Asymptotics of ruin probabilities for risk processes under optimal reinsurance policies: the small claim case

Hanspeter Schmidli*

Laboratory of Actuarial Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

Abstract
We consider a classical risk model with the possibility of reinsurance. Moreover, in one of the models also investment into a risky asset is possible. The insurer follows the optimal strategy. In this paper we find the Cramér-Lundberg approximation in the small claim case and prove that the optimal strategy converges to the asymptotically optimal strategy as the capital increases to infinity.

1991 Mathematical Subject Classification: Primary 60F10; Secondary 60G35, 65K10

Key words: ruin probability, optimal control, Cramér-Lundberg approximation, adjustment coefficient, light tails, martingale methods, change of measure, geometric Brownian motion, Hamilton-Jacobi-Bellman equation

1. Introduction

Let $S_t = \sum_{i=1}^{N_t} Y_i$ be the aggregate claims process of an insurance portfolio, where $\{N_t\}$ is a Poisson process with rate $\lambda$. The claim sizes $\{Y_i\}$ are iid, strictly positive and independent of the claim arrival process. We denote by $Y$ a generic random variable, by $M_Y(r) = \mathbb{E} [\exp \{ rY \}]$ its moment generating function and by $G(y)$ its distribution function. All stochastic quantities are defined on a large enough complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

*Part of the research was done while the author visited the Department ‘Matematica per le Decisioni’ at the University of Firenze. The Author thanks for the hospitality at the department.
The insurer follows a strategy \((A(u), b(u))\) of feedback form, where \((A(u), b(u)) \in \mathcal{A} \subset [0, \infty) \times [0, 1]\). The following cases had been investigated in [3], [8], [10]:

\[
\begin{align*}
\mathcal{A} &= [0, \infty) \times \{1\}, & \text{no reinsurance,} \\
\mathcal{A} &= \{0\} \times [0, 1], & \text{no investment,} \\
\mathcal{A} &= [0, \infty) \times [0, 1], & \text{investment and reinsurance.}
\end{align*}
\]

\(A(u)\) denotes the amount invested into a risky asset, modelled as a geometric Brownian motion

\[dZ_t = \mu Z_t dt + \sigma Z_t dW_t,\]

where \(\{W_t\}\) is a standard Brownian motion independent of \(\{S_t\}\). We assume here that all economic quantities are discounted. In particular, the claim sizes increase with inflation and the amount “not invested” is put on a bank account or invested in a riskless bond. It is even possible to borrow money at the same rate. The latter can be interpreted that the portfolio under consideration has a debt to the capital resources of the company. The parameters fulfill \(\mu, \sigma > 0\).

\(b(u)\) is the retention level in proportional reinsurance, i.e. if a claim \(Y\) occurs at the time where the surplus is \(u\) (prior to the claim payment) then the insurer pays \(b(u)Y\) and the reinsurer pays \((1 - b(u))Y\). In order to get this reinsurance cover the insurer has to pay a continuous premium at rate \(c(b(u))\). As in [10] we assume that \(c(b)\) is strictly decreasing, \(c(1) = 0\), and that \(c < c(0) < \infty\), where \(c\) is the rate at which the insurer gets premiums. We have chosen here proportional reinsurance for simplicity. Other types of reinsurance can be treated similarly.

Note that the asymptotic behaviour of the ruin probability is completely different to the behaviour in the case where a certain fraction of the surplus has to be invested into the risky asset. If investment is possible we always get an exponential bound (even for large claims, see [11]), whereas in the case where a constant fraction has to be invested the ruin probability always decays like a power function, see [5].
In this paper we work with the natural filtration \( \{\mathcal{F}_t\} \) of \( \{(S_t, W_t)\} \), i.e. the smallest right continuous filtration such that \( \{(S_t, W_t)\} \) is adapted. Note that we cannot complete the filtration because we want to change the measure later. The filtration has to be right continuous in order that the ruin time defined below is a stopping time.

Under the chosen strategy the surplus process is

\[
dX_t = (c - c(b(X_t)) + \mu A(X_t)) \, dt + \sigma A(X_t) \, dW_t - b(X_{t-}) \, dS_t, \quad X_0 = u. \tag{1}
\]

The time of ruin is \( \tau^{A,b} = \inf\{t \geq 0 : X_t < 0\} \) and the ruin probability \( \psi^{A,b}(u) = \mathbb{P}[\tau^{A,b} < \infty] \). The control function is \( \psi(u) = \inf_{A} \psi^{A,b}(u) \). In order that \( \psi(u) < 1 \) we have to assume that \( c > \lambda \mathbb{E}[Y] \) in the case without investment. If investment is possible the positive safety loading can be achieved by investment.

The following result has been proved in [8] and [10].

**Proposition 1.** Suppose there is an increasing function \( \delta(u) \) solving the Hamilton-Jacobi-Bellman equation

\[
sup_{(A,b) \in A} \frac{1}{2} \sigma^2 A^2 \delta''(u) + (c - c(b) + \mu A) \delta'(u) + \lambda(\mathbb{E}[\delta(u - bY)] - \delta(u)) = 0, \tag{2}
\]

where \( \delta(u) = 0 \) for \( u > 0 \). If investment is possible suppose that \( \delta(u) \) is twice continuously differentiable on \((0, \infty)\). Then \( \delta(u) \) is bounded and \( \delta(u) = \delta(\infty)(1 - \psi(u)) \). Moreover, the arguments \((A(u), b(u))\) in (2) maximising the left hand side determine the optimal strategy \( \{(A(X_t), b(X_{t-}))\} \). If no investment is possible then there exists a solution to (2). If investment is possible and the claim sizes have a bounded density then there exists a twice continuously differentiable increasing solution to (2).

\[\square\]

Note that if investment is possible then any increasing solution to (2) is concave, yielding that \( \psi(u) \) is convex.
We therefore suppose that $\psi(u)$ is twice continuously differentiable. Then $\psi(u)$ solves
\[
\inf_{(A,b) \in A} \frac{1}{2} \sigma^2 A^2 \psi''(u) + (c - c(b) + \mu A) \psi'(u) + \lambda [\mathbb{E}[\psi(u - b Y)] - \psi(u)] = 0 \quad (3)
\]
where we let $\psi(u) = 1$ for $u < 0$. In the following sections we investigate the asymptotic behaviour of $\psi(u)$ as $u \to \infty$ as well as the asymptotic behaviour of the strategies $(A(u), b(u))$. Part of this problem has been solved in [4]. There the case without reinsurance had been considered. Several technical problems did not appear in this case that show up in the case with reinsurance. We therefore assume here always that reinsurance is possible. We will in this paper only consider the small claim case, i.e. exponential moments of the claim size distribution exist. The large claim case had been considered in [11].

2. Lundberg bounds and the change of measure formula

We assume that the distribution tail $1 - G(y)$ is decreasing exponentially fast. More specifically, we assume that $M_Y(R) < \infty$ and $M'_Y(R) < \infty$ for $R > 0$ defined below. These conditions are exactly the conditions needed for $\psi^{0,1}(u) \sim C e^{-Ru}$ in the classical risk model, see [6], and in the case without reinsurance, see [4].

We start by defining the Lundberg exponent $R$. Let $r = R(A, b)$ be the strictly positive solution to
\[
\lambda (M_Y(b r) - 1) - (c - c(b) + \mu A) r + \frac{1}{2} \sigma^2 A^2 r^2 = 0. \quad (4)
\]
$R(A, b)$ is the Lundberg exponent in the case of a constant strategy $(A, b)$. The Lundberg exponent for our problem is $R = \sup_{(A,b) \in A} R(A, b)$. This is, we maximise the Lundberg exponent in order to obtain an asymptotically optimal constant strategy. In the case without investment this problem is discussed in [12]. Note that the function on the left hand side of (4) is positive at $r = R$, and therefore the optimal parameters $(A^*, b^*)$ minimise (4) at $r = R$. 

4
Example 1. Consider the case where investment and proportional reinsurance is possible. The reinsurer uses an expected value principle, i.e. \( c(b) = (1 + \theta)(1 - b)\lambda \mathbb{E}[Y] \). Taking the derivative with respect to \( A \) in (4) gives an equation for \( \partial/(\partial A) R(A, b) \) by the implicit function theorem. Letting this derivative to be zero gives \( RA = \mu/\sigma^2 \). Taking the derivative with respect to \( b \) yields \( \lambda M_Y'(bR) = (1 + \theta)\lambda \mathbb{E}[Y] \). Because \( M_Y'(r) \) is a continuous increasing function a value for \( bR \) can be obtained. Plugging in these values into (4) yields \( R \), and therefore \( b \) and \( A \). It is possible that \( b > 1 \). In this case one has to set \( b = 1 \) and then to solve

\[
\lambda(M_Y(r) - 1) - cr - \frac{1}{2}\mu^2/\sigma^2 = 0.
\]

If the claim sizes are exponentially distributed with parameter \( \alpha \) we get \( bR = \alpha(1 - \sqrt{1/(1 + \theta)}) \). This yields

\[
R = \frac{\lambda(\sqrt{1 + \theta} - 1)^2 + \frac{1}{2}\mu^2/\sigma^2}{\lambda(1 + \theta)/\alpha - c},
\]

which is positive because \( c < c(0) \). The optimal \( b \) is now easily obtained.  

We denote the asymptotically optimal constant strategy by \( (A^*, b^*) \), that is \( R = R(A^*, b^*) \). From the considerations above it is clear that \( A^* = \mu/(R\sigma^2) \) is unique. Waters [12] gives conditions under which also \( b^* \) is unique.

We now prove an upper Lundberg bound.

**Proposition 2.** There exists a constant \( 0 < c_+ \leq 1 \) such that \( \psi(u) \leq c_+ e^{-Ru} \).

**Proof.** Choose the constant strategy that maximises the Lundberg coefficient. If investment is possible the result follows from [1]. If no investment is possible, the result is the Lundberg inequality for the classical risk process, see [6]. □

For simplicity we suppose that the process is stopped at ruin, i.e. \( dX_t = 0 \) for \( t > \tau \). Consider now the process

\[
M_t = \exp \left\{ -R(X_{t\wedge \tau} - u) - \int_0^{t\wedge \tau} \theta(X_s) \, ds \right\},
\]
where
\[
\theta(u) = \lambda(M_Y(b(u)R) - 1) - (c - c(b(u)) + \mu A(u))R + \frac{1}{2}\sigma^2 A^2(u)R^2.
\]

The function \(\theta(u)\) is positive and zero exactly if \((b(u), A(u))\) are the values maximising the Lundberg coefficient. We will later change the measure and readily get the lower bound.

**Lemma 1.** The process \(\{M_t\}\) is a martingale with mean value 1.

**Proof.** The result follows in the same way as in [4].

The martingale \(\{M_t\}\) can be used to change the measure on \(\mathcal{F}_t\). We denote the measure by \(\mathbb{P}^*\), that is \(\mathbb{P}^*[A] = \mathbb{E}[M_t; A]\). It turns out that the measure is independent of \(t\). It will become clear from the lemma below that \(\mathbb{P}^*\) can be extended to \(\mathcal{F}\). However, the two measures are singular on \(\mathcal{F}\). Let \(T\) be a stopping time. Then for \(A \in \mathcal{F}_T \cap \{T < \infty\}\), \(\mathbb{P}^*[A] = \mathbb{E}[M_T; A]\). For details see [7] or [6].

**Lemma 2.** Under the measure \(\mathbb{P}^*\), the process \(\{X_t\}\) is a jump diffusion process with location dependent parameters. The claim intensity is \(\lambda^*_u = \lambda M_Y(b(u)R)\), the claim size distribution (that is the distribution of \(Y\), the jump size is \(b(u)Y\)) is \(dG^*_u(y) = e^{b(u)Ry} dG(y) / M_Y(b(u)R)\), the drift parameter is \(c^*_u = c - c(b(u)) + \mu A(u) - \sigma^2 A^2(u)R\) and the diffusion parameter is \(\sigma^*_u = \sigma\). Moreover, \(\mathbb{P}^*[\tau < \infty] = 1\), and therefore
\[
\psi(u) = \mathbb{E}^*\left[\exp\left\{RX_\tau + \int_0^\tau \theta(X_s)\right\}\right] e^{-Ru}.
\]

In particular, the expected value is bounded by \(c_+\).

**Proof.** That \(\{X_t\}\) remains a strong Markov process follows by direct verification similarly as in [9]. Calculation of the generator yields the result similarly as in [9]. The infinitesimal drift of the process is
\[
c - c(b(u)) + \mu A(u) - \sigma^2 A^2(u)R - \lambda b(u)M_Y(b(u)R).
\]
This is minus the derivative of (4) with respect to $r$. Because (4) is convex in $r$, has a zero at zero and $R(A(u), b(u))$ and $R \geq R(A(u), b(u))$ the derivative must be strictly positive. This means that the process $\{X_t\}$ has a negative drift, implying that $\mathbb{P}^*[\tau < \infty] = 1$. 

We can now easily find a lower Lundberg bound. The following result had been proved in [2] and [4] in the case of no reinsurance.

**Proposition 3.** There exists a constant $c_- > 0$ such that $\psi(u) \geq c_- e^{-Ru}$.

**Proof.** The expected value in (5) is bounded from below by $\mathbb{E}^*[e^{RX_\tau}]$. Conditioning on $X_{\tau-}$ yields

$$\mathbb{E}^*[e^{RX_\tau} \mid X_{\tau-} = y] = \mathbb{E}^*[e^{R(y-b(y)Y)} \mid Y > y/b(y)] = \frac{e^{Ry} \int_{y/b(y)}^{\infty} dG(z)}{\int_{y/b(y)}^{\infty} e^{Rb(y)z} dG(z)}.$$

Taking the infimum over all $y$ (we omit the condition $G(y/b(y)) < 1$) one obtains

$$\mathbb{E}^*[e^{RX_\tau}] \geq \inf_{y \geq 0, b \in (0,1]} \frac{1 - G(y)}{\int_{y}^{\infty} e^{Rb(z-y)} dG(z)} \geq \inf_{y \geq 0} \frac{1 - G(y)}{\int_{y}^{\infty} e^{R(z-y)} dG(z)} = c_-.$$

That $c_- > 0$ follows from the classical case, see [6]. 

3. The Cramér-Lundberg approximation

In this section we consider formally the case where investment and reinsurance is possible. If we let $\mu = 0$ in the calculations below then the optimal strategies are $A(u) = A^* = 0$ and therefore the case with no investment follows. We only need to consider the two cases separately if properties of the second derivative are used because we do not assume existence of the second derivative in the case without investment.
Taking the infimum over $A$ in (3), the Hamilton-Jacobi-Bellman equation reads
\[\inf_{b} - \frac{\mu^2}{2\sigma^2} \psi'(u)^2 + (c - c(b))\psi'(u) + \lambda \left( \int_{0}^{u/b} \psi(u-by) \, dG(y) + 1 - G(u/b) - \psi(u) \right) = 0.\]

(6)

$b$ can be replaced by $b(u)$. Let $f(u) = \psi(u)e^{R_u}$. Then
\[\frac{-\mu^2}{2\sigma^2} \frac{(Rf(u) - f'(u))^2}{R^2f(u) - 2Rf'(u) + f''(u)} - (c - c(b(u)))(Rf(u) - f'(u))
+ \lambda \left( \int_{0}^{u/b(u)} f(u - b(u)y)e^{Rb(u)y} \, dG(y) + (1 - G(u/b(u)))e^{R_u} - f(u) \right) = 0.\]

Note that $Rf(u) - f'(u) > 0$ and $R^2f(u) - 2Rf'(u) + f''(u) > 0$ by the corresponding properties of $\psi(u)$.

From the definition of $R$ we have
\[\lambda(M_Y(b^*R) - 1)f(u) - (c - c(b^*))Rf(u) - \frac{\mu^2}{2\sigma^2}f(u) = 0.\]

Taking the difference to the above equation yields
\[\frac{\mu^2}{2\sigma^2} \frac{f''(u)f(u) - f'(u)^2}{R^2f(u) - 2Rf'(u) + f''(u)} + (c - c(b^*))f'(u) + (c(b(u)) - c(b^*))(Rf(u) - f'(u))
+ \lambda \left( \int_{0}^{u/b(u)} f(u - b(u)y)e^{Rb(u)y} \, dG(y) + (1 - G(u/b(u)))e^{R_u} ight)
- \lambda M_Y(b^*R)f(u) = 0.\]

(8)

Let $g(u) = Rf(u) - f'(u) = -\psi'(u)e^{R_u}$. Note that $g(u) > 0$ and $g'(u) < Rg(u)$. From (8) it follows that $f'(u)$ is bounded also from below, and therefore $g(u)$ is bounded. Indeed, consider a point where $u$ is large, $f'(u)$ is very small and $f''(u) = 0$. Then $b = b^*$ would yield a negative value. Because the infimum has to be taken at $b = b(u)$ this is not possible. Equation (7) reads then
\[\frac{-\mu^2}{2\sigma^2} \frac{g^2(u)}{Rg(u) - g'(u)} - (c - c(b(u)))g(u) + \lambda \int_{0}^{u} g(u - y)e^{R_y(1 - G(y/b(u)))} \, dy
+ \lambda(1 - \psi(0))e^{R_u(1 - G(u/b(u)))} = 0.\]

Replacing $b(u)$ by $b^*$ yields
\[\frac{-\mu^2}{2\sigma^2} \frac{g^2(u)}{Rg(u) - g'(u)} - (c - c(b^*))g(u) + \lambda \int_{0}^{u} g(u - y)e^{R_y(1 - G(y/b^*))} \, dy
+ \lambda(1 - \psi(0))e^{R_u(1 - G(u/b^*))} \geq 0.\]
From the definition of $b^*$ we get

$$
- \frac{\mu^2}{2\sigma^2 R} \frac{g(u)g'(u)}{Rg(u) - g'(u)} + \lambda \int_0^u (g(u - y) - g(u))e^{Ry(1 - G(y/b^*))} dy
+ \lambda g(u) \int_u^\infty e^{Ry(1 - G(y/b^*))} dy + \lambda(1 - \psi(0))e^{Ru(1 - G(u/b^*))} \geq 0.
$$

Note that the last two terms tend to zero as $u \to \infty$.

Let $\zeta = \lim_{u \to \infty} g(u)/R$. Similarly as in [4] it follows that $\lim_{u \to \infty} f(u) = \lim_{u \to \infty} g(u)/R = \zeta$ and we can find arbitrarily large intervals on which $g(u)$ and $f(u)$ are close to $R\zeta$ and $\zeta$, respectively.

This enables to prove our main result.

**Theorem 1.** There exists a constant $\zeta \in (0, 1]$ such that $\lim_{u \to \infty} \psi(u)e^{Ru} = \zeta$.

**Proof.** The proof follows similarly as in [4]. That $\zeta \in (0, 1]$ follows from Proposition 3. \hfill \Box

### 4. Convergence of the strategies

Consider first the case with reinsurance. Choose $0 < \varepsilon < R\zeta/2$ and let $\delta = \varepsilon^2/(4R^2\zeta)$. Choose $u_0$ such that for all $u > u_0$, $|f(u) - \zeta| < \delta$. Suppose there is $u_1 > u_0$ such that $g(u_1) < R\zeta - 2\varepsilon$. Then $g(u) < R\zeta - \varepsilon$ for all $u_1 \leq u \leq u_1 + \varepsilon/(R^2\zeta)$ because $g'(u) < Rg(u)$. From

$$
f'(u) = Rf(u) - g(u) > R(\zeta - \delta) - (R\zeta - \varepsilon) = \varepsilon - R\delta
$$

it follows that

$$
f\left(u_1 + \frac{\varepsilon}{R^2\zeta}\right) > \zeta - \delta + (\varepsilon - R\delta)\frac{\varepsilon}{R^2\zeta} > \zeta + \delta
$$

which is a contradiction. Since $\varepsilon$ is arbitrary $\lim_{u \to \infty} g(u) = R\zeta$. From

$$
R\zeta = \lim_{u \to \infty} g(u) = R\zeta - \lim_{u \to \infty} f'(u)
$$
it follows that \( \lim_{u \to -\infty} f'(u) = 0 \).

Because the second derivative is used in this argument the prove cannot be used for the case without investment. From \( f'(u) = Rf(u) - g(u) \) it follows immediately from \( \zeta = \lim_{u \to -\infty} f(u) = \lim_{u \to -\infty} g(u)/R \) that \( \lim_{u \to -\infty} f'(u) = 0 \). Let \( u_n \) be a sequence tending to infinity such that \( f'(u_n) \) converges to \( \eta \geq 0 \) and \( b(u_n) \) converges to \( b_0 \). In the limit (7) is \((\mu = 0)\)

\[
(c - c(b_0))\eta - (c - c(b_0))R\zeta + \lambda\zeta(M_Y(b_0R) - 1) = 0.
\]

Because \( R(0, b_0) \leq R \) it follows that \( \eta \leq 0 \). Thus \( \lim f'(u) = 0 \).

In [4] it is shown that the strategy \( A(u) \) converges in the case without reinsurance. In [8, 10] it was conjectured that the strategy \( b(u) \) converges to the asymptotically optimal \( b^* \). This is motivated by the optimal rate \( e^{-Ru} \) which corresponds to the strategy \((A^*, b^*)\). It is clear that, if \( b(u) \) converges, the limit must be \( b^* \). We now prove convergence of the strategy \((A(u), b(u))\).

**Theorem 2.** Suppose that \( b^* \) is uniquely defined. Then \( \lim_{u \to -\infty} b(u) = b^* \). Suppose investment is possible. Then \( \lim_{u \to -\infty} A(u) = A^* \).

**Proof.** Let \( \{u_n\} \) be a sequence tending to infinity such that \( b(u_n) \) and \( f''(u_n) \) converge (in the case without investment convergence of \( f''(u_n) \) in not necessary). Denote the limit of \( b(u_n) \) by \( b_0 \) and the limit of \( f''(u_n) \) by \( \kappa \). The limit of (7) is

\[
-\frac{\mu^2}{2\sigma^2} \frac{R^2\zeta^2}{R^2\zeta + \kappa} - (c - c(b_0))R\zeta + \lambda(M_Y(b_0R) - 1)\zeta = 0.
\]

Replacing \( b_0 \) by \( b^* \) yields

\[
-\frac{\mu^2}{2\sigma^2} \frac{R^2\zeta}{R^2\zeta + \kappa} - (c - c(b^*))R + \lambda(M_Y(b^*R) - 1) \leq 0.
\]

By the definition of the Lundberg exponent

\[
\frac{\mu^2}{2\sigma^2} \leq \frac{\mu^2}{2\sigma^2} \frac{R^2\zeta}{R^2\zeta + \kappa}.
\]
Replace in (7) \( b(u_n) \) by \( b^* \) and let then \( n \to \infty \). This yields

\[
- \frac{\mu^2}{2\sigma^2} \frac{R^2 \zeta}{R^2 \zeta + \kappa} - (c - c(b^*))R + \lambda(M_Y(b^* R) - 1) \geq 0.
\]

Thus

\[
\frac{\mu^2}{2\sigma^2} \geq \frac{\mu^2}{2\sigma^2} \frac{R^2 \zeta}{R^2 \zeta + \kappa}.
\]

In particular, \( \kappa = 0 \), i.e. \( \lim f''(u) = 0 \). This yields for the trading strategy

\[
\lim_{u \to \infty} A(u) = \lim_{u \to \infty} \frac{\mu}{\sigma^2} \frac{R f(u) - f'(u)}{R f(u) - 2 R f'(u) + f''(u)} = \frac{\mu}{R \sigma^2} = A^*.
\]

The limit of (7) is then

\[
- \frac{\mu^2 \zeta}{2\sigma^2} - (c - c(b_0))R \zeta + \lambda(M_Y(b_0 R) - 1) \zeta = 0.
\]

Because \( b_0 \) is unique we have \( b_0 = b^* \).

**Remark.** If \( b^* \) is not unique \( b_0 \) can be any of the points where \( R(A^*, b) \) is maximal. In order to find the limit of \( b(u) \) (if a limit exists) one needs to determine close to which point of maximisation \( b(u) \) lies for large \( u \).

**References**


