/S_0)).$$

Applying Theorem 2.1, the value of the option is $\pi(S_0) = V_*(0, 0)$ with $\lambda = \alpha - r$ and $b = \sigma^{-1} \log(K/S_0)$. The length of the exercise window ΔT_* is defined as $\Delta T_* = \infty$ if $\mu < 0$ satisfying $\mu^2 \geq \alpha - r$ and otherwise ΔT_* is the root of equation (2.2). The optimal starting time of the exercise window is

$$T_* = (T - \Delta T_*)^+.$$

If $T_* = 0$, then the option is optimally exercised the first time the stock prices S_t reaches the strike price K , and if $T_* > 0$, then the optimal exercise strategy is to defer exercising until the remaining time is $T - T_* = \Delta T_*$ and then exercise the first time the stock prices S_t reaches the strike price K (see Figure 1), that is,

$$\tau_* = \inf\{T_* < t \leq T : S_t = K\} \quad (\inf \emptyset = T).$$

The value of the employee option with $b = \sigma^{-1} \log(K/S_0)$ is

$$(4.1) \quad \pi(S_0) = \begin{cases} L_b^{(\alpha-r)}(T) + e^{(\alpha-r)T} H_b(T) \mathbf{1}_{(-\infty, 0)}(b) & \text{if } T_* = 0 \\ \int_{T_*}^T e^{(\alpha-r)t} h_b^{(T_*)}(t) dt + e^{(\alpha-r)T} \left(\Phi\left(\frac{\mu T_* - b}{\sqrt{T_*}}\right) - \int_{T_*}^T \bar{h}_b^{(T_*)}(t) dt \right) & \text{if } T_* > 0. \end{cases}$$

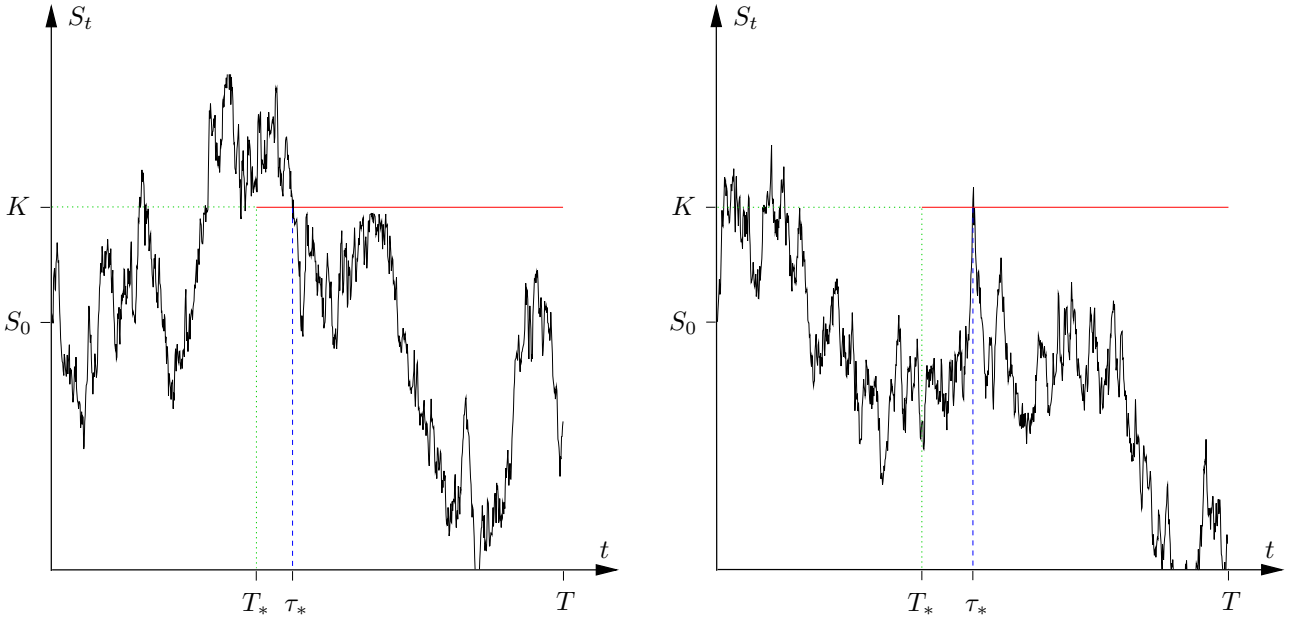


Figure 1. Drawings of the optimal exercise strategy

Example 4.1. This example studies in detail some scenarios in order to give an intuitive description of the employee option and its valuation formula. The length of the exercise window ΔT_* is approximated numerically by solving equation (2.2). Then the option value $\pi(S_0)$ is computed by valuation formula (4.1).

The stock does not pay any dividend, $\delta = 0$. The first case of scenarios is with the interest rate, r , fixed and the volatility of the value of the firm, σ , varying and vice versa in the second case. More specific, the first case is with scenarios where the interest rate, r , is fixed at 3% and σ takes one of the following three values: 20%, $\sqrt{2}r \approx 24.5\%$ or 30% (then $\mu = 5\%$, 0% or -5%). Fixing the volatility at 20%, the second case is three scenarios where the short interest rate is equal to 3%, 2% or 1% (again $\mu = 5\%$, 0% or -5%). Let the time to maturity be five years, $T = 5$. Suppose the initial stock price, S_0 , is equal to 100 and the strike price, K , is equal to 103 so that the option starts out-of-the-money.

Figure 2 presents calculated options prices for the two cases of scenarios. The value of the option increases with higher employee benefit rate, obviously from the construction of the payoff of the option. The left drawing of Figure 2 demonstrates that increasing the volatility results in a lower value of the option, when the optimal starting times of the exercise window are greater than 0. This is caused by the fact that increasing the volatility the longer time one wait to exercise and thus it is optimal to exercise at an earlier time and thus the employee account will have increased when exercising, also the (risk neutral) expected value. A similar picture (see the right drawing of Figure 2) is produced when the interest rate increases.

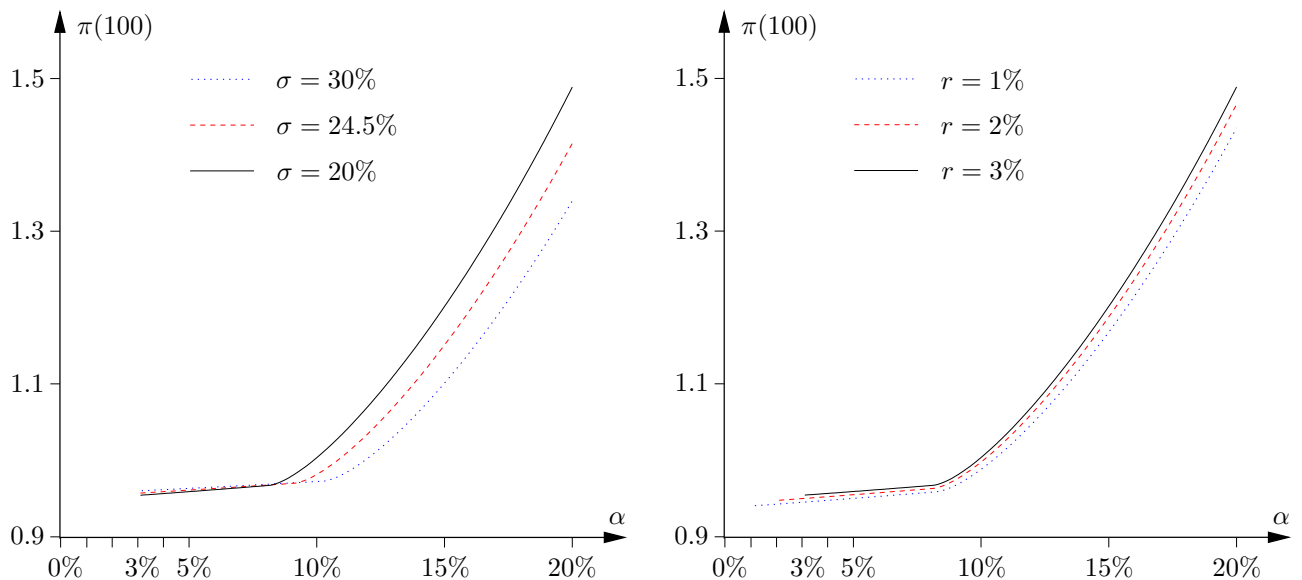


Figure 2. Drawings of option values, $\pi(S_0)$.

To end this example, Figure 3 displays the value of the option and some “obvious” boundaries in the case the volatility is equal to 20% and the interest rate is equal to 3%. The upper boundary is the discounted maximal value of the payoff. One lower boundary is the value of the European style of the option and the other is the value using the strategy to exercise once the stock price reaches the strike price.

Remark 4.2. A fixed strike price might not be the best measure of the performance of the company. By specifying a benchmark as the option plan’s strike price it is possible to obtain a better measure of positive performance and reward truly superior performs. Johnson & Tian [5], see also Jørgensen [6], suggested a model for a benchmark with this idea. Below is a short presentation of the construction of the benchmark (for details see the above two papers).

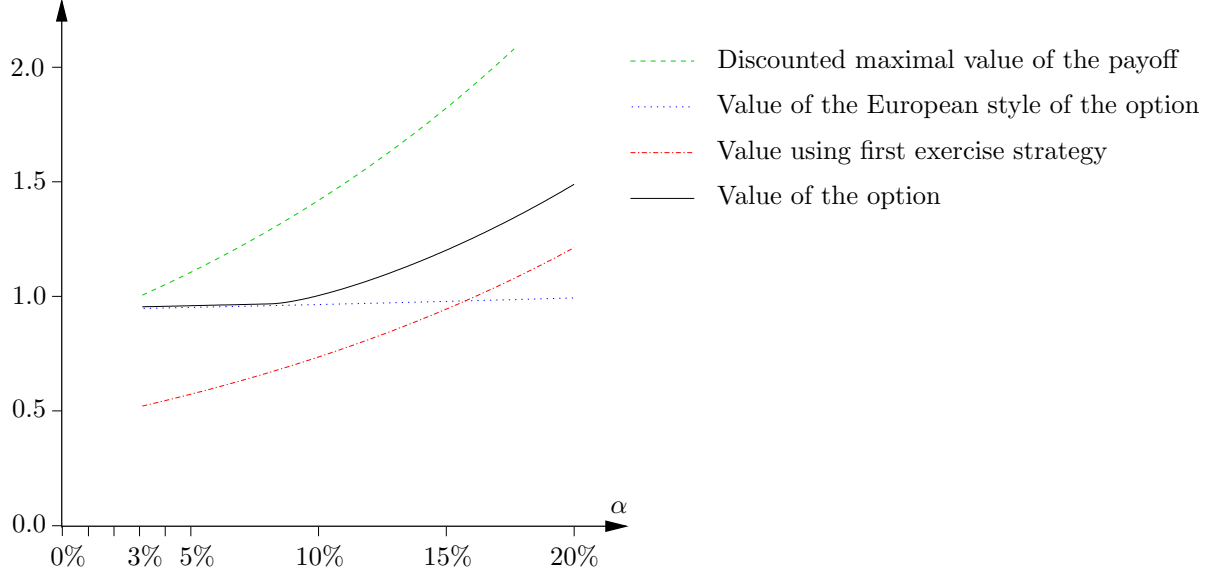


Figure 3. Drawings of $\pi(S_0)$ and different bounds.

Given an appropriate index I_t , the dynamics (under the physical measure) of the stock price and the index are

$$dS_t = (\tilde{\mu} - \delta)S_t dt + \sigma S_t dW_t, S_0 > 0 \quad \text{and} \quad dI_t = (\tilde{\mu}^{(I)} - \delta^{(I)})I_t dt + \sigma^{(I)}I_t dW_t^{(I)}, I_0 > 0$$

where W_t and $W_t^{(I)}$ are two correlated Brownian motions with $dW_t dW_t^{(I)} = \rho dt$. Moreover, $\tilde{\mu}$, $\tilde{\mu}^{(I)}$, σ , $\sigma^{(I)}$, δ , and $\delta^{(I)}$ are the drift, the volatility and the dividend rate of the stock and the index respectively. Since the idea is to reward positive performance relative to the index, define the relation between the two drifts by $\tilde{\mu} = r + \rho(\sigma/\sigma^{(I)})(\tilde{\mu}^{(I)} - r)$. This is an additional model assumption that has an interpretation in the framework of incentive compensation as remarked in Johnson & Tian [5]. The benchmark is constructed as

$$H_t = S_0^{-1} e^{\beta t} \mathbf{E}(S_t | I_t)$$

which measure the stock against the index, where β is a constant that can adjust the expected growth rate of the benchmark. Given that the benchmark H_t , the payoff of the option is

$$\tilde{\gamma}_t = e^{\alpha t} \mathbf{1}_{[KH_t, \infty)}(S_t) = e^{\alpha t} \mathbf{1}_{[K, \infty)}(S_t/H_t)$$

where $K > 0$ control the moneyness of the option – the option is granted in-the-money if $S_0 > K$, at-the-money if $S_0 = K$, or out-of-the-money if $S_0 < K$.

The valuation formula of the option with fixed strike price can be applied. Indeed, due to the above extra model assumption the martingale measure is uniquely determined (and hence there is a unique price for the option). It can be shown that under this measure the dynamics of H_t and S_t/H_t are geometric Brownian motions and that the price of the option is $\pi(S_0)$ with

$$\mu = -\frac{\beta}{\sigma\sqrt{1-\rho^2}} - \frac{\sigma\sqrt{1-\rho^2}}{2} \quad \text{and} \quad b = \frac{K}{S_0\sigma\sqrt{1-\rho^2}}.$$

Further, the optimal exercise strategy is on the form

$$\tau_* = \inf\{T_* < t \leq T : S_t = KH_t\} \quad (\inf \emptyset = T).$$

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