A Two-Account Model of Pension Saving Contracts

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Abstract

Saving contracts studied in the literature and available on the pension market share certain characteristics. An overview implies a unified formalization that exposes, at the same time, the common characteristics and the important differences. This article presents such a formalization in terms of two interacting accounts and specifies a series of examples from the literature and the market. We solve the valuation problem by derivation of a partial difference-differential equation and implementation of a numerical finite difference procedure. Illustrations are included.

Keywords: Stochastic differential equation, contingent claims, with-profit insurance, partial differential equation, finite difference method.

1 Introduction

The pension market is undergoing several changes in these years due to product development and modernization and globalization of accounting rules. There is a need for general formalizations of the products and the methods of valuation in order to compare the features and values across countries, across insurance companies, and across products. A too abstract and general formalization may fail to show the differences and a too specific formalization may fail to illustrate the similarities. We seek a compromise which, at the same time, illustrates the similarities and the differences in the structural design of different products. Furthermore, the aim is to provide a model that is useful in practical valuation problems.

Product developments in the pension market includes variations of traditional with-profit contracts and unit-link insurance contracts. These products differ primarily in the investment strategies and allocation of capital gains to the contract holders. However, they also seem to share some important characteristics, primarily concerning smoothing of capital gains. Some of these products are very concretely accounted for by two interacting accounts. One (market/real) account earns capital gains in the market in accordance with some specified investment strategy and is changed, at discrete time points, according to another (technical/guaranteed) account. This other (technical/guaranteed) account earns artificial capital gains and is changed, at the same discrete time points, according to the first account. Upon retirement one of the two accounts is paid out. Two such products recently introduced on the Danish pension market are treated below. One of them, Tidspension (Time Pension), was described and discussed in connection with its introduction on the market in Jørgensen and Nielsen (2002).

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While certain new products have the two-account system stipulated in the contract, it is less obvious that also traditional with-profit insurance may be accounted for by such two accounts. However, several formalizations of with-profit life insurance presented in the literature are examples of or very closely related to our two-account model. Below we consider the constructions suggested by Hansen and Miltersen (2002), Miltersen and Persson (2003), and Devolder and Dominguez-Fabian (2005). In addition we consider a formalization that, to the best of the authors’ knowledge, comes very close to the Danish practice of with-profit life insurance.

Parallel to the product development, accounting rules are modernized and globalized. The life insurance and pension business is going from accounting by technical book values to accounting by market values. Marking to market naturally invites financial economists to contribute to the understanding and implementation of valuation methods since they have always valued on a market basis. Financial economists have taught actuaries how this machinery works in different media ranging from daily newspapers to actuarial journals - all contributions are very much appreciated! A state-of-the-art exposition of a financial economist’s view on mark-to-market valuation and market value accounting in life insurance and pension is presented by P. L. Jørgensen (2004). See also his fairly exhaustive list of references.

The product development, the modernization of accounting rules and the contribution by financial economists are processes which are connected by involved cause and effect relationships, and which will go on in the future. This is to the benefit of the insurance companies, the policy holders, the actuarial literature - and the financial economists. In this article we present a common ground where the understanding of certain pension saving products and some specific formalizations in the literature merge.

If the underlying stochastic processes are Markovian, a (conditional expected) value can be characterized by the solution to a partial difference-differential equation. There are various techniques at hand for solving such. We implement an explicit finite difference scheme which basically means that we replace the partial differential equation by a partial difference equation. This replacement is handled with care such that the solution to the difference equation is ‘close to’ the solution of the differential equation. Numerous articles approach financial valuation by finite difference methods. We refer here to Jensen et al. (2001) who actually evaluate life insurance liabilities by these methods.

In Section 2 we present the general setup of the two-account pension saving contract. In Section 3 we specify the dynamics of these accounts such that several examples in the literature and in the market come out. In Section 4 we derive the partial difference-differential equation characterizing the value of the general contract and derive certain results for some special cases. A numerical procedure is presented in Section 5 and values in some of the examples from Section 3 are illustrated. Section 6 concludes with a comment on a possible third account which could bring further details to the examples.

2 The General Contract

In this section we formalize the contract which is the objective of our study.

Consider a contract issued at time 0 and terminating at time \( n \). The payments connected to the contract are formalized by a lump sum benefit \( \Phi \) at time \( n \) and a contribution process \( C \) such that
\(C(t)\) are the accumulated contributions over \([0, t]\). The contract is accounted for by two processes \(X\) and \(Y\) which are introduced below. The terminal benefit is a function of the value of these two processes at termination and, with an abuse of notation, we use the letter \(\Phi\) for the function as well, such that

\[
\Phi = \Phi(X(n), Y(n)).
\]

The two accounts are updated at specific deterministic time points and we therefore take as given a partition of the time horizon into \(m\) intervals by introducing \(m + 1\) time points \(t_0, t_1, \ldots, t_m\) such that \(0 = t_0 < \ldots < t_m = n\). We speak of these time points briefly as \textit{updates} and introduce a deterministic process counting the number of updates until time \(t\) defined by

\[
\varepsilon(t) = \sum_{t_i \leq t} 1.
\]

The \textit{primary account} \(X\) is assumed to be invested in an investment portfolio \(S\) with instantaneous return \(dS(t)/S(t-).\) A process of contributions \(C\) adds to the account. In Section 4 we specify a particular form of \(C\) and \(S\) and derive a partial differential equation that characterizes the value of future payments. In addition to the capital gains and the contributions, a \textit{primary claim premium} \(\pi(t)\) is subtracted from and a \textit{primary claim} \(\phi(t)\) is added to the primary account at updates. These processes are specified later. Thus, the dynamics of \(X\) are given by

\[
\begin{align*}
  dX(t) &= X(t-) \frac{dS(t)}{S(t-)} + dC(t) + (\phi(t) - \pi(t)) d\varepsilon(t), \\
  X(0-) &= 0.
\end{align*}
\]

The \textit{secondary account} \(Y\) is assumed to earn an artificial, deterministic, instantaneous return rate \(r^*(t)\). The contributions in \(C\) are added to the account. In addition to the artificial capital gains and the contributions, a \textit{secondary claim} \(\delta(t)\) is added to \(Y\) at updates. This process is specified later. The dynamics of \(Y\) are given by

\[
\begin{align*}
  dY(t) &= Y(t-) r^*(t) dt + dC(t) + \delta(t) d\varepsilon(t), \\
  Y(0-) &= 0.
\end{align*}
\]

We also introduce two state processes \(\bar{X}\) and \(\bar{Y}\) which keep track of the two accounts at the latest update. These state processes are piecewise constant and follow the dynamics

\[
\begin{align*}
  d\bar{X}(t) &= \left( X(t) - X \left( \max_{i} t_i : t_i < t \right) \right) d\varepsilon(t), \\
  \bar{X}(0-) &= 0, \\
  d\bar{Y}(t) &= \left( Y(t) - Y \left( \max_{i} t_i : t_i < t \right) \right) d\varepsilon(t), \\
  \bar{Y}(0-) &= 0.
\end{align*}
\]

The dynamics of the two accounts at the updates are primarily specified by the primary claim premium \(\pi\), the primary claim \(\phi\), and the secondary claim \(\delta\). These are taken to be functions of \((t, X(t-), Y(t-), \bar{X}(t-), \bar{Y}(t-))\). Abusing notation we also use the letters \(\pi, \phi,\) and \(\delta\) for these functions such that

\[
\begin{align*}
  \pi(t) &= \pi(t, X(t-), Y(t-), \bar{X}(t-), \bar{Y}(t-)), \\
  \phi(t) &= \phi(t, X(t-), Y(t-), \bar{X}(t-), \bar{Y}(t-)), \\
  \delta(t) &= \delta(t, X(t-), Y(t-), \bar{X}(t-), \bar{Y}(t-)).
\end{align*}
\]

3
These three functions specify how the two accounts are connected. Of course we have certain forms of this connection in mind for our pension contracts. Therefore we now present types of general claims $\pi$, $\phi$, $\delta$, and terminal benefits $\Phi$ that the reader should think of. These types are supposed to capture the main examples in the market and in the literature on pension contracts. The point is that specific pension contracts can then be constructed as special combinations of the presented claims. In the next section we specify further these claims into examples from the market and from the literature.

It is clear that one of the functions $\phi$ or $\pi$ is superfluous since we can always formalize $\phi - \pi$ in one function. When we choose to specify two functions, it is because we have some very specific forms of these functions in mind. We think of a stream of positive premiums $\pi$ which is paid partly to cover the stream of positive claims $\phi$. The stream of claims $\delta$ added to the secondary account may in general be positive or negative although in most of the examples in the next section they are positive.

From now on we drop the arguments of all four functions such that we write, e.g., $\pi = \kappa x$ instead of $\pi (t, x) = \kappa (t) x$ for a deterministic function $\kappa$. This makes the specifications easier to read. In general we speak of excess of accounts when we consider the primary account over the secondary account and of excess of returns when we consider the return on the primary account over the return on the secondary account since the last update. The topscript in $\cdot^+$ denotes that only the positive part is taken into account.

The primary claim premium $\pi$ can take various forms. We think of $\pi$ as being a fraction of one of the accounts, of the excess of accounts, or of the excess of returns, i.e. as one of the following,

$$
\pi = \kappa x, \\
\pi = \kappa y, \\
\pi = \kappa (x - y)^+, \\
\pi = \kappa (x - \bar{x} - (y - \bar{y}))^+.
$$

The primary claim $\phi$ is a contingent claim added to the primary account. The primary claim $\phi$ adjusts the primary account such that it is larger than or equal to the secondary account at updates. The primary claim is based on the primary account after subtraction of the primary claim premium. Then we have that

$$
\phi = (y - (x - \pi))^+.
$$

The secondary claim $\delta$ is a contingent claim added to the secondary account. The secondary claim $\delta$ pulls the secondary account towards the primary account at updates. This pulling can happen in different ways: It can be based on excess of accounts or on excess of return and it can be upwards only ($\delta > 0$) or in both directions. This means that the secondary claim is one of the following,

$$
\delta = (\alpha (x - \pi) - \beta y)^+, \\
\delta = \alpha (x - \pi) - \beta y, \\
\delta = \alpha (x - \bar{x} - (y - \bar{y}))^+, \\
\delta = \alpha (x - \bar{x} - (y - \bar{y})).
$$
The constants $\alpha \leq 1$ and $\beta \geq 1$ specify the power of the pulling.

The benefit $\Phi$ is the terminal payoff presented to the contract holder. This is either one of the accounts or the larger of the two accounts, i.e. one of the following,

$$
\Phi = x, \\
\Phi = y, \\
\Phi = \max(x, y).
$$

We emphasize that $\Phi$, in contrast to the other three functions, is defined as a function of the accounts at termination and not prior to termination. This has as consequence that the updates taking place at time $n$ are accounted for before calculating the terminal benefit.

3 Examples of Pension Contracts

The claims presented at the end of the previous section can be combined in different ways in order to form a series of contracts which exist in the market and/or have been studied in the literature. In this section we present such a series of contracts, that includes modern contracts present on the Danish pension market, contracts recently studied in the actuarial literature, and contracts present on the UK and US markets. For each contract, the non-specified functions are set to zero.

- A **Partial look back guarantee unit-link insurance** was introduced on the Danish market in 2001 (by Danica Pension under the name *Danica Link*). In the Danish market version $r^* = 0$ but it could of course be something else. The secondary account is updated such that it is the maximum of the secondary account before the update and a fraction $\alpha$ of the primary account. At termination the maximum of the two accounts is paid out. For this construction of claims, the policy holder pays in each period a deterministic fraction $\kappa$ of the primary account. Thus,

$$
\pi = \kappa x, \\
\delta = (\alpha (x - \pi) - y)^+, \\
\Phi = \max(x, y).
$$

- A **Linear regulator unit-link insurance** was introduced on the Danish market in 2002 (by Codan Pension, now SEB Pension, under the name *Tidspension*). In the Danish market version $r^*$ is the ten-year yield but it could be something else. The secondary account is pulled towards the primary account, independently of the sign of the difference. At termination the secondary account is paid out. For the artificial return on the secondary account $r^*$ the policy holder pays in each period a constant fraction $\kappa$ of the primary account. Thus,

$$
\pi = \kappa x, \\
\delta = \alpha (x - \pi - y), \\
\Phi = y.
$$

- The **Danish market with-profit life insurance** has been available for decades and the products have recently been managed more and more concretely in accordance with the following.
The primary account is set equal to the secondary account (the technical reserve) if, prior to update, it is lower. If it is larger than the secondary account, including a possible loading, the secondary account is pulled towards the primary account. At termination the secondary account is paid out. For the construction of claims, the policy holder pays in each period a fraction of the primary account. Thus,
\[
\pi = \kappa x, \\
\phi = (y - (x - \pi))^+, \\
\delta = \alpha (x - \pi - \beta y)^+, \\
\Phi = y.
\]

It has been advocated for by the Danish regulatory authorities that the primary claim premium can not exceed the excess of returns. Thus, actually, in a given period,
\[
\pi = \min (\kappa x, x - \pi - (y - \bar{y}))^+. 
\]

The return \( r^* \) on the secondary account represents an interest rate guarantee, i.e. a minimum return on the account between updates. Under Danish legislation, it is sufficient that the primary account at updates is set equal to the present market value of the guarantee. Denoting the value of the guarantee at termination by \( \tilde{y} \), this can be written as
\[
\phi = \left( \tilde{y} E[e^{\int_0^t r']} - (x - \pi) \right)^+, 
\]
where
\[
\tilde{y} = ye^{\int_0^t r^*}.
\]

In the Danish market the contract is often based on a benefit guarantee rather than an interest rate guarantee. In this case \( \tilde{y} \) is calculated at time 0 and represents a minimum terminal benefit. Since the rate of return between updates is not guaranteed, negative secondary claims can be allowed. Hence,
\[
\delta = \alpha (x - \pi - \beta y), \\
\Phi = \max(y, \tilde{y}).
\]

- *Hansen and Miltersen (2002)* have studied a variation of the Danish market with-profit life insurance where, on one hand, there is no primary claim but where, on the other hand, the larger of the two accounts is paid out at termination. Thus the primary claim premium, which is specified as a fraction of the secondary account, pays for the option-like terminal benefit. The secondary account is pulled towards the primary account as in the Danish market with-profit life insurance above but with a slightly changed power. Thus,
\[
\pi = \kappa y, \\
\delta = (\alpha (x - \pi) - \beta y)^+, \\
\Phi = \max (x, y). 
\]

This specification corresponds to what Hansen and Miltersen speak of as the direct method. Their indirect method cannot be captured by our model since this makes essential use of a
third account which, in principle, keeps track of the primary claim premiums paid in the past. However, introducing a third account in our model is relatively simple, as described in Section 6. The coefficients of the primary and secondary accounts would then be influenced by this third account in a specific way as well. Although we cannot formalize their indirect method with only two accounts, we can present the underlying idea, namely that the structure of the primary claim premium is similar to the structure of the secondary claim. Thus, for given constants $\alpha'$ and $\beta'$, their indirect method more or less corresponds to letting

$$\pi = (\alpha' x - \beta' y)^+.$$ 

- *Miltersen and Persson (2003)* have studied a contract where, as in *Hansen and Miltersen (2002)*, there is no primary claim but where the larger of the two accounts is paid out and the primary claim premium pays for the option-like terminal benefit. However, in contrast to *Hansen and Miltersen (2002)*, the primary claim premium and the secondary claims are formulated in terms of the excess of returns, i.e.

$$\begin{align*}
\pi &= \kappa (x - \pi - (y - \pi))^+, \\
\delta &= \alpha (x - \pi - (y - \pi))^+, \\
\Phi &= \max(x, y).
\end{align*}$$

- *Devolder and Domínguez-Fabian (2005)* have studied the three classical participation schemes, *reversionary bonus, cash bonus* and *terminal bonus*. Their reversionary bonus formalization is different from the Danish market with-profit life insurance in two directions: First the primary claim is not calculated as a fraction of the primary account but as a fraction of the excess of accounts. Second the secondary account is pulled all the way, upwards, to the primary account at every update. For *reversionary bonus* their specification of claims is given by

$$\begin{align*}
\pi &= \kappa (x - y)^+, \\
\phi &= (y - (x - \pi))^+, \\
\delta &= (x - \pi - y)^+, \\
\Phi &= \max(x, y).
\end{align*}$$

For *cash bonus* the only difference is that instead of adding $\delta$ to $y$, this is paid out cash to the contract holder. For *terminal bonus*, all adjustments take place only at the end such that the coefficients above only hold for $t = n$.

It is easy to realize that, due to the specification of $\phi$ and $\delta$, we have that the two accounts are equal at updates since $X(t) = Y(t) = \max(X(t-)-\pi, Y(t-)) + \Delta C(t)$. Therefore also $X = Y$ and there is no difference between working with *excess of accounts* and *excess of returns*. Due to the specification of $\pi$ we can furthermore write that

$$\begin{align*}
Y(t) &= \max \left( X(t-)-\kappa (X(t-)-Y(t-))^+, Y(t-) \right) \\
&= Y(t-)+(1-\kappa)(X(t-)-Y(t-))^+.
\end{align*}$$
Thereby an alternative specification is given by ($\pi = 0$)

\[
\delta = (1 - \kappa) (x - y)^+, \\
\phi = y - x + (1 - \kappa) (x - y)^+, \\
\Phi = y.
\]

This specification then holds for all $t$ for reversionary bonus, with $\delta$ paid out for cash bonus, or only for $t = n$ in case of terminal bonus.

- The UK market with-profit life insurance works such that the primary claim premium is paid only at termination and there is no primary claim. The terminal claim premium pays for the option-like terminal benefit since the larger of the two accounts is paid out. The only action during the term of the contract is that the secondary account is increased by a fraction of the excess of returns. Thus,

\[
\pi (n) = \kappa (x - y)^+, \\
\delta = \alpha (x - x^* - (y - y^*))^+, \\
\Phi = \max (x, y).
\]

In general the UK market with-profit life insurance is a bit more involved in the sense that not only the excess of returns in the present year but the excess of returns in the most recent $\nu$ update periods, in some specific sense, are taken into account, where $\nu$ is a positive integer. How this is done is explained thoroughly in Ballotta et al. (2006). We could implement this construction easily by introducing a sequence of state variables corresponding to $\pi$ and $\gamma$ remembering the accounts at the most recent $\nu$ updates.

- The US marketed products universal life and variable life work, basically, in the same way as the reversionary bonus scheme above, see Cummins, Miltersen and Persson (2003).

4 Valuation of the Contract

In this section we present a partial differential equation that characterizes the value of the contracts presented in the previous sections for specific contribution and portfolio processes $C$ and $S$. These two processes are the residual processes which determine the accounts and which have not been specified so far.

We assume that lump sum contributions to the accounts take place at updates only. Furthermore we assume that the lump sum payments and the rate of continuous payments at time $t$ are deterministic functions of the two accounts prior to time $t$ such that we can write

\[
dC (t) = c (t, X (t^-), Y (t^-)) dt + \Delta C (t, X (t^-), Y (t^-)) d\zeta (t)
\]

for sufficiently regular functions $c$ and $\Delta C$.

The portfolio $S$ is assumed to consists of bonds and stocks for which market prices are easily observable. The value of the portfolio under the valuation measure is assumed to evolve according to the stochastic differential equation

\[
dS (t) = r (t) S (t) dt + \sigma S (t) dW^s (t), \\
S (0) = s_0.
\]
Here $\sigma$ is constant and $W^r(t)$ is a standard Brownian motion. The risk free rate of interest is assumed to be stochastic and is described by the Vasicek model, i.e.

$$dr(t) = (b - ar(t))dt + \gamma W^r(t).$$

Here $a$, $b$, and $\gamma$ are constants and $W^r(t)$ is a standard Brownian motion. The Brownian motions $W^x(t)$ and $W^r(t)$ are assumed to have a correlation factor of $\rho$. The Vasicek rate of interest has the property of mean reversion, meaning that it tends to revert to the mean level $b/a$. A disadvantage of the model is that the rate of interest has a positive probability of attaining negative values.

In general we are interested in calculating the financial value of future payments, i.e.

$$V(t, r, x, y, \pi, \gamma) = E_{t, r, x, y, \pi, \gamma} \left[ e^{-\int_t^r r(t) dt - \int_t^r rW^r(s)} \Phi - \int_t^r e^{-\int_t^s r(t) dt} dC(s) \right],$$

where $E_{t, r, x, y, \pi, \gamma}$ denotes conditional expectation under the valuation measure given that $r(t) = r$, $X(t) = x$, $Y(t) = y$, $\bar{X}(t) = \pi$, $\bar{Y}(t) = \gamma$. We now present a partial differential equation that fully characterizes the reserve. First introduce the differential and difference operators for the function $U(t, r, x, y, \pi, \gamma)$,

$$DU = U_t + U_r(b - ar) + U_x x + U_y y + \frac{1}{2} U_{rr} r^2 + \frac{1}{2} U_{xx} x^2 + U_{xy} x y \rho,$$

$$\Delta U = U(t, r, x + \Delta C + \phi - \pi, y + \Delta C + \delta),$$

$$U(n, x, y) = \Phi(x, y),$$

**Proposition 1** If there exists a function $U$, such that

$$DU(t, r, x, y, \pi, \gamma) = rU(t, r, x, y, \pi, \gamma) + c(t, x, y),$$

$$\Delta U(t, r, x, y, \pi, \gamma) = \Delta C(t, x, y),$$

then

$$U = V.$$

**Proof.** In this proof we abbreviate $A(t) = (r(t), X(t), Y(t), \bar{X}(t), \bar{Y}(t))$. Consider the function $U$ taken on the state variables, i.e. $U(t) = U(t, A(t))$. For this process we have that

$$e^{-\int_t^r U(t)} = -\int_t^r d \left( e^{-\int_t^s r} U(s) \right) + e^{-\int_t^n U(n)}$$

$$= -\int_t^r e^{-\int_t^s r} (dU(s) - r(s) U(s) ds) + e^{-\int_t^n U(n)}$$

$$= -\int_t^r e^{-\int_t^s r} \left( U_r(s) \gamma dW^r(s) + U_x(s) \sigma S(s) dW^x(s) \right)$$

$$- \int_t^r e^{-\int_t^s r} \left( DU(s) - r(s) U(s) - c(s) \right) ds$$

$$- \int_t^r e^{-\int_t^s r} \left( U(s) - U(s) - \Delta C(s) \right) ds$$

$$+ e^{-\int_t^s r} U(n) - \int_t^r e^{-\int_t^s r} dC(s).$$
By the assumptions on $U$ the second last and third last lines disappear and we are left with

$$e^{-\int_0^r U(t)} = e^{-\int_0^r U(n)} - \int_t^n e^{-\int_s^r dC(s)}$$

Now we multiply by $e^{\int_0^r}$ on both sides and take conditional expectation given $A(t) = (r, x, y, \pi, \gamma)$. Since the last term is an integral over martingale increments, we reach at

$$U(t, r, x, y, \pi, \gamma) = E_{t, r, x, y, \pi, \gamma} \left[ e^{-\int_0^r U(n)} - \int_t^n e^{-\int_s^r dC(s)} \right] = V(t, r, x, y, \pi, \gamma).$$

We end this section with a proposition that studies the value of the future payments in the cases where the terminal benefit is one of the accounts. Then one can decompose the value into the present value of this account and a future flow of changes to this account. In particular, if this future flow of changes has the value zero, the value of the future payments is simply given by the account itself. A very simple way to obtain that the value of the future changes is zero is to set the coefficients such that the value of the local change is zero. E.g. in case of the primary account one could construct the primary claim and the primary claim premium such that for all $i$,

$$E_{t_i, i, A(t_i - 1)} \left[ e^{-\int_{t_{i-1}}^{t_i} \varphi(t_i) - \pi(t_i)} \right] = 0.$$

One could then speak of the natural claim premium. In the proposition such a condition together with $\Phi = x$ leads to a value of future payments equal to the primary account itself.

**Proposition 2** If $\Phi = x$, then

$$V(t, r, x, y, \pi, \gamma) = x + E_{t, r, x, y, \pi, \gamma} \left[ \int_t^n e^{-\int_s^r \varphi(s) - \pi(s)} ds \right].$$

If furthermore

$$E_{t_i, i, A(t_i - 1)} \left[ e^{-\int_{t_{i-1}}^{t_i} \varphi(t_i) - \pi(t_i)} \right] = 0,$$

then for $t \in \{t_0, t_1, \ldots, t_m\}$,

$$V(t, r, x, y, \pi, \gamma) = x. \quad (2)$$

If $\Phi = y$, then

$$V(t, r, x, y, \pi, \gamma) = y + E_{t, r, x, y, \pi, \gamma} \left[ \int_t^n e^{-\int_s^r \delta(s) ds} \right].$$

If furthermore

$$E_{t_i, i, A(t_i - 1)} \left[ \int_{t_{i-1}}^{t_i} e^{-\int_s^r \delta(s) ds} \right] = 0,$$

then for $t \in \{t_0, t_1, \ldots, t_m\}$,

$$V(t, r, x, y, \pi, \gamma) = y. \quad (5)$$
Proof. If $\Phi = x$, we have that

\[
V(t,r,x,y,x,y) = E_{t,r,x,y,x,y} \left[ e^{-\int_t^s r} X(n) - \int_t^s e^{-\int_t^s r} dC(s) \right]
\]

\[
= x + E_{t,r,x,y,x,y} \left[ \int_t^s d \left( e^{-\int_t^s r} X(s) \right) - \int_t^s e^{-\int_t^s r} dC(s) \right]
\]

\[
= x + E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} dX(s) - r(s)e^{-\int_t^s r} X(s) ds - \int_t^s e^{-\int_t^s r} dC(s) \right]
\]

\[
= x + E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} \left( X(s) - r(s) \frac{dS(s)}{S(s-r)} - r(s) ds + (\phi(s) - \pi(s)) dz(s) \right) \right]
\]

\[
= x + E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} (\phi(s) - \pi(s)) dz(s) \right].
\]

If furthermore (2) holds, we have that

\[
E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} (\phi(s) - \pi(s)) dz(s) \right] = 0,
\]

such that (3) follows. If $\Phi = y$, we have that

\[
V(t,r,x,y,x,y) = E_{t,r,x,y,x,y} \left[ e^{-\int_t^s r} Y(n) - \int_t^s e^{-\int_t^s r} dC(s) \right]
\]

\[
= y + E_{t,r,x,y,x,y} \left[ \int_t^s d \left( e^{-\int_t^s r} Y(s) \right) - \int_t^s e^{-\int_t^s r} dC(s) \right]
\]

\[
= y + E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} dY(s) - r(s)e^{-\int_t^s r} Y(s) ds - \int_t^s e^{-\int_t^s r} dC(s) \right]
\]

\[
= y + E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} (\delta(s) ds - (r(s) - r^*) Y(s) ds) \right].
\]

If furthermore (4) holds, we have that

\[
E_{t,r,x,y,x,y} \left[ \sum_{i=1}^{n} E_{i-1,A(t_{i-1})} \left[ \int_{t_{i-1}}^{t_i} e^{-\int_{t_{i-1}}^{t_i} r} (\delta(s) ds - (r(s) - r^*) Y(s) ds) \right] \right]
\]

\[
= E_{t,r,x,y,x,y} \left[ \int_t^s e^{-\int_t^s r} (\delta(s) ds - (r(s) - r^*) Y(s) ds) \right] = 0,
\]

such that (5) follows. \hfill \blacksquare

5 Numerical Valuation

There are several numerical methods available by which to valuate the contracts in the present model. In the following we have chosen to use the explicit finite difference method. The basic
premise is that differential mathematics are based on partial derivatives defined as difference quotients over infinitesimal intervals. Difference quotients are used to construct a discrete version of the partial differential equation. Defining a sufficiently fine grid in the state space of the contract, it is then possible to approximate values recursively throughout the grid starting with a priori known boundary conditions at the time of termination.

The claims and the claim premium necessitate a division of time into periods between updates, in which the finite difference method can be applied separately. The behavior of the accounts in these periods can be described by time and the stochastic changes in the primary account and the risk-free rate of interest. Since the state processes and the secondary account are deterministic between updates, it is only necessary to take these into consideration at the updates. By carefull definition of the connections between the results in separate time periods, it is possible to construct a scheme for each of the various products presented earlier.

We define a grid on the three-dimensional state space consisting of time \( t \) between updates, the value of the primary account \( x \) from 0 to an upper limit \( x_{\text{max}} \) and the risk-free rate of interest \( r \) between chosen limits \( r_{\text{min}} \) and \( r_{\text{max}} \). Using interval lengths \( \Delta t \), \( \Delta x \) and \( \Delta r \) and indexing the grid point in each direction by \( l = 0, \ldots, L \), \( k = 0, \ldots, K \) and \( q = 0, \ldots, Q \) respectively, we define the function

\[
W(t_i + l \Delta t, r_{\text{min}} + q \Delta r, k \Delta x, y, x, y)
\]

on the grid in period \( i + 1 \). Within each period we omit the \( i \) and all nondeterministic variables and use the shorter notation

\[
W_{l,k,q}^i.
\]

The following proposition can then be interpreted as a discrete version of Proposition 1.

**Proposition 3** Assume that certain convergence conditions on \( \Delta t \), \( \Delta x \) and \( \Delta r \) are fulfilled. Let the function \( W \) be constructed on the grid such that

\[
\begin{align*}
W_{l,k,q}^{i-1} &= \Delta t \left[ \left( \frac{1}{\Delta t} - r - \frac{\sigma^2 k^2}{\Delta x^2} - \frac{\sigma^2 k^2}{\Delta r^2} \right) W_{l,k,q}^i + \left( \frac{\sigma^2 k^2}{2 \Delta x^2} - \frac{\sigma^2 k^2}{2 \Delta r^2} \right) W_{l-1,k,q}^i \\
&\quad + \left( \frac{\sigma^2 k^2}{2 \Delta x^2} + \frac{\sigma^2 k^2}{2 \Delta r^2} \right) W_{l+1,k,q}^i + \left( \frac{\sigma^2 k^2}{4 \Delta x^2} - \frac{\sigma^2 k^2}{4 \Delta r^2} \right) W_{l,k,q-1}^i \\
&\quad + \left( \frac{\sigma^2 k^2}{4 \Delta x^2} + \frac{\sigma^2 k^2}{4 \Delta r^2} \right) W_{l,k,q+1}^i \\
&\quad + \frac{\beta \sigma k^2}{4 \Delta r^2} \left( W_{l,k-1,q-1}^i + W_{l,k+1,q+1}^i - W_{l,k+1,q-1}^i - W_{l,k-1,q+1}^i \right) \right].
\end{align*}
\]

If \( W \) fulfills

\[
c(t, x, y) = W(t_{i+1}, r, x + c + \phi - \pi, y + c + \delta, x + c + \phi - \pi, y + c + \delta) = W(t_i + L \Delta t, r, x, y, x, y)
\]

then

\[
W(t, r, x, y, x, y) \approx V(t, r, x, y, x, y).
\]

We have applied this method to three of the products presented earlier. For each of these we have approximated the value of the contract over a ten-year period with yearly updates, where the
only payment made by the contract holder is a contribution of 1 at time 0. We have chosen the parameters for each contract type so that the value of the contract equals the contribution at time 0, i.e. \( V(0) = 1 \). The corresponding parameters depend on the parameters for the financial market. We have let the volatility \( \sigma \) of the portfolio process and the correlation factor \( \rho \) reflect a typical proportion of stocks and bonds for each contract type in the Danish market. The calculations are based on interest rate parameters \( (\alpha, b, \gamma) = (0.213, 0.007445, 0.015) \).

Applying the above method leads to approximations to the value of each contract for each combination of time, short rate of interest and primary account value in the grid. We have plotted the values for each contract following the same randomly generated underlying portfolio process over the ten years. All values plotted are calculated given that the short rate of interest at each point in time is equal to 0.035. Thus we let the primary account vary over time and hold the short rate of interest fixed.

In Figure 1 we have plotted the partial look back guarantee unit-link insurance. Based on the Danish market version of this product, we have chosen \( \alpha = 0.95 \) and \( r^* = 0 \). Our calculations have shown that \( \kappa = 0.01 \) fulfills the condition \( V(0) = 1 \), when \( (\sigma, \rho) = (0.112, 0.089) \). These parameters reflect an assumed proportion of stocks in the underlying portfolio of 45%.

![Figure 1: Partial look back guarantee unit-link insurance](image)

This product pays out the primary account plus a put option. The strike may be increased over time, and a premium is paid from the primary account yearly. This explains why the value of the contract is close to the value of primary account when it is 'in-the-money'. When the primary account moves 'out-of-the-money', the value of the contract tends to the value of the secondary account discounted by interest. As for a put-option, the value of the contract would always be higher than the primary account, were it not for the yearly premium. The contract is made fair by a choosing \( \kappa \) in the primary claim premium.

In Figure 2 we have shown the linear regulator unit-link insurance. Inspired by the version in the Danish market, we have set \( \alpha = 0.2, \kappa = 0 \) and used portfolio parameters \( (\sigma, \rho) = (0.164, 0.255), \)
reflecting a stock proportion of 70%. The condition \( V(0) = 1 \) can then be fulfilled by careful choice of the rate of return \( r^* \) on the secondary account. Not surprisingly we found that this could be done by choosing a \( r^* \) close to the short rate of interest 0.035.

Figure 2: Linear regulator unit-link insurance

In this product the primary account is not affected by claims or premiums. The secondary account, which is paid out at termination, contains a smoothing of the two accounts over time. Therefore the value of the contract is always between the values of the two accounts, moving closer to the secondary account as time nears termination.

The final example is the Danish market with-profit life insurance with interest rate guarantee and where the primary claim is based on the present value of the secondary account. Results are shown in Figure 3. We have set the guaranteed rate of interest such that \( r^* = 0.015 \), which is the current level in the Danish market. We have chosen \( \alpha = 0.2 \) and portfolio parameters \((\sigma, \rho) = (0.084, -0.153)\), reflecting a stock proportion of 30%. The condition \( V(0) = 1 \) can be fulfilled by choosing \( \kappa = 0.004 \).

It is anticipated that most of the excess of accounts is paid out over time, and therefore the value of the contract is pulled towards the primary account when it is higher than the secondary account. Since the secondary account has a guaranteed rate of return, the value of the contract only moves slightly below this as the primary account drops.

At the updates 5 and 4 years prior to termination, the primary account is set equal to the secondary account. Because of these increments, the primary account ends with a significantly higher value than in the previous two examples. This increase of the primary account is paid for by the financial institution, and it is primarily for the risk of this happening, that the primary claim premium is paid.

When comparing these three examples it is important to keep in mind, that they are plotted for the same underlying portfolio process. This particular path was arbitrarily chosen for its ability to show the various properties of these three contracts and does not reflect the different portfolio
The primary account
The secondary account
The value of the contract

Figure 3: Danish market with-profit life insurance compositions in the market versions of the products.

6 A Third Account

So far we have presented a pension contract model consisting of two contracts which affect each other at predetermined updates through claims and a claim premium. Careful definition of the claims, the claim premium and a terminal benefit have been used to construct a number of pension contracts.

However, some pension contracts in the market and the literature contain characteristics which cannot be represented adequately without defining a tertiary account in addition to $X$ and $Y$. It would be mathematically trivial to expand our model with such a tertiary account and a corresponding tertiary claim. The existing claims, the claim premium and the terminal benefit could then easily take this new process into account as well.

In the present article we have chosen to restrict our model to two accounts in order to keep the presentation clear and concise. The addition of a third account would add very little to the basic idea of our model, and it would only be used in two of our examples in section 3. Nevertheless, we now outline its role in these two cases.

In the Danish market with-profit life insurance, the primary claim premium $\pi = kx$ is typically limited by the excess of returns as described in Section 3. Danish legislation permits keeping track of primary claim premiums not claimed due to this limitation, and subsequently claiming them when they can be contained in the excess of returns. Similarly, it is permitted to keep track of primary claims and reclaiming them when they can be contained in the excess of accounts.

The primary claims and the unclaimed parts of primary claim premiums are accumulated on a so-called shadow account. In our model we would let this shadow account constitute the tertiary
account, which could then affect the claims and the claim premium. It is important to note here that the term *shadow account* is used very differently in Danish accounting practice compared with recent developments in international accounting standards.

In their variation of the Danish market with-profit life insurance, *Hansen and Miltersen (2002)* describe an indirect method of payment collection. There the primary claim premiums are collected in a separate account, which in turn affects the following secondary claims and primary claim premiums. As in their model, we could naturally let the tertiary account keep track of this accumulated value of primary claim premiums and adjust our secondary claim and primary claim premium accordingly.

**References**


