WORKSHOP ON STRING TOPOLOGY AND RELATED TOPICS

NOTES TAKEN BY RICHARD HEPWORTH

ABSTRACT. These are the notes from the talks in the *Workshop on string topology and related topics*, which is taking place in Copenhagen April 15th – 19th 2013.

Note-taker's note: I have tried to replicate, as closely as possible, what happened in the talks. I particular I have incorporated some of the comments that the speakers made but perhaps did not write down. There will be many errors and infidelities. The reader should bear this in mind. My apologies to the authors for any misrepresentations.

CONTENTS

1. David Chataur — Loop products and closed geodesics I

We will talk about how to do some computations of homology of free loop spaces. I don't want to give you some motivation on why we compute the homology and cohomology of free loop spaces — Nancy will give motivation from Riemannian geometry. For me it is just fun. I don't think that I will convince you of this by computing spectral sequences, but I will try.

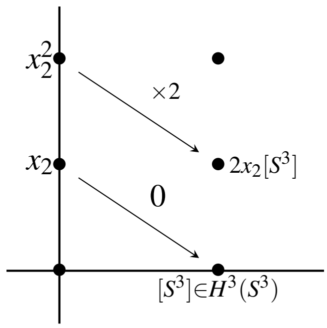
1) How to compute $H_*(LM,R)$? First, how to compute $H^*(LM,R)$? (Before Chas-Sullivan, it was better to compute cohomology.) At first sight $H^*(-)$ is easier to compute because of the cup product.

(a). Consider the fibration $\Omega_m M \to LM \xrightarrow{ev_0} M$ in which ev_0 is $\gamma \mapsto \gamma_0$, and apply the Serre spectral sequence. Need to know $H^*(\Omega_m M)$.

Example. $H^*(\Omega_m S^n)$ computed by Serre.

$$\Omega_m M \to P_m M = \{\gamma \colon I \to M,\ \gamma_0 = m\} \xrightarrow{ev_1} M$$

$P_m M$ is contractible. Taking $n = 3$ we get the following.



For S^{2n+1} what you get is

$$H^*(\Omega_m S^{2n+1};\mathbb{Z}) = \Gamma(x_{2n}) = \mathbb{Z}\left[x_{2n}, \frac{x_{2n}^2}{2}, \dots, \frac{x_{2n}^k}{k!}, \dots\right] \subset \mathbb{Q}[x_{2n}],$$

the algebra of divided powers of x_2. This is infinitely generated. So you can imagine what happens if you add cells — you will have lots of divided power algebras. This phenomenon explains why it is not so easy to compute the cohomology of the based loop space, and hence of the free loop space.

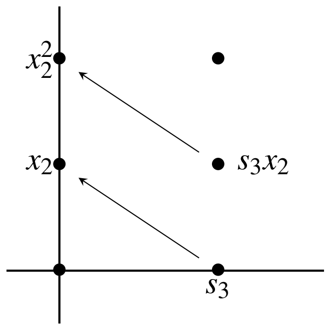
b). $HH_*(-,-)$, rational homotopy theory, Eilenberg-Moore spectral sequence. We do not know the algebra structure of $H^*(L(\mathbb{C}P^n \# \mathbb{C}P^n;\mathbb{Q})$ for $n \geqslant 3$. Thanks to the work of Pascal Lambrechts (Topology, '90s) we know the Betti numbers, but not the algebra structure.

1) Second, the homology $H_*(LM;R)$. Let us look at Serre spectral sequences. We need to know $H_*(\Omega_m M;R)$. It is an algebra: product is Pontryagin product given by composition of loops. Let us play the same game: we want to compute the homology, and we want to compute it along with the Pontryagin product. Later on we will see that Morse theory is a good way to do that. Just now we will do it with the James construction. The answer is that $H_*(\Omega_m S^n) \cong_{alg} T(x_{n-1})$. To get this, replace $\Omega_m M$ by Moore loops in order to make the product associative.

$$\Omega_m^M M = \{\gamma\colon [0,T] \to M \mid T \geqslant 0,\ \gamma(0) = \gamma(T)\}$$

This is an associative monoid. Look at the map $\lambda\colon S^{n-1} \to \Omega_m^M S^n$, (it sends x to a based loop that intersects the 'equator' once, at the point x) and take the adjoint. Extend λ to a multiplicative map $\mu\colon J(S^{n-1}) \to \Omega_m^M S^n$ where $J(-)$ is the free topological monoid. Now μ is a weak homotopy equivalence and taking homology of $J(S^{n-1})$ you get $T[x_{n-1}]$.

What is the spectral sequence for $\Omega S^3 \to LS^3 \to S^3$? (We have to forget that S^3 is a Lie group, because that would make it simpler!)



Here we have an algebra in fact. Because if you use Pontrjagin product and intersection product this thing becomes an algebra. So if we have a product on homology of free loop space that has a product that combines these two, then here you only need to compute one differential because then the remainder are determined by the algebra structure. But the differential of s_3 is zero because there is a section of $LS^3 \to S^3$ given by the constant loops. So now you only need to compute the differential of s_3x_2.

Now we will see how to compute homology of free loop spaces using the spectral sequence of Cohen, Jones and Yan.

Chas -Sullivan loop product. It mixes Pontryagin product and intersection product. Consider the following maps.

$$\begin{array}{ccccc} LM & \xleftarrow{comp} & \mathrm{Map}(\infty, M) & \xrightarrow{\widetilde{\Delta}} & LM \times LM \\ & & \downarrow & & \downarrow {\scriptstyle ev_0 \times ev_0} \\ & & M & \xrightarrow[\Delta]{} & M \times M \end{array}$$

For finite dimensional closed oriented manifolds you have Gysin maps. Here we want to do the same thing to $\widetilde{\Delta}_!$.

$$H_*(LM \times LM; R) \xrightarrow{\widetilde{\Delta}_!} H_{*-\dim(M)}(\mathrm{Map}(\infty, M); R)$$

This can be done in several ways. Chas-Sullivan used intersection theory of cycles, by making cycles transverse to the diagonal and then taking intersections. This can be made precise, as in a very nice note by Francois Laudenbach. There are other techniques involving tubular neighbourhoods. So we get

$$\begin{aligned} H_i(LM;R)\otimes H_j(LM;R) &\xrightarrow{\times} H_{i+j}(LM\times LM;) \\ &\xrightarrow{\widetilde{\Delta}_!} H_{i+j-\dim(M)}(\mathrm{Map}(\infty,M)) \\ &\xrightarrow{comp_*} H_{i+j-\dim(M)}(LM) \end{aligned}$$

and this is the Chas-Sullivan product. In the fibre this product is the Pontrjagin product, and in the base it is the intersection product. The product is in fact commutative.

Now from a purely computation point of view if you look at the Serre spectral sequence, it is compatible with this product.

Theorem (Cohen, Jones, Yan)**.** *If M is a closed, simply connected smooth manifold of dimension d then the Serre spectral sequence*

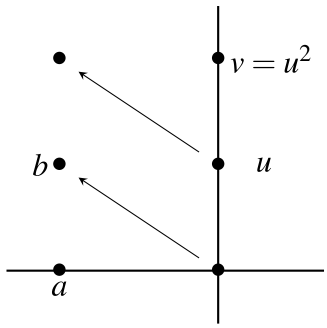
$$\mathbb{E}_{p,q}=\mathbb{H}_p(M;H_q(\Omega_m M))\Longrightarrow \mathbb{H}_*(LM)$$

of

$$\Omega_m M\to LM\xrightarrow{ev_0} M$$

is multiplicative.
Notation: $\mathbb{H}_*(M;R)=H_{*+d}(M;R)$ and $\mathbb{H}_*(LM;R)=H_{*+d}(LM;R)$.

You have a shift in degree in the product, so you need to shift the degree in the spectral sequence to make it multiplicative. We shift the homology of the base to place it in negative degrees. Let's draw the spectral sequence with the degree-shift.



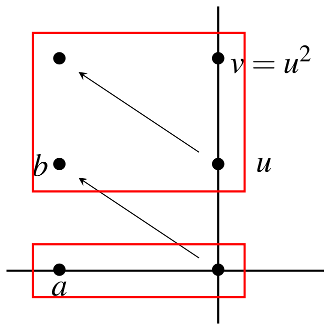
Differentials out of the bottom row are zero because there is a section. So we only need to know the differential of u. For odd spheres:

$$\mathbb{H}_*(LS^n)\cong_{alg} H_*(\Omega_m S^n)\otimes H_*(S^n).$$

And for even spheres:

$$\mathbb{H}_*(LS^n;\mathbb{Z})\cong\frac{\Lambda_{\mathbb{Z}}(b)\otimes\mathbb{Z}[a,v]}{(a^2,ab,2av)}$$

where $|a| = -n$, $|b| = -1$, $|v| = 2n-2$., There are two blocks and the pattern repeats in higher degrees.



These two blocks are explained by Morse theory. It will become clearer with $\mathbb{C}P^n$. First, we can compute $H_*(\Omega\mathbb{C}P^n)$ using the fibration

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

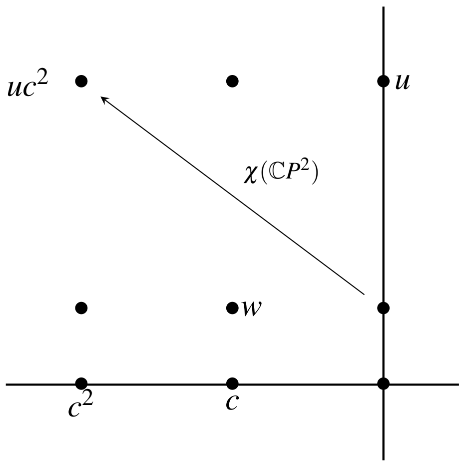
which gives

$$\Omega S^{2n+1} \to \Omega\mathbb{C}P^n \leftrightarrows S^1.$$

So as a topological space $\Omega\mathbb{C}P^n$ is homeomorphic to $\Omega S^{2n+1} \times S^1$, and as an algebra

$$H_*(\Omega\mathbb{C}P^n;\mathbb{Z}) \cong H_*(\Omega S^{2n+1}) \otimes H_*(S^1).$$

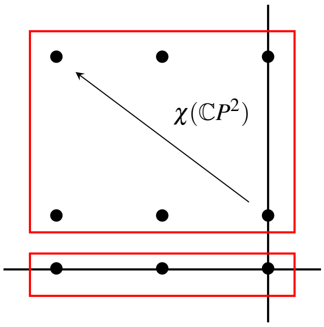
Now we have the spectral sequence (in the case $n = 2$).



So the algebra is

$$\mathbb{H}_*(L\mathbb{C}P^n;\mathbb{Z}) \cong \frac{\Lambda(w) \otimes \mathbb{Z}[c,u]}{(c^{n+1}, (n+1)c^n u, wc^n)}.$$

where $|w| = -1$, $|c| = -2$, $|u| = 2n$. This breaks into two blocks. It contains these two blocks.



The first block is the homology of $\mathbb{C}P^n$ which is the constant loops. The second block is isomorphic to

$$\mathbb{H}_*(U_1 T\mathbb{C}P^n;\mathbb{Z})$$

where U_1 denotes the unit tangent bundle. In the second part of the talk I have to explain why Morse theory is useful here.

2) Morse Theory.

a) Critical points and homology. Let X be a compact manifold, $f\colon X \to \mathbb{R}$ smooth. Given $[\alpha] \in H_i(X)$ we define

$$Cr([\alpha]) = \inf\{a \in \mathbb{R} \mid [\alpha] \in \mathrm{Im}(H_i(X^{\leqslant a}) \to H_i(X))\}$$

where $X^{\leqslant a} = f^{-1}(]-\infty, a])$. Alternatively

$$Cr([\alpha]) = \inf(\sup(f(\beta)))$$

where the infimum is over $\beta \in C_i(X)$ such that $[\beta] = [\alpha]$.

Theorem (Birkhoff, 1930)**.** *$Cr([\alpha])$ is a critical value of f.*

In analysis this is called a min-max principle. What Nancy and Mark Goresky did in their paper was to use this.

Definition. A critical point $p \in X$ is nondegenerate if the Hessian $H_{f,p}$ (a bilinear form) is nondegenerate. If every critical point is nondegenerate then we say that f is *Morse*.

Lemma (Morse Lemma)**.** *Every critical point p has a neighbourhood U_p such that for $x \in U_p$ we have $f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$ where $n = \dim(M)$ and k is called the* index *of p with respect to f.*

Let b be a value of f. Then the *level homology* is

$$\check{H}_*(X^{\leqslant b}, X^{<b}) = \lim_{\varepsilon \to 0^+} H_*(X^{\leqslant b+\varepsilon}, X^{<b}).$$

If b is a regular value then

$$\check{H}_*(X^{\leqslant b}, X^{<b}) = H_*(X^{\leqslant b}, X^{<b}) = 0.$$

If $X^{=b}$ contains exactly one nondegenerate critical point then

$$\check{H}_j(X^{\leqslant b}, X^{<b}) \cong H_j(X^{\leqslant b}, X^{=b}) \cong \begin{cases} \mathbb{Z} & j = k = \mathrm{index}(b), \\ 0 & j \neq q. \end{cases}$$

If you put all this together then you have a Morse spectral sequence. Let us suppose that the critical values are $b_0 < \cdots < b_i < \cdots$. I suppose that $X^{=b_i}$ contains only nondegenerate critical points. We can filter $C_*(X)$ as

MG: There are finitely many if X is compact.

$$C_*(X^{\leqslant b_0}) \subset C_*(X^{\leqslant b_1}) \subset \cdots$$

and then you have a spectral sequence with

$$E^1_{i,j} \cong \bigoplus H_j(X^{\leqslant b_i}, X^{<b_i})$$

Morse theory and Poincaré duality. How do you read Poincaré duality in Morse theory? Here $X = M$ closed oriented and $\dim(X) = d$ and $f\colon M \to \mathbb{R}$ a Morse function.

Theorem.
$$H_*(X^{[a,b]}, X^{=a}) \cong^{PD} H_{d-*}(X^{[a,b]}, X^{=b}).$$
Using excision, this is
$$H_*(X^{\leqslant b}, X^{\leqslant a}) \cong H_{d-*}(X^{\geqslant a}, X^{\geqslant b}).$$

Poincaré duality follows by reversing the filtration, or in other words negating f.

Morse theory and loop spaces. M is a Riemannian manifold. If you want to do Morse theory with free loop spaces, you have a way to avoid analysis, but we won't avoid it. We work with

$$\Lambda M = H^1(S^1, M)$$

which is the completion of the space of piecewise smooth loops with respect to

$$\langle \gamma, \gamma' \rangle_1 = \int_{S^1} d_g(\gamma(t), \gamma'(t))^2 dt + \int_{S^1} \langle \dot\gamma(t), \dot\gamma'(t) \rangle_g dt$$

where d_g is the sup-metric on M determined by g. So functions converge in L^2 as well as their derivatives.

Proposition. $C^\infty(S^1, M) \subset C^\infty(S^1, M)_{piecewise} \subset H^1(S^1, M) \subset C^0(S^1, M)$ *are all weak homotopy equivalences.*

We take the energy

$$E\colon \Lambda M \longrightarrow \mathbb{R}, \qquad \gamma \longmapsto \int_{S^1} \|\dot\gamma(t)\|^2 dt.$$

Critical points of E are the closed geodesics. We want to do Morse theory with the square root of the energy. We have finite dimensional models (Milnor / Morse). Let $\rho <$ injectivity radius of M. Let N be an integer and set $a = \sqrt{N}\rho$.

$$\mathcal{M}^a_N = \left\{ (x_0, \ldots, x_N) \in M^{N+1} \mid x_0 = x_N, \ \sum_{i=1}^N |x_i - x_{i-1}|^2 < \rho^2 \right\}$$

Since between two points x_i and x_{i+1} there is only one geodesic segment, you can do the following.

Proposition. *Define*
$$i\colon \mathcal{M}^a_N \longrightarrow \Lambda^{\leqslant a} M, \qquad (x_0, \ldots, x_N) \mapsto ((x_0 \to \cdots \to x_N)$$
Then i is a homotopy equivalence.

2. Richard Hepworth — String topology for stacks I

Theorem (Preview)**.** *Let $\mathfrak{X}$ be an oriented (Hurewicz) stack of dimension d. Then $H_*(L\mathfrak{X})$ is a non-unital Batalin-Vilkovisky algebra.*

- Taking $\mathfrak{X} = M$ a manifold, $H_*(L\mathfrak{X}) = H_*(LM)$.
- Taking $\mathfrak{X} = [\mathrm{pt}/G]$, G a connected Lie group, $H_*(L\mathfrak{X}) = H_*(LBG)$.

Stacks on Diff. Diff is the category of smooth manifolds.

Definition. A *category fibred in groupoids over Diff* is a functor $\pi\colon \mathfrak{X} \to \mathrm{Diff}$ such that:

- For every diagram
$$U \to V$$
in Diff, every partial lift
$$\bar{V}$$
in fX extends to a lift
$$\bar{U} \to \bar{V}.$$
Call such a lift $\bar{V}|U$.
- For every diagram
$$\begin{array}{ccc} U & \longrightarrow & W \\ & \searrow \quad \nearrow & \\ & V & \end{array}$$
in Diff, every partial lift
$$\begin{array}{ccc} \bar{U} & \longrightarrow & \bar{W} \\ & \quad \nearrow & \\ & \bar{V} & \end{array}$$
extends uniquely to a lift
$$\begin{array}{ccc} \bar{U} & \longrightarrow & \bar{W} \\ & \searrow \quad \nearrow & \\ & \bar{V} & \end{array}$$

Example. Let G be a Lie group. Set $\mathfrak{X}$ = category of principal G-bundles $P \to U$. Call it $[\mathrm{pt}/G]$.

Example. Let X be a manifold. Set $\mathfrak{X}$ = category of morphisms $U \to X$. Call it X.

Example. Let G be a Lie group acting smoothly on a manifold X. Set $\mathfrak{X}$ = category of pairs $(P \to U, P \to X)$ with $P \to U$ a principal G-bundle and $P \to X$ equivariant. Call it $[X/G]$.

[Notation: U, V and W will always denote manifolds, and X and G will always belong to one of these examples.]

A category $\mathfrak{X}$ fibred in groupoids over Diff is a *stack on Diff* if for every U and every open cover $\{U_i\}$ of U, we have:

- Given A, B over U, $\phi_i\colon A|U_i \to B|U_i$ for all i, such that $\phi_i|U_i \cap U_j = \phi_j|U_i \cap U_j$, there is a unique $\phi\colon A \to B$ such that $\phi|U_i = \phi_i$ for all i.

- Given
  - A_i over U_i for all i;
  - $\phi_{ij}\colon A_i|U_i \cap U_j \to A_j|U_i \cap U_j$ for all i, j;
  - cocycle condition on triple intersections

  there is A over U with $\lambda_i\colon A|U_i \to A_i$ for all i, such that the λ_i afford the ϕ_{ij}.

The examples above are all stacks.

Stacks, functors over Diff and natural transformations over the identity on Diff form a strict 2-category. The 2-morphisms are all invertible.

Equivalence, not isomorphism, is the correct notion of sameness for stacks.

Weak pullbacks. Given a diagram $\mathfrak{X} \xrightarrow{F} \mathfrak{Z} \xleftarrow{G} \mathfrak{Y}$ of stacks, define

$$\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$$

to be the category of triples (x, y, θ) where x is an object of $\mathfrak{X}$, y is an object of $\mathfrak{Y}$, and $\theta\colon F(x) \to G(y)$ is an isomorphism in $\mathfrak{Z}$ over an identity of Diff. This fits into a square

$$\begin{array}{ccc} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} & \longrightarrow & \mathfrak{Y} \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{Z} \end{array}$$

Yoneda. Let $\mathfrak{X}$ be a stack and X a manifold. Consider:

- $\mathfrak{X}(X) =$ the fibre of $\mathfrak{X}$ over X. Here X is being considered as an object of Diff. It is a groupoid.
- $\hom(X, \mathfrak{X})$, the groupoid of morphisms of stacks. Here X is being considered as a stack.

There is an equivalence of groupoids

$$\hom(X, \mathfrak{X}) \longrightarrow \mathfrak{X}(X), \qquad F \longmapsto F(X \xrightarrow{=} X).$$

For example:

- $\hom(X, Y) \cong Y(X) = \{X \to Y\}$.
  Stacks are generalised manifolds.
- $\hom(X, [\mathrm{pt}/G]) \simeq [\mathrm{pt}/G](X) =$ the groupoid of principal G-bundles on X.
  Stacks are 'moduli spaces' / 'classifying spaces'.

Differentiable stacks and Lie groupoids. A stack $\mathfrak{X}$ on Diff is a *differentiable stack* if there is a morphism $X \to \mathfrak{X}$ such that in every diagram

$$\begin{array}{ccc} U \times_{\mathfrak{X}} X & \longrightarrow & U \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow \\ X & \longrightarrow & \mathfrak{X} \end{array}$$

$U \times_{\mathfrak{X}} X$ is (equivalent to) a manifold and $U \times_{\mathfrak{X}} X \to U$ is a surjective submersion. We call such an $X \to \mathfrak{X}$ an *atlas*.

Exercise. If $\mathfrak{X} = X$ then $X \xrightarrow{=} X$ is an atlas.

Exercise. If $\mathfrak{X} = [\mathrm{pt}/G]$ then $\mathrm{pt} \to [\mathrm{pt}/G]$ giving trivial bundles is an atlas. Indeed, given $U \to [\mathrm{pt}/G]$ classifying a bundle $P \to U$, we have $\mathrm{pt} \times_{[\mathrm{pt}/G]} U \simeq P$.

Given an atlas $X \to \mathfrak{X}$ we obtain:

- a manifold of objects X
- a manifold of morphisms $X \times_{\mathfrak{X}} X$
- source and target submersions $X \times_{\mathfrak{X}} X \rightrightarrows X$, $(x,y,\theta) \mapsto x,y$.
- a composition map $(X \times_{\mathfrak{X}} X) \times_X (X \times_{\mathfrak{X}} X) \to X \times_{\mathfrak{X}} X$, $((x,y,\theta),(y,z,\phi)) \mapsto (x,z,\phi\theta)$
- an identities map $X \to X \times_{\mathfrak{X}} X$, $x \mapsto (x,x,1_x)$
- an inverses map $X \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} X$, $(x,y,\theta) \mapsto (y,x,\theta^{-1})$.

This is a *Lie groupoid*. (Exercise: define this notion and check that the above is an example.)

Every Lie groupoid $\mathbb{X} = X_1 \rightrightarrows X_0$ has a stack of torsors (exercise: define it) called $[X_0/X_1]$. There is an atlas $X_0 \to [X_0/X_1]$ recovering $\mathbb{X}$. There is an equivalence $\mathfrak{X} \simeq [X/X \times_{\mathfrak{X}} X]$. This underpins an equivalence between the 2-category of differentiable stacks and a certain 2-category of Lie groupoids.

Vector bundles. A *vector bundle* on a stack $\mathfrak{X}$ is

- a morphism $\mathfrak{E} \to \mathfrak{X}$
- for each $U \to \mathfrak{X}$, the structure of a vector bundle on $U \times_{\mathfrak{X}} \mathfrak{E} \to U$

such that, given a triangle

$$\begin{array}{ccc} U & \longrightarrow & \mathfrak{X} \\ \downarrow & \Downarrow \nearrow & \\ V & & \end{array}$$

the induced $U \times_{\mathfrak{X}} \mathfrak{E} \to V \times_{\mathfrak{X}} \mathfrak{E}$ is fibrewise a linear isomorphism over $U \to V$. Every differentiable stack has a *tangent stack*, but it is not always a vector bundle.

Topological stacks. Replacing Diff by Top, we obtain the theory of topological stacks, up to the word 'submersion', which you can omit, or replace with something else.

There is a 2-functor

$$(\text{Diff stacks}) \longrightarrow (\text{Top stacks})$$

sending U (a manifold) to U (the underlying space) and preserving pullbacks.

Definition. A *classifying space* for a topological stack $\mathfrak{X}$ is a space $B\mathfrak{X}$ and a morphism $B\mathfrak{X} \to \mathfrak{X}$ such that for every $U \to \mathfrak{X}$ the induced map $B\mathfrak{X} \times_{\mathfrak{X}} U \to U$ is a weak equivalence.

Theorem (Noohi). *Every topological stack admits a classifying space.*

- Let $B\mathfrak{X} \to \mathfrak{X}$, $B'\mathfrak{X} \to \mathfrak{X}$ be classifying spaces. We obtain canonical weak equivalences $B\mathfrak{X} \leftarrow B\mathfrak{X} \times_{\mathfrak{X}} B'\mathfrak{X} \to B'\mathfrak{X}$.
- Given $B\mathfrak{X} \to \mathfrak{X}$ and $B\mathfrak{Y} \to \mathfrak{Y}$ classifying spaces, and $f\colon \mathfrak{X} \to \mathfrak{Y}$, we obtain a zig-zag
$$B\mathfrak{X} \xleftarrow{w.e.} B\mathfrak{X} \times_{\mathfrak{Y}} B\mathfrak{Y} \longrightarrow B\mathfrak{Y}.$$

We define the *homology* of a stack $\mathfrak{X}$ to be $H_*(\mathfrak{X}) := H_*(B\mathfrak{X})$ for any choice of classifying space. Applying $H_*(-)$ to the previous points, this is well-defined up to canonical isomorphism. Extend any other weak-homotopy invariant construction to stacks in an analogous way.

Example. $B[\mathrm{pt}/G] = BG$.

Example. $BX = X$.

Bivariant theories. A setting for homology, cohomology, and umkehr maps.

We consider a category $\mathcal{C}$ with:

- A class of *confined* morphisms containing all identity morphisms and closed under composition.
- A class of *independent squares*,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

  all of them pullbacks, closed under vertical and horizontal pasting, containing all squares in which a parallel pair of arrows are identities, and such that if f' is confined, so is f, and such that if g' is confined, so is g.

A *bivariant theory* on $\mathcal{C}$ consists of:

- For every morphism $f\colon X \to Y$, a graded abelian group $T(X \xrightarrow{f} Y)$.
- For every $X \xrightarrow[\text{confined}]{f} Y \xrightarrow{g} Z$ a *pushforward* $f_*\colon T(gf) \to T(f)$.
- For every independent square above, a *pullback* $g^*\colon T(f') \to T(f)$.
- For every $X \xrightarrow{f} Y \xrightarrow{g} Z$ a product $\circ\colon T(f) \otimes T(g) \to T(gf)$.

We write $a \in T(X \xrightarrow{f} Y)$ as $X \xrightarrow[f]{a} Y$. (There should be a circle around a here!) These data must satisfy a host of axioms that will be omitted here.

A bivariant theory induces:

- Cohomology $T^*(X) = T(X \xrightarrow{=} X)$,
  cup-product (the product),
  pullbacks (given by pullback).
- Homology $T_*(X) = T^{-*}(X \to \mathrm{pt})$,
  cap-product (the product),
  push-forwards by confined morphisms (given by pushforwards).

It also induces *Gysin homomorphisms* associated to an element $\theta \in T(X \xrightarrow{f} Y)$:

- $\theta^*\colon T_*(Y) \to T_*(X)$,
  $a \mapsto \theta \circ a$.
- $\theta_*\colon T^*(X) \to T^*(Y)$,
  $a \mapsto f_*(a \circ \theta)$.
  (Only if f is confined.)

These satisfy many natural properties, again omitted.

Example. Taking $\mathcal{C}$ to be closed manifolds with all morphisms confined and all pullback squares independent.

$$T(X \xrightarrow{f} Y) = H^{*+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n - i(X))$$

where $i\colon X \to Y \times \mathbb{R}^n$ is an embedding lifting f.

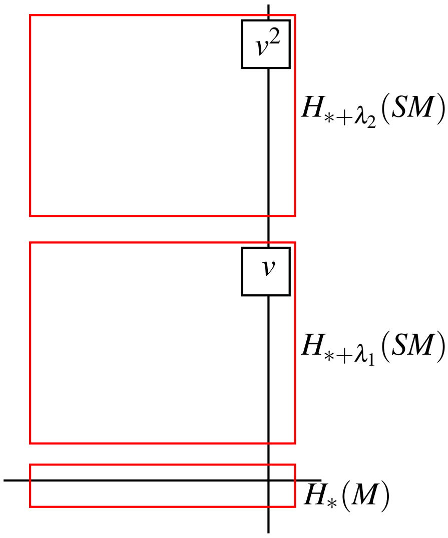
3. Nancy Hingston — Loop products and closed geodesics II

This is all joint work with Mark Goresky.

$M =$ simply connected oriented manifold with a metric all of whose geodesics are closed with the same minimal period. Examples include S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$. (Are there ∞ many closed goedesics for *any* metric?)

Take the square root of the energy as our Morse function on ΛM.

There is a Morse-Bott nondegenerate critical manifold of closed geodesics, and it is isomorphic to SM, the unit sphere in the tangent bundle. Let λ_1 be the index of the number of closed geodesics. (For the examples we have λ_1 equal to $n-1$, 1 and 3 respectively.)



Explicit generators for $M = S^{odd} = S^3$.

- $U \in C_0(\Omega S^3)$, the constant loop at $* \in M$.
- $A = \{$circles great and small beginning at $*$ with velocity v $\} \in C_2(\Omega M)$, here v is fixed.
- $B = \{$circles beginning at $*\} \in C_4(\Omega M)$.
- $E = \{$all constant loops$\} \in C_3(\Lambda M)$
- $C = \{$all circles$\} \in C_7(\Lambda M)$

Products:

- Pontrjagin product $[u] \cdot_{PP} [A] = [A]$. In fact $[U] =$unit.
- $[A] \cdot_{PP} [A] \neq 0$, $H_*(\Omega S^3) = \mathbb{Z}[A]$.
- $[A]$ non-nilpotent, $[A]^{\cdot_{PP} m} \neq 0$.

What are the critical levels? For any fixed metric on any M

$$Cr[X \cdot_{PP} Y] \leqslant Cr(X) + Cr(Y).$$

In the standard metric on S^3:

- $Cr[A] = 2\pi$.
- $Cr[B] = 2\pi$.
- What is $Cr[A] \cdot_{PP} [A]$? Exercise: $[A] \cdot_{PP} [A] = [B]$.
  $Cr[A]^{2m-1} = Cr[A^{2m}] = 2m\pi$.

So $[A]$ is non-nilpotent. But $[A]$ is *level-nilpotent*, meaning that $Cr[A]^k < kCr[A]$ for some k.

Chas-Sullivan product:

- $[C] \cdot_{CS} [E] = [C]$ because $[E]$ is the unit.
- $[A] \cdot_{CS} [A] = 0$ because the Chas-Sullivan product $H_*(\Omega M) \cdot_{CS} H_*(\Omega M) = 0$.
- $[C] \cdot_{CS} [U] = [B]$
- $[C] \cdot_{CS} [C] \cdot_{CS} \cdots \cdot_{CS} [C] \neq 0$, non-nilpotent.

A basic property of the Chas-Sullivan product, for any M and any metric, $Cr([X] \cdot_{CS} [Y]) \leqslant Cr[X] + Cr[Y]$. So $[C]$ is non-nilpotent and it is also level-non-nilpotent.

Bott, Samelson and Morse knew these generators using broken geodesics.

Geometry and Poincaré duality in ΛM told me that there has to be a product in $H^*(\Lambda M)$ of degree $n-1$, i.e. the Chas-Sullivan product is

$$H_k(\Lambda M) \otimes H_j(\Lambda M) \longrightarrow H_{j+k-n}(\Lambda M)$$

and the cohomology product should look like

$$H^k(\Lambda M) \otimes H^j(\Lambda M) \longrightarrow H^{j+k+n-1}(\Lambda M)$$

Geometry. What does the Chas-Sullivan product pick up? What does the geometry look like when the Chas-Sullivan product is nontrivial?

The search for closed geodesics. Given a compact Riemannian manifold. Look for periodic 'closed' geodesics. This search goes back to Poincaré, Birkhoff and Morse. This is what Morse invented Morse theory for.

Morse theory.

$$f = \sqrt{E} \colon \Lambda M \longrightarrow \mathbb{R}$$

The critical points of f are exactly the closed geodesics on M.

- $H_*(\Lambda M)$ corresponds to critical points of f via $X \mapsto Cr(X)$.
- $H_k(\Lambda M)$ corresponds to critical points of index k.

Use $H_*(\Lambda M)$ to get a lower bound for the number of critical points of $\sqrt{E}$, which is the number of closed geodesics.

We observe that $H_*(\Lambda M)$ is nontrivial in many dimensions. Does this give us lots of closed geodesics? No. The difficulty is the iterates.

Iterates. If γ is a closed geodesic on M then so are γ^2, γ^3 and so on where $\gamma^m(t) = \gamma(mt)$. So what is only one closed geodesic in fact looks like a whole army inside the free loop space. Each homology class somehow comes from a closed geodesic and we want to count closed geodesics by counting homology classes, but we see that that will give the wrong result.

Question. Is there an algebraic operation on $H_*(\Lambda M)$ that corresponds to iteration, $\gamma \mapsto \gamma^m$? The Chas-Sullivan powers $[X]$, $[X] \cdot [X]$ and so on model iteration of closed geodesics when the index growth is minimal.

Theorem (Bott, 1953?)**.** $n = \dim(M)$

$$m \cdot \mathrm{Index}(\gamma) - (m-1)(n-1) \leqslant \mathrm{Index}(\gamma^m) \leqslant m \cdot \mathrm{Index}(\gamma) + (m-1)(n-1)$$

The index grows approximately linearly. If the second inequality was an equality then we call it maximal index growth*, and* minimal index growth *is when the second inequality is an equality.*

Example. The ellipsoid

$$x^2/a^2 + y^2/b^2 + u^2/c^2 + v^2/d^2 = 1$$

with $a < b < c < d$, all approximately equal to 1. There are $\binom{4}{2} = 6$ 'short' closed geodesics, the intersections with the standard coordinate planes.

Shortest γ_s has index 2. The indices of its iterates are 2, 6, 10 and so on, which is maximal growth.

Longest γ_l has index 6. Its iterates have indices 6, 10, 14, 18 and so on, which is minimal growth.

In the free loop space these all represent a circle. So the homology classes have dimension 7, 11, 15, 19, where in dimension 7 the homology class is $[C]$, which is v in the first diagram.

Chas-Sullivan product models iteration in the case of minimal growth. The cohomology product models iteration in the case of maximal growth.

Poincaré Duality on ΛM. *Idea of the cohomology product.*

$$H^*(\Lambda^{\leqslant b}, \Lambda^{\leqslant a}) \cong H_{N-*}(\Lambda^{\geqslant a}, \Lambda^{\geqslant b}).$$

The right hand side is generated by 'Morse cochains'. The products are

$$H^j(\Lambda) \otimes H^k(\Lambda) \longrightarrow H^{j+k+n-1}(\Lambda),$$
$$H^j(\Omega) \otimes H^k(\Omega) \longrightarrow H^{j+k+n-1}(\Omega).$$

The first is Poincaré dual to the Chas-Sullivan product. The second is Poincaré dual to the Pontrjagin product. Poincaré duality works best for loops of constant speed.

Easier to define the associated coproduct V on the homology of the free loopspace. It is related to the Goresky-Hingston product $\cdot_{GH}$ as follows. Let $X \in H_*(\Lambda)$, let $y, z \in H^*(\Lambda)$, and write $[-,-]$ for the Kronecker product. Then $VX \in H_*(\Lambda) \otimes H_*(\Lambda)$.

$$[VX, y \otimes z] = [X, y \cdot_{GH} z]$$

The Chas-Sullivan product had to do with the diagram

$$\Lambda \times \Lambda \longleftarrow F \longrightarrow \Lambda$$

where $F = \mathrm{Map}(\infty, M)$. Reversing this, get a coproduct

$$\Lambda \longleftarrow F \longrightarrow \Lambda \times \Lambda$$

called V_t, almost trivial on homology. Set

$$F_{[0,1]} = \{(\gamma, s) \in \Lambda \times [0,1] \mid \gamma(0) = \gamma(s)\}.$$

Then we get

$$\Lambda \times [0,1] \longleftarrow F_{[0,1]} \longrightarrow \Lambda \times \Lambda$$

or rather

$$\big(\Lambda \times [0,1], \partial(\Lambda \times [0,1])\big) \longleftarrow \big(F_{[0,1]}, \partial F_{[0,1]}\big) \longrightarrow \big(\Lambda \times \Lambda, \partial(\Lambda \times \Lambda)\big)$$

Given $A \in H_*(\Lambda, \Lambda^0)$, form $A \times [0,1]$, take an umkehr by the first map, then apply the second. The coproduct has degree $-n+1$.

The resulting *cohomology product*

$$H^i(\Lambda, \Lambda^0) \otimes H^j(\Lambda, \Lambda^0) \longrightarrow H^{i+j+n-1}(\Lambda, \Lambda^0)$$

satisfies the inequality

$$Cr(x \cdot_{GH} y) \geqslant Cr(x) + Cr(y).$$

(The cup product does not have this property!) We say that $x \in H^*(\Lambda)$ is *level-nilpotent* if $Cr(x^{\cdot_{GH} m}) > m \cdot Cr(x)$ for all m.

Some rephrased theorems that illustrate the principle of Poincaré duality in action. This one due to Bott:

(1) M compact, oriented, dimension n, metric with all closed geodesics non-degenerate as critical points. (This is a generic condition on the metric.) Then every homology class in $H_*(\Lambda)$ is level-nilpotent, and every cohomology class in $H^*(\Lambda)$ is also level-nilpotent.

And now these ones due to myself, which apply in the non-nilpotent case. Let γ be an isolated closed geodesic of length L.

(1) Assume that γ has non-nilpotent level homology. Then for any $\varepsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there is a closed geodesic with length in $(mL, mL+\varepsilon)$. It follows that M has infinitely many closed geodesics.
(2) Assume that γ has non-nilpotent level cohomology. Then for any $\varepsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large there is a closed geodesic with length in $(mL - \varepsilon, mL)$.

Just as we were finishing this up I went to a talk by Eliashberg at Princeton. He asked this question.

Question of Eliashberg, 2007. Given M metric. Define

$$d(t) = \max\{k \mid \text{Image}[H_k(\Lambda^{\leqslant t}) \to H_k(\Lambda)] \neq 0\}.$$

Does there exist C independent of the metric so that $d(t_1 + t_2) \leqslant d(t_1) + d(t_2) + C$?

The answer is 'yes', provided that $H^*(\Lambda, \Lambda^0)$ is finitely generated as a ring with the cohomology product. (This holds for spheres and projective spaces.)

4. Luc Menichi — Eilenberg-Moore spectral sequence and string topology

Joint work with Kunbayashi and Naito.

I. String topology of manifold. M^m. LM =free loops on M. Chas-Sullivan loop product

$$H_p(LM)\otimes H_q(LM)\xrightarrow{\bullet} H_{p+q-m}(LM).$$

Cohen-Jones homotopical definition:

$$\begin{array}{ccccc} LM & \xleftarrow{\text{comp}} & LM\times_M LM & \xrightarrow{\widetilde{\Delta}} & LM\times LM \\ & & \downarrow & & \downarrow \\ & & M & \xrightarrow[\Delta]{} & M\times M \end{array}$$

Using Thom-Pontrjagin, Cohen and Jones defined:

$$H_{p+q}(LM\times LM)\xrightarrow{\widetilde{\Delta}} H_{p+q-m}(LM\times_M LM)\xrightarrow{\text{comp}_*} H_{p+q-m}(LM).$$

II. Felix-Thomas extension to Gorenstein spaces.

Definition. An augmented differential graded algebra A is a Gorenstein algebra of dimension $m\in\mathbb{Z}$ if:

$$\dim \operatorname{Ext}^l_A(\mathbb{F},A)=\begin{cases}0 & \text{if } l\neq m,\\ 1 & \text{if } l=m\end{cases}$$

Definition. A space M is a Gorenstein space if the singular cochain $C^*(M)$ is a Gorenstein algebra.

Example.
1. Closed oriented manifold M, $m=\dim(M)>0$.
2. BG connected compact Lie group, $m=-\dim(G)$.
3. If G acts on M, $EG\times_G M$, $m=\dim(M)-\dim(G)$.

Theorem (Felix-Thomas)**.**
1. *Let M be a connected Gorenstein space of dimension m. Then*
$$\operatorname{Ext}^*_{C^*(M^2)}(C^*(M),C^*(M))\cong H^{*-m}(M).$$
Write $\Delta^!\in\operatorname{Ext}^m_{C^*(M^2)}(C^*(M),C^*(M)$ for the element corresponding to 1.
2. *There exists a unique*
$$\widetilde{\Delta}^!\in\operatorname{Ext}^m_{C^*(LM\times_M LM)}(C^*(LM\times_M LM),C^*(LM\times LM))$$
such that the following square commutes.

$$\begin{array}{ccc} C^*(LM\times_M LM) & \overset{\widetilde{\Delta}^!}{\dashrightarrow} & C^{*+m}(LM\times LM) \\ {\scriptstyle ev_*}\uparrow & & \uparrow \\ C^*(M) & \xrightarrow[\Delta^!]{} & C^{*+m}(M\times M) \end{array}$$

The dual of the loop coproduct is given by

$$C^*(LM) \xrightarrow{\text{comp}_*} C^*(LM \times_M LM) \xrightarrow{\widetilde{\Delta}^!} C^{*+m}(LM \times LM)$$

Proof.

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \text{ fibration}} \\ N & \xrightarrow[f]{} & B \end{array}$$

Suppose we have $f^! \in \operatorname{Ext}^d_{C^*(B)}(C^*(N), C^*(B))$. Let $\varepsilon \colon P \xrightarrow{\simeq} C^*(N)$ a right $C^*(B)$-free resolution of $C^*(N)$. Then there exists a unique right $C^*(E)$-linear map such that

$$\begin{array}{ccccc} C^*(X) & \xleftarrow{EM} & P \otimes_{C^*(B)} C^*(E) & \overset{g^!}{\dashrightarrow} & C^{*+d}(E) \\ {\scriptstyle ev_*}\uparrow & & \uparrow{\scriptstyle p \mapsto p\otimes 1} & & \uparrow{\scriptstyle p^*} \\ C^*(N) & \underset{\simeq}{\xleftarrow{\varepsilon}} & P & \xrightarrow[f^!]{} & C^{*+d}(B) \end{array}$$

Applying homology, EM gives an iso, the Eilenberg-Moore isomorphism.

$$\operatorname{Tor}_*^{C^*(B)}(C^*(N), C^*(E)) \underset{\cong}{\xrightarrow{H^*(EM)}} H^{-*}(X). \qquad \square$$

Recall that by filtering $P \otimes_{C^*(B)} C^*(E)$ we obtain the cohomological Eilenberg-Moore spectral sequence.

III. A Tor-description of the loop product. Suppose

$$\begin{array}{ccccc} X & \xrightarrow{g} & E & \longrightarrow & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p \text{ fib}} & & \downarrow{\scriptstyle o \text{ fib}} \\ N & \xrightarrow[f]{} & B & \longrightarrow & Z \end{array}$$

then we can show that

$$\begin{array}{ccc} H_*(X) & \xrightarrow{g^!} & H^{*+d}(E) \\ \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\ \operatorname{Tor}^{C^*(Z)}(C^*(N), C^*(Y)) & \xrightarrow[\operatorname{Tor}(f^!,1)]{} & \operatorname{Tor}^{C^*(Z)}(C^*(B), C^*(Y)) \end{array}$$

in particular in the case of

$$\begin{array}{ccccc} LM \times_M LM & \xrightarrow{\widetilde{\Delta}} & LM \times LM & \longrightarrow & M^I \times M^I \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle ev\times ev} \\ M & \xrightarrow[\Delta]{} & M \times M & \xrightarrow[\Delta\times\Delta]{} & M^2 \times M^2 \end{array}$$

we obtain

$$\begin{array}{ccc} H_*(LM \times_M LM) & \xrightarrow{\widetilde{\Delta}^!} & H^{*+d}(LM \times LM) \\ \uparrow \cong & & \uparrow \cong \\ \mathrm{Tor}^{C^*(M^2\times M^2)}(C^*(M), C^*(M^I \times M^I)) & \xrightarrow[\mathrm{Tor}(\Delta^!,1)]{} & \mathrm{Tor}^{C^*(M^2\times M^2)}(C^*(M\times M), C^*(M^I \times M^I)) \end{array}$$

Now we can get a description of the dual of the loop product in terms of Tor.

IV. Rational isomorphism with Hochschild. Let M be a 1-connected Gorenstein space Denote by $A(M)$ the commutative algebra of polynomial differential forms on M introduced by Sullivan.

$$A(M) \longleftrightarrow C^*(M, \mathbb{Q})$$

Theorem (KMN)**.** *The Eilenberg-Moore isomorphism*

$$H_p(LM; \mathbb{Q}) \cong HH^{-p}(A(M), A(M)^\vee)$$

with Hochschild cohomology of $A(M)$ with coeffients in $A(M)^\vee$ is an isomorphism of graded algebras with respect to the loop product, and to the following cup-product of the Hochschild cohomology of a commutative Gorenstein algebra.

Let A be a 1-connected commutative Gorenstein algebra of dimension m. The proof of Felix-Thomas in the case $A = C^*(M)$ shows that $\mathrm{Ext}^*_{A\otimes A}(A, A\otimes A) \cong H^{*-m}(A)$. Let $\Delta^!$ be a generator of $\mathrm{Ext}^m_{A\otimes A}(A, A\otimes A)$. By taking duals,

$$(\Delta^!)^\vee \in \mathrm{Ext}^m_{A\otimes A}(A^\vee \otimes A^\vee, A^\vee).$$

Since A is commutative, $(\Delta^!)^\vee$ induces an element

$$\mu \in \mathrm{Ext}^m_{A\otimes A}(A^\vee \otimes_A A^\vee, A^\vee).$$

By definition, the cup product is

$$HH^p(A, A^\vee) \otimes HH^q(A, A^\vee) \xrightarrow{\otimes_A} HH^{p+q}(A, A^\vee \otimes_A A^\vee) \xrightarrow{HH^*(A,\mu)} HH^{p+q+m}(A, A^\vee)$$

In the Poincaré duality case:

Corollary (F-T-V, Merkulov)**.** $H_{p+m}(LM; \mathbb{Q}) \cong HH^{-p}(A(M), A(M))$.

IV. Eilenberg-Moore spectral sequences.

$$\begin{array}{ccccc} LM & \xrightarrow{\widetilde{\Delta}} & M^I & \xleftarrow{\simeq} & M \\ \downarrow & & \downarrow & \swarrow \Delta & \\ M & \xrightarrow[\Delta]{} & M \times M & & \end{array}$$

Consider the EMSS in homology

$$E^2_{p,*} = HH^{-p}(H^*(M), H_*(M)) \Longrightarrow H_*(LM).$$

Theorem. *The EMSS is multiplicative. The product on the E^2-term is the cup product induced by*

$$H((\Delta^!)^\vee).$$

(Here $\Delta^!$ is the product induced by F-T for the Gorenstein space M.) Suppose that M is a closed 1-connected manifold (or just Poincaré duality space).

$$E^2_{p,*} = HH^{-p,*}(H^*(M), H^*(M)) \Longrightarrow H_*(LM)$$

5. Kate Poirier — Compactified combinatorial string topology

This does not mean string topology of finite sets. This is joint work in progress — old work with Nathaniel Rounds and new work with Gabriel Drummond-Cole.

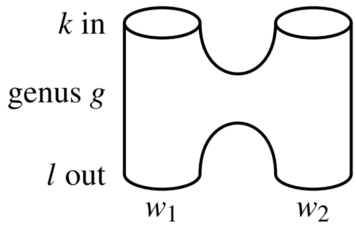
Problem of Sullivan: Describe the compactification of the moduli space of Riemann surfaces which is appropriate for string topology.

Spoiler: Something like Bödigheimer's harmonic compactification.

Know: $H_*(\mathcal{M})$ acts on $H_*(LM)$. (Godin.)

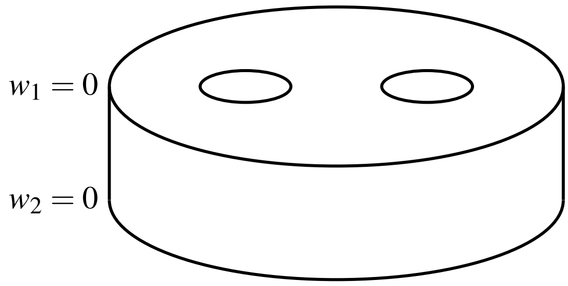
Want: $C_*(\overline{\mathcal{M}})$ acts on $C_*(LM)$, inducing the structure on homology.

Definition. Let $\mathcal{M}(g,k,l)$ be the moduli space of Riemann surfaces with genus g and $k+l$ boundary components.

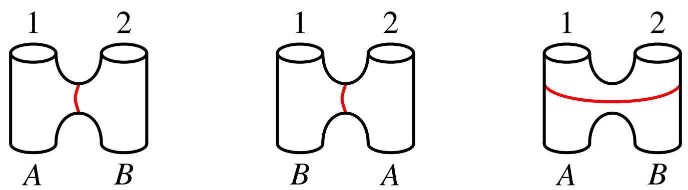


There are weights w_i on the output with $\sum w_i = 1$. We will also have surfaces with punctures, which we think of as boundary at ∞.)

Example. $\mathcal{M}(0,2,2) \simeq \mathcal{M}_{0,4} \times \Delta^1$. (Here we have punctures, not boundary, though the talk will blur the distinction *throughout*.) The moduli space $\mathcal{M}_{0,4}$ of the 4-punctured sphere is the 3-punctured sphere, so we can draw $\mathcal{M}(0,2,2)$ like this:

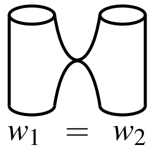


(Think of this picture as solid, containing the top and bottom boundary but not the rest.) The three given loops in $\mathcal{M}_{0,4}$, namely the two small ones in the 'interior' and the large one around the 'outside' correspond to Dehn twist about the curves, respectively:



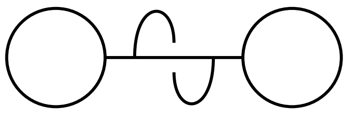
'Compactify' by:

- Taking the closure of this picture.
- Inserting a DM stratum:



Definition. A *fatgraph* is a graph together with a cyclic ordering of the half-edges adjacent to each vertex.

(For the orientation we take the chalkboard orientation.) The graph can be thickened, replacing each vertex by a disc and each arc by a strip, in order to make a surface. Here what I have done is produced a pair of pants. In the resulting surface I think of the boundary components as what we call the boundary cycles of the graph.



This surface has genus 0 and 4 boundary components.

I will use special kinds of fat graphs to define string topology operations at the chain level.

Definition. A *string diagram of type* (g,k,l) is a sequence of metric fat graphs

$$\Gamma_0 \subset \cdots \subset \Gamma_N$$

constructed inductively so that:

- Γ_0 is k disjoint circles (length 1);
- Γ_{n+1} is constructed from Γ_n by adjoining metric trees by attaching their leaves to Γ_n;
- together with 'spacing parameters' $(s_1,\dots,s_{N-1}) \in [0,1]^{N-1}$.

It must be such that Γ_N has genus g and $k+l$ boundary cycles, with k of them in Γ_0. The metric trees must satisfy some condition on lengths. (In the previous pictures Γ_0 will be the two circles, Γ_1 will be the first diagram, and Γ_2 is the second diagram.)

Proposition. *The space $\mathcal{S}(g,k,l)$ of string diagrams of type (g,k,l) is a finite cell complex. Cells are labelled by the combinatorial type.*

Definition. A string diagram is called *simple* if

- $N = 1$
- $\overline{\Gamma_1 - \Gamma_0}$ is a forest.

(Cohen, Godin and others call this a Sullivan diagram.) In the example fat graphs, the first was simple but the second was not.

Proposition. *Simple diagrams form a union of open cells. This is noncompact in general.*

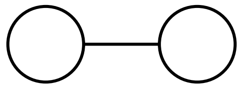
Theorem (In progress with G.C.Drummond-Cole)**.** *The cellular chains of $\mathcal{S}$ act on the singular chains of LM. (Here M is a d-dimensional closed, oriented, compact Riemannian manifold with injectivity radius ε.)*

The string topology operation associated to $\Gamma \in \mathcal{S}(g,k,l)$.

$$\mu_\Gamma \colon C_*(LM^k) \longrightarrow C_{*-|\chi|d}(LM^l)$$

We will define $\mu_\Gamma(\sigma)$ where σ is a generator $\sigma \colon \Delta^n \to LM^k$ and extend linearly.

Example. Here we take $\Gamma \in \mathcal{S}(0,2,1)$.



Then

$$\mu_\Gamma \colon C_*(LM \times LM) \longrightarrow C_{*-d}(LM)$$

And $\sigma \colon \Delta^n \to LM \times LM$, which we think of as $\sigma(t) \colon S^1 \sqcup S^1 \to M$ for $t \in \Delta^n$. We'll define

$$g_{\sigma,\Gamma} \colon C_*(\Delta^n) \longrightarrow C_{*-d}(LM)$$

and set $\mu_\Gamma(\sigma) = g([\Delta^n])$, where $[\Delta^n]$ is the fundamental chain.

The construction has four ingredients:

(1) Let N_ε be a neighbourhood of $\Delta \colon M \to M \times M$.
(2) $U \in C^d(N_\varepsilon, N_\varepsilon - N_{\frac{\varepsilon}{2}})$ representing the Thom class of the diagonal.
(3) An evaluation map
$$ev_\Gamma \colon \Delta^n \longrightarrow M \times M$$
given by evaluating $\sigma(t)$ at the chord endpoints.
(4) Let $S_\varepsilon = ev_\Gamma^{-1}(N_\varepsilon)$.

Observe that $\sigma(t)$ sends chord endpoints into an ε-ball in M. Define g in three steps:

(1) The composite:
$$C_*(\Delta) \longrightarrow C_*(\Delta, \Delta - S_{\frac{\varepsilon}{2}}) \xrightarrow{s} C_*(S_\varepsilon, S_\varepsilon - S_{\frac{\varepsilon}{2}}) \xrightarrow{-\cap ev_\Gamma^*(U)} C_{*-d}(S_\varepsilon)$$
Here s is an explicit chain homotopy inverse to i_*, and Hatcher has a formula for it.
(2) heart: $S_\varepsilon \to \mathrm{Map}(\Gamma, M)$, $t \mapsto \begin{cases} \sigma(t) & \text{on circles,} \\ \text{geodesic segment} & \text{on chords.} \end{cases}$.
(3) $\mathrm{Map}(\Gamma, M) \xrightarrow{out} LM$.

Then

$$g = out_* \circ \mathrm{heart}_* \circ (- \cap ev_\Gamma^*(u)) \circ s \circ j.$$

6. Gregory Ginot — String topology for stacks II

Goals of the project "String topology for stacks"

- Relate string topology with orbifold cohomology of Chen and Ruan. This is an algebra $(H^*(\Lambda\mathfrak{X});\cup)$ with a strange grading, for $\mathfrak{X}$ an 'almost complex' orbifold.
- Have a common framework for string topology operations encompassing
  - closed oriented manifolds;
  - classifying spaces of Lie groups;
  - "commutative families" of groups over a fixed (closed oriented) manifold M.

I) Bivariant theory for stacks.

Theorem. *Fix a commutative ring k. There exists a bivariant theory for topological stacks, denoted $H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ such that*

- $H^*(\mathfrak{X} \xrightarrow{=} \mathfrak{X}) = H^*(\mathfrak{X})$
- $H^n(\mathfrak{X} \to pt) = H_{-n}(\mathfrak{X})$ *for all n*

and which induces the usual operations of cup and cap product, pullback in cohomology and pushforward in homology.

(Now the rest of the bivariant theory is essentially a tool that allows you to form Gysin maps.)

Definition. Let $p\colon \mathfrak{E} \to \mathfrak{X}$ be a vector bundle of rank n. It is *orientable* if there exists a class $\tau \in H^n(\mathfrak{E}, \mathfrak{E} - \mathfrak{X})$ such that the map

$$H^i(\mathfrak{X}) \longrightarrow H^{n+i}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}), \qquad c \mapsto p^*(c) \cup \tau$$

is an isomorphism. We call τ an *orientation*.

Proposition. *Let $p\colon \mathfrak{E} \to \mathfrak{X}$ and $q\colon \mathfrak{F} \to \mathfrak{X}$ be vector bundles over $\mathfrak{X}$ of ranks n and m. Let $\mathfrak{K}$ be a closed substack of $\mathfrak{X}$. Then there exists canonical isomorphisms fitting inside a commutative diagram.*

$$\begin{array}{ccc} H^i(\mathfrak{X}, \mathfrak{X} - \mathfrak{K}) & \xrightarrow{\simeq} & H^{n+i}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \\ \downarrow \simeq & & \downarrow \simeq \\ H^{i+m}(\mathfrak{F}, \mathfrak{F} - \mathfrak{K}) & \xrightarrow{\simeq} & H^{i+m+n}(\mathfrak{E} \oplus \mathfrak{F}, \mathfrak{E} \oplus \mathfrak{F} - \mathfrak{K}) \end{array}$$

What this means is that the cohomology $H^i(\mathfrak{X}, \mathfrak{X} - \mathfrak{K})$ can be computed by replacing $\mathfrak{X}$ with a vector bundle over $\mathfrak{X}$, and the result doesn't depend on how you do that.

Recall the bivariant theory for closed manifolds, that for $f\colon X \to Y$ a smooth map gives

$$H^i(X \xrightarrow{f} Y) = H^{i+n}(\mathbb{R}^n \times Y, \mathbb{R}^n \times Y - i(X))$$

where $i\colon X \to \mathbb{R}^n \times Y$ is given by $x \mapsto (\varphi(x), f(x))$ for φ an embedding. Two ideas:

- Allow all vector bundles over $\mathfrak{Y}$.
- Probe $\mathfrak{X}$ by relatively (compared to $\mathfrak{Y}$) small stacks $\mathfrak{K}$.

Assume that we have a stack map $f\colon \mathfrak{X} \to \mathfrak{Y}$ that satisfies the following property:

$(*)$ (Bounded proper.) There exists a closed embedding φ

$$\begin{array}{ccc} & & \mathfrak{E} \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

where $p\colon \mathfrak{E} \to \mathfrak{Y}$ is a metric, oriented vector bundle over $\mathfrak{Y}$ and $\varphi(\mathfrak{X})$ lies in the unit vector bundle of $\mathfrak{E}$.

Definition (Partial definition)**.** If f is bounded proper then set

$$H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) = H^{i+\dim(\mathfrak{E})}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}).$$

By the earlier definition this does not depend on the choice of embedding, but it does not always work.

For general maps $f\colon \mathfrak{X} \to \mathfrak{Y}$ we start by defining a category $C(f)$.

- Object of $C(f)$: map of stacks $a\colon \mathfrak{K} \to \mathfrak{X}$ such that $f \circ a$ has property $(*)$.
- Morphisms of $C(f)$: (homotopy classes of) maps

$$\begin{array}{ccc} \mathfrak{K} & \xrightarrow{a} & \mathfrak{X} \\ \downarrow & \overset{b}{\nearrow} & \\ \mathfrak{L} & & \end{array}$$

Definition. The bivariant theory for an arbitrary morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ is

$$H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) = \operatorname{colim}_{C(f)} \left(H^{i+\dim(\mathfrak{E})}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \right).$$

I leave it to you to define the maps in the colimit. There should be a proposition that the partial definition from before is an instance of this definition.

Pullback construction: Given a 2-cartesian square of stacks

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

we get a map

$$H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \longrightarrow H^i(\mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}')$$

as follows. Take

$$\begin{array}{ccccc} & & & & \mathfrak{E} \\ & & \overset{\varphi}{\nearrow} & & \downarrow \\ \mathfrak{K} & \xrightarrow[a]{} & \mathfrak{X} & \longrightarrow & \mathfrak{Y} \end{array}$$

Define $\mathfrak{K}' = \mathfrak{K} \times_{\mathfrak{Y}} \mathfrak{Y}'$ and define $\mathfrak{E}' = p^*\mathfrak{E} = \mathfrak{E} \times_{\mathfrak{Y}} \mathfrak{Y}'$. Then a induces a map

$$\begin{array}{ccccc} & & & & \mathfrak{E}' \\ & & \overset{\varphi'}{\nearrow} & & \downarrow \\ \mathfrak{K}' & \xrightarrow[a']{} & \mathfrak{X}' & \longrightarrow & \mathfrak{Y}' \end{array}$$

So we obtain

$$\begin{array}{ccc} H^{i+\dim(\mathfrak{E})}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) & \longrightarrow & H^{i+\dim(\mathfrak{E}')}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}') \\ & & \downarrow \\ & & H^i(\mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}') \end{array}$$

Pushforward:

$$H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}) \xrightarrow{f_*} H^i(\mathfrak{Y} \xrightarrow{g} \mathfrak{Z})$$

for any maps. (In other words any map is confined.)

Cup-product: The product

$$H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \otimes H^j(\mathfrak{Y} \xrightarrow{f} \mathfrak{Z}) \xrightarrow{\cup} H^{i+j}(\mathfrak{X} \xrightarrow{g \circ f} \mathfrak{Y})$$

is defined for *(strongly) adequate maps* $g\colon \mathfrak{Y} \to \mathfrak{Z}$.

Example. Some strongly adequate maps:

(1) $f\colon \mathfrak{X} \to K$ is strongly adequate for K a compact topological space.
(2) $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ equivalences. (In particular you get a cup-product.)
(3) $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ diagonal satisfying the condition $(*)$.
(4) Assume that X and Y are smooth manifolds with an action of a compact Lie group G. Let $f\colon X \to Y$ be G-equivariant. Then

$$[X/G] \xrightarrow{[f/G]} [Y/G]$$

is strongly adequate.

To check that the cohomology groups associated to a stack are the ordinary cohomology groups is trivial. And to check that the homology groups associated to a stack are the ordinary homology groups is essentially Alexander duality.

II) Orientation.

Definition. A *normally nonsingular* or *nns* map is a (representable) map $f\colon \mathfrak{X} \to \mathfrak{Y}$ such that there is a factorisation

$$\begin{array}{ccc} \mathfrak{E} & \longrightarrow & \mathfrak{F} \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

with $\mathfrak{E} \to \mathfrak{X}$ and $\mathfrak{F} \to \mathfrak{Y}$ vector bundles, with $\mathfrak{F}$ oriented, and with the top arrow an open embedding. It is said to be *oriented* if $\mathfrak{E}$ is oriented. Define $\operatorname{codim}(f) = \dim(\mathfrak{F}) - \dim(\mathfrak{E})$.

Example.

- $X \to Y$ an embedding of manifolds, then it is an nns map.

$$\begin{array}{ccc} \mathrm{Tub} & \longrightarrow & Y \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & Y \end{array}$$

- Let $p\colon \mathfrak{E} \to \mathfrak{X}$ be an oriented vector bundle. Then it is nns oriented:

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{=} & \mathfrak{E} \\ {\scriptstyle =}\uparrow & & \downarrow{\scriptstyle p} \\ \mathfrak{E} & \xrightarrow[p]{} & \mathfrak{X} \end{array}$$

Example (Main example)**.** Let X, Y be smooth manifolds acted upon by a compact Lie group G. Let $f\colon X \to Y$ be equivariant. Assume in addition that there exists a linear representation V of G together with an equivariant embedding $X \hookrightarrow V$. (*Mann Theorem*: If $H_*(X)$ is finitely generated in every degree for all i, then you have this property.) Then:

- The map $[X/G] \to [Y/G]$ is nns.
- If X, Y oriented and G is orientation-preserving then it is nns-oriented.

Sketch of proof: You can assume that V is oriented. (If not then take $V \oplus V$.) Then $[V/G] \to [\mathrm{pt}/G]$ is an oriented vector bundle. So we form $[V/G] \times_{[\mathrm{pt}/G]} [Y/G]$ which is an oriented vector bundle over $[Y/G]$. We obtain a diagram

$$\begin{array}{ccc} & & [V/G] \times_{[\mathrm{pt}/G]} [Y/G] \\ & \nearrow & \downarrow \\ [X/G] & \xrightarrow[f]{} & [Y/G] \end{array}$$

and this factors through an nns diagram by choosing a G-equivariant tubular neighbourhood. $\square$

Definition (Orientation)**.** Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a strongly adequate nns map. A class $\theta \in H^{\mathrm{codim}(f)}(f\colon \mathfrak{X} \to \mathfrak{Y})$ is called a *strong orientation* if for all $g\colon \mathfrak{Z} \to \mathfrak{X}$, multiplication by θ induces an isomorphism

$$H(\mathfrak{Z} \xrightarrow{g} \mathfrak{X}) \xrightarrow{\cup\theta} H(\mathfrak{Z} \xrightarrow{f\circ g} \mathfrak{Y}).$$

A stack $\mathfrak{X}$ is said to be *oriented* if $\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X}$ is strongly oriented.

Proposition. *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a strongly adequate oriented nns map, then there is a canonical strong orientation $\theta \in H^{\mathrm{codim}(f)}(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$.*

Example. If X, Y are smooth manifolds, X compact, G compact acting in an orientation-preserving way on X and Y, and $f\colon X \to Y$ is G-equivariant, then $[X/G] \to [Y/G]$ is oriented. So $[\mathrm{pt}/G]$ is always orientable.

Proposition. *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is an nns map between oriented stacks, then it is canonically strongly oriented.*

III) Mapping stacks.

Definition. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological stacks. Set

$$\mathrm{Map}(\mathfrak{X},\mathfrak{Y})(T) = \hom(T \times \mathfrak{X}, \mathfrak{Y}).$$

Proposition. *The mapping stack* $\mathrm{Map}(\mathfrak{X},\mathfrak{Y})$ *is a stack over Top.*

Proposition. *If* $\mathfrak{X} \simeq [X_0/X_1]$ *with* $X_1 \rightrightarrows X_0$ *being "compact", then* $\mathrm{Map}(\mathfrak{X},\mathfrak{Y})$ *is a topological stack.*

So the good news is that the circle, figure-eight and so on are compact.

Definition. $L\mathfrak{X} = \mathrm{Map}(S^1, \mathfrak{X})$ which is a topological stack. (In general very far from being differentiable.)

Example. If $\mathfrak{X} = [\mathrm{pt}/G]$ with G connected then $L\mathfrak{X} \simeq [\mathrm{pt}/LG]$. If G is discrete then $L\mathfrak{X} \simeq [G/G]$ (adjoint action).

Theorem. *Let* $\mathfrak{X}$ *be a (Hurewicz, which includes differentiable) oriented stack of dimension d. Then* $H_*(L\mathfrak{X})$ *is a dimension-d Frobenius algebra. (Meaning that we have the loop product and loop coproduct, but no units or counits in general.)*

$$\begin{array}{ccccc} L\mathfrak{X} & \longleftarrow & L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} & \longrightarrow & L\mathfrak{X} \times L\mathfrak{X} \\ & & \downarrow & & \downarrow \\ & & \mathfrak{X} & \xrightarrow[\Delta]{} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

(The fact that $L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \simeq \mathrm{Map}(\infty, \mathfrak{X})$ follows because $\mathfrak{X}$ is Hurewicz. A pushout of spaces is not necessarily a pushout in stacks. Here the relevant pushout is ∞. However for Hurewicz stacks what you want still holds.)

Take $\theta \in H^d(\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X})$ to be your strong orientation. Get

$$p^*(\theta) \in H^d(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \to L\mathfrak{X} \times L\mathfrak{X})$$

If

$$[x] \in H_i(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}) \simeq H^{-i}(L\mathfrak{X} \times L\mathfrak{X} \to \mathrm{pt})$$

then

$$\Delta^![x] = p^*(\theta) \cup [x] \in H^{-i+d}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X} \to \mathrm{pt}) \simeq H_{i-d}(L\mathfrak{X} \times_{\mathfrak{X}} L\mathfrak{X}) \to H_{i-d}(L\mathfrak{X}).$$

Example. If G is a compact connected Lie group, then

$$H_*(L[\mathrm{pt}/G], \mathbb{R}) \cong H_*([G^{\mathrm{ad}}/G]) \cong S(\mathfrak{g}^*) \otimes^G (\Lambda^*(\mathfrak{g})^G \cong S(x_1, \dots, x_l) \otimes \Lambda^*(y_1, \dots, y_l).$$

The loop product is null, however the coproduct is counital.

7. Daniela Egas — Higher string operations using radial slit configurations I

Bödigheimer: "Configuration models of moduli space of Riemann surfaces with boundary".

I will give the leisurely introduction to what Sander will do on Friday. Sander uses a modification of Bödigheimer's model of moduli space. It behaves really nicely. Because my talk will be leisurely I will give explanations using pictures.

Part I. Configuration models.

A. Radial slit configurations. The idea is we will take a bunch of annuli, cut them up, and then glue them together along the cuts.

Fix integers $h \geqslant 0$ (minus the Euler characteristic of the result), $n, m \geqslant 1$ (the number of incoming & outgoing boundary circles). The space of possibly-degenerate preconfigurations is

$$\mathrm{PRad}_h^{\deg}(n,m) \subset (\bigsqcup_{i=1}^{n} \mathbb{C})^{2h} \times S_{2h} \times S_{2h} \times \{0,1\}^{2h} \times (1,\infty) \times (\bigsqcup_{i=1}^{n} \mathbb{C})^{m}.$$

An element is $L = (\zeta, \lambda, \omega, \theta, R, P)$. Given $R \in (1,\infty)$ we will write

$$\mathbb{A}_R = \{z \in \mathbb{C} \mid 1 \leqslant z \leqslant R\}$$

and

$$\mathbb{B} = \mathbb{A}_R^1 \sqcup \cdots \sqcup \mathbb{A}_R^n.$$

*image*1

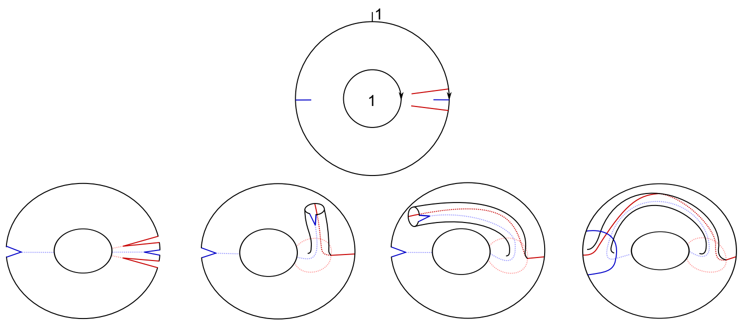
Then $L \in \mathrm{PRad}_h^{\deg}(n,m)$ if the following hold:

(1) The element $\zeta \in (\bigsqcup_{i=1}^{n} \mathbb{C})^{2h}$ is written $\zeta = (\zeta_1, \ldots, \zeta_{2h})$ where the ζ_i are the *slits*. We demand $\zeta_i \in \mathbb{B}$. The *slit segments* are:

*image*2

(2) $\lambda \in S_{2h}$ is the *slit pairing*.
- We assume λ has h disjoint cycles of length 2. "Pairing slits 2 by 2".
- We demand that $|\zeta_i| = |\zeta_{\lambda(i)}|$ so we are pairing slits of the same modulus.

Each slit has a negative and positive side, determined by the orientation of the plane. We pair the positive bank of each slit with the negative bank of its pair. In the following case we get a pair of pants.



Here is a more interesting case:

*image*4

A configuration is *generic* when no slit points ζ_i lie on the same radial segment. If a configuration is non-generic then this process we have defined so far is ambiguous. Let's see why.

*image*5

If I only say where the points are then I don't know if I should place 1 before 3 or vice versa. In one case you get a pair of pants with three legs, and in the other you get a surface of genus 1 with two boundary components. This problem is why we have ω.

(3) $\omega \in S_{2h}$ is such that:
- $\omega = \omega_1 \cdots \omega_n$
- $\omega_i = (j,k,l,\ldots)$ is a cycle and $\zeta_j, \zeta_k, \zeta_l \ldots \in \mathbb{A}^i_R$
- ω_i respects the weak cyclic ordering coming from the argument.

I think of slits lying in the same radial segment as being epsilon apart. This ω tells me what ordering to place those slits in.

There is still a problem. You might not have noticed it yet, but there is still a possible ambiguity in the resulting surface.

Definition. The *exceptional case* is when there is an annulus with more than 1 slit, and all of its slits are at the same radial segment .

*image*6

The two possible orderings are not distinguished by ω, which is $(1\,2)$ in each case.

(4) $\theta \subset \{1,2,\ldots,2h\}$ satisfies $\theta = \emptyset$ in the nonexceptional case, and in the exceptional case it contains all except the 'last' of the slits.

*image*7

This fixes a genuine ordering; I only need it when all slits lie on the same radial segment.

(5) We demand that $\lambda \circ \omega$ consists of exactly m cycles. Note this will divide $\partial_{out}\mathbb{B}$ into m sections $O_1,\ldots,O_m$ which intersect only at points.

*image*8

In the first case $m = 3$, and in the second case $m = 1$.

(6) $P \in (\bigsqcup_{i=1}^n \mathbb{C})^m$ *parametrization points*. We demand that $P_i \in O_i$, so there is exactly one P_i in each O_i.

B. The space Rad. We have seen that

$$\mathrm{PRad}^{\mathrm{deg}} \in L \longmapsto F(L)$$

where $F(L)$ is the surface obtained by cut-and-paste. Is $F(L)$ a surface? No.

Proposition (Bödigheimer)**.** *$F(L)$ is a "degenerate" surface if and only if at least one of the following holds.*

(1) *$\zeta_i \in \partial_{out}\mathbb{B} \cup \partial_{in}\mathbb{B}$ for some i. (Finitely many identified points.)*

(2) *There is i such that $\zeta_i = \zeta_{\lambda(i)}$, and no $\zeta_k \geqslant |\zeta_i| = |\zeta_{\lambda(i)}|$ where k is between i and $\lambda(i)$.*

*image*9

Anything not of this form is in fact a surface. (Bödigheimer proves it by giving charts.)

Definition. PRad $\subset$ PRad$^{\deg}$ where PRad is the subspace of nondegenerate configurations.

Is it possible that $F(L_1) \sim F(L_2)$ when $L_1 \neq L_2$? Yes, sometimes.

Definition. $\equiv_1$ generated by jumps of parametrization points and jumps of a short slit along a long slit.

image

Definition. $\equiv_2$ is defined by $L_1 \equiv L_2$ if $L_1 \equiv_1 L_2$ or L_1 is obtained from L_2 by relabelling ζ_i and modifying the remaining data accordingly.

Definition.

$$\mathrm{LRad} = \mathrm{PRad}/\equiv_1, \qquad \mathrm{Rad} = \mathrm{PRad}/\equiv_2$$

The latter is Bödigheimer's space of *radial slit configurations*.

Theorem (Bödigheimer, Kupers (who protests))**.**

$$\mathrm{Rad}_h(n,m) \simeq \bigsqcup_{[\Sigma]} B\Gamma_\Sigma$$

where Γ_Σ is the mapping class group of Σ and Σ ranges over all cobordisms with n incoming, m outgoing, and $\chi = -h$.

Definition.

$$\begin{array}{ccc} \mathrm{PRad} & \longrightarrow & \mathrm{PRad}^{\deg} \\ {\scriptstyle \equiv_2}\downarrow & & \downarrow{\scriptstyle \equiv_2} \\ \mathrm{Rad} & \longrightarrow & \overline{\mathrm{Rad}} \end{array}$$

$\overline{\mathrm{Rad}}$ is the *harmonic compactification*.

C. Composition and disjoin union. A very nice thing about this model is that composing and taking disjoint unions is very nice. We define

$$P\sqcup\colon \mathrm{PRad}_h(n,m) \times \mathrm{PRad}_{h'}(n',m') \longrightarrow \mathrm{PRad}_{h+h'}(n+n',m+m')$$

by

$$(L,L') \longmapsto L''$$

where L'' is obtained by disjoint union, and we scale the radius to the max of R and R'. And we define

$$P\#\colon \mathrm{PRad}_h(n,m) \times \mathrm{PRad}_{h'}(m,k) \longrightarrow \mathrm{PRad}_{h+h'}(n,k)$$

by

$$(L,L') \longmapsto L''.$$

We use the following picture of the annuli:

$$\mathbb{A}_{R\circ R'} = \mathbb{A}_R \cup \mathbb{A}_R^{R\cdot R'}$$

where $\mathbb{A}_R^{R\cdot R'}$ has inner radius R and outer radius $R \cdot R'$.

On L we divide up the boundary compondents $O_1, \ldots, O_m$. Each O_i is divided into $a_{i1} \cdots a_{il}$ by cutting at every P_i and at every radial segment. On L' we split $\mathbb{A}_R^i$ starting at the marked point so that the inner radius is cut proportionally to the lengths of the a_{ij}. Label the sections $F_{i1}, \ldots, F_{il}$.

images

Now I will take the piece F_{ij} and scale (in the normal and in the angular sense) it so that it fits onto the piece a_{ij}.

We get L'' where we take first the slits of L, then the slits of L' (rotated and scaled) and where the marked points are carried over from L', and where the remaining data is read from L and L', *except* where you have to cut along some slits, and there you can make any choice and it doesn't make a difference.

Remark. These operations are associative, and the cylinder is a unit for the composition up to homotopy.

D. Fat graphs. From a radial slit configuration you get a fat graph:

$$L \in \mathrm{PRad} \longmapsto \Gamma$$

where

$$\Gamma_{in} = \partial_{in}\mathbb{B}, \qquad \Gamma_{out} = O_1 \cup \cdots \cup O_m / \simeq,$$

and

$$\tilde{\Gamma} = \partial_{in}\mathbb{B} \cup \{S'_k - S_k | k = 1, \ldots, 2h\}$$

(here $S'_k - S_k$ is a whole radial segment, minus a radial slit) and then

$$\Gamma = \tilde{\Gamma} / \simeq$$

where $\simeq$ is gluing according to λ.

images

All the combinatorial data in the slit configuration tells you how to equip this graph with a fat structure.

Part II. Operation of a generic configuration.

$$\Gamma_{in} \hookrightarrow \tilde{\Gamma} \xrightarrow{\text{identify } 2h \text{ points}} \Gamma \hookrightarrow F(L) \hookleftarrow \Gamma_{out} \cong \coprod_{i=1}^{n} S^1$$

Let M be a manifold, oriented of dimension d.

$$\mathrm{Map}(\coprod_{i=1}^{n} S^1, M) = LM^n \to \mathrm{Map}(\tilde{\Gamma}, M) \leftarrow \mathrm{Map}(\Gamma, M) \to LM^m$$

We want to create a wrong-way map for the middle arrow. The simplest case is $h = 1$. We assume that Γ comes from a generic configuration (no slits in the same segment).

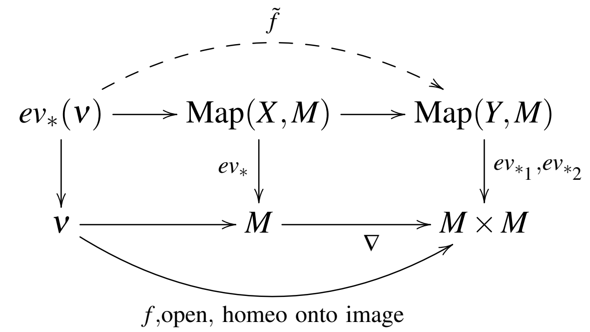
$$Y \xrightarrow{P} X = Y / *_1 \sqcup *_2$$

Toy model

image

Can define $\phi \colon Y \to [0,1]$ such that $\phi^{-1}(1) = \{*_1, *_2\}$ and $\phi^{-1}(0,1] = U_1 \cup U_2$ with $U_i \subset Y$ open and deformation retracting onto $*_i$.

A. Tubular neighbourhoods. We have $p\colon Y \to X$ which is identification of two points.



with ∇ of codimension d. Here

$$ev_*(\nu) = \big\{(g,v) \mid g\colon X \to M,\ v \in \nu_{g(*)}\big\}$$

The preimage of ν in Map(Y,M) is $\{h\colon Y \to M \mid h(*_1), h(*_2) \in \nu\}$. Say $g(*) = \tilde{m}$ and $v \in \nu_{\tilde{m}}$. Now $f(\tilde{m},v) = (m,n)$. We want to make the following assignments

$$(g\colon X \to M) \mapsto (p(g)\colon Y \to M) \mapsto (\tilde{g}\colon Y \to M)$$

such that

$$(* \mapsto \tilde{m}) \mapsto (*_1 \mapsto \tilde{m}, *_2 \mapsto \tilde{m}) \mapsto (*_1 \mapsto m, *_2 \mapsto n)$$

$$\Theta_v\colon M \times M \times I \longrightarrow M \times M, \qquad (\tilde{m},\tilde{m},t) \mapsto \begin{cases} (\tilde{m},\tilde{m}) & \text{if } t = 0, \\ (m,n) = f(\tilde{m},v) & \text{if } t = 1. \end{cases}$$

$$\psi_v(y) = \Theta_v(g(y), g(y), \phi(y)) = \begin{cases} (g(y), g(y)) & \text{if } y \notin U_1 \cup U_2, \\ (m,n) & \text{if } y = *_1,\ y = *_2. \end{cases}$$

$$\tilde{f}(g,v)(y) = \begin{cases} g(y) & y \notin U_1 \cup U_2 \\ \pi_1(\psi_v(y)) & y \in U_1 \\ \pi_2(\psi_v(y)) & y \in U_2 \end{cases}$$

Want Θ_v to change continuously with v and g.

Definition. Let $\nu \to M$ be a vector bundle. A *propagating flow* on ν is a map

$$\chi\colon \nu \longrightarrow \chi_c(\nu),$$

where $\chi_c(\nu)$ is compactly supported vector fields on ν. It is such that for every $v \in \nu$, the result $\chi(v)$ (which is a vector field on ν) sends tv to v for $0 \leqslant t \leqslant 1$. In particular it flows $(\tilde{m},0) \to (\tilde{m},v)$ in time 1.

Proposition (Stacey, Godin)**.** *The space of propagating flows is nonempty and contractible.*

Now choose a propagating flow χ, and use f to turn this into a map X with values in $\chi_c(M \times M)$ by extending by 0. Then take the flow to obtain a map

$$\theta\colon \nu \times I \to \mathrm{Diff}_c(M \times M).$$

Define Θ_v to be $\theta(v,-)$.

8. DMITRY PAVLOV — NATURAL OPERATIONS ON HOCHSCHILD COMPLEXES

Outline:

(1) dg props, A_∞-props
(2) (co)Hochschild complexes for functors
(3) operations on Hochschild complexes
(4) computation of the chain complex of operations
(5) example: cap product

1. dg props, A_∞-props.

Definition. Ch $= \mathbb{Z}-$graded chain complexes.

Definition. A dg-category is a category enriched in Ch.

Definition. A dg prop is a symmetric monoidal dg category with $\otimes$ on objects given by $(\mathbb{N}, +, 0)$. (So the objects are the natural numbers, the product is just addition, and the unit is 0.)

Example. SymFrob$_h$: morphisms $m \to n$ is chains on the moduli space of open bordisms.

Definition. A_∞ (non-unital version) is the dg-prop

(1) Construct a prop enriched in $\mathbb{Z}$-graded sets. Free symmetric monoidal category on one object x, and for any $n \geqslant 2$ a generating morphism $m_n\colon x \otimes \cdots \otimes x \to x$ of degree $n-2$, which we visualise as a tree with n incoming vertices and one outgoing vertex.

*image*1

This is the morphism $m_2 \circ (m_2 \otimes m_3)\colon x^{\otimes 5} \to x$. We think of m_2 as a more-or-less ordinary product on an algebra, but it is only associative up to a homotopy which is given by m_3, and so on.
(2) Replace each set of morphisms by the free $\mathbb{Z}$-graded abelian group.
(3) To define the differential we need only define it for the generators, and there is it given by

$$d(m_n) = \sum m_{k+l+1} \circ (1_{x^{\otimes k}} \otimes m_{n-k-l} \otimes 1_{x^{\otimes l}})$$

where the sum is over all $k \geqslant 0$ and $l \geqslant 0$ such that $k+l+2 \leqslant n$.

A remark: for unital A_∞-algebras one has to add an additional generator $1 \to x$.

Definition. A_∞^{cyclic} is A_∞ together with an additional object y, and morphisms $l_n\colon x \otimes \cdots \otimes x \to y$ where l_n has degree $n-1$.

*image*3

We think of this as a morphism from three things to a 'white' vertex. The differential is given by

$$d(l_n) = \sum \pm \text{ cyclically split off a tree from } l_n$$

So dl_3 is the sum (with signs) of the following six objects.

*image*4

Definition. $\mathcal{L}\colon A_\infty^{\text{op}} \to \text{Ch}$ is the functor $m \mapsto \text{Hom}(m, y)$ where Hom is taken in A_∞^{cyclic}

Remark. $\mathcal{L}(m) = \bigoplus_{n\geqslant 1} \mathrm{Mor}_{A_\infty}(m,n)\otimes L_n$ where $L_n = \mathbb{Z}[n-1]$.

Definition. An A_∞-*prop* is a strong symmetric monoidal functor $i\colon A_\infty \to \mathcal{E}$.

(1) For any such, and any $\Phi\colon \mathcal{E} \to \mathrm{Ch}$ (not necessarily monoidal) we define the *Hochschild chain complex* $\mathcal{C}\Phi\colon \mathcal{E} \to \mathrm{Ch}$ on objects by

$$\mathcal{C}\Phi(m) = ((\Phi\circ(-+m)\circ i)\otimes_{A_\infty}\mathcal{L}).$$

And on a morphism $f\colon m\to n$ by

$$\mathcal{C}\Phi(f) = ((\Phi\circ(-+f)\circ i)\otimes_{A_\infty}\mathcal{L})$$

(2) Given $\Psi\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Ch}$, the *coHochschild comples* of Ψ is defined on objects by

$$\mathcal{D}\Psi(m) = \mathrm{Hom}_{A_\infty}(\mathcal{L},\Psi\circ(-+m)\circ i)$$

and on morphisms by

$$\mathcal{D}\Psi(f) = \mathrm{Hom}_{A_\infty}(\mathcal{L},\Psi\circ(-+f)\circ i).$$

Definition. In the last definition the $\otimes_{A_\infty}$ means the following. Given $F\colon C\to \mathrm{Ch}$ and $G\colon C^{\mathrm{op}}\to\mathrm{Ch}$, where C is a dg-category, then

$$F\otimes_C G = \int_C F(-)\otimes G(-)$$

which is a chain complex.

Summary:

$$\mathcal{C}\colon \mathrm{Fun}(\mathcal{E},ch)\to\mathrm{Fun}(\mathcal{E},ch)$$
$$\mathcal{D}\colon \mathrm{Fun}(\mathcal{E}^{\mathrm{op}},ch)\to\mathrm{Fun}(\mathcal{E}^{\mathrm{op}},ch)$$

Remark.

$$(C\Phi)(m) = \bigoplus_{n\geqslant 1}\Phi(n+m)\otimes L_n = \bigoplus_{n\geqslant 1}\Phi(n+m)[n-1]$$
$$(D\Phi)(m) = \prod_{n\geqslant 1}\mathrm{Hom}(L_n,\Psi(n+m)) = \prod_{n\geqslant 1}\Psi(n+m)[1-n]$$

Remark. Take $i\colon A_\infty\to\mathcal{E}$ to be the identity on A_∞. Let Φ be symmetric monoidal (so it is basically an A_∞-algebra). Then:

(1) $C\Phi(0) =$ Hochschild complex of A_∞ algebra $\Phi(1)$.
(2) $C^n\Phi(0) = (C\Phi(0))^{\otimes n}$. I will explain momentarily why it is interesting to

8.1. operations on Hochschild complexes.

Definition. Fix $i\colon A_\infty\to\mathcal{E}$.

(1) Define a functor $C_{\mathcal{E}}^{m,n}\colon \mathrm{Fun}(\mathcal{E},\mathrm{Ch})\longrightarrow \mathrm{Ch}$ by $\Phi\longmapsto (C^m\Phi)(n)$.
(2) Define the dg-category of *formal operations* $\mathrm{Nat}_{\mathcal{E}}$ as follows. Objects = $\mathbb{N}^2$. The complex of morphisms $(m_1,n_1)\to(m_2,n_2)$ is

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{E},\mathrm{Ch})}(C_{\mathcal{E}}^{m_1,n_1},C_{\mathcal{E}}^{m_2,n_2}).$$

(3) Define the dg-category $\mathrm{Nat}^{\otimes}$ of *natural operations* by substituting $\mathrm{Fun}^{\otimes}$ for Fun.
(4) There is a canonical functor $\mathrm{Nat}_{\mathcal{E}}\to\mathrm{Nat}_{\mathcal{E}}^{\otimes}$.

Definition. (1) Define the *representable functor* $\mathcal{E}(p,-)\colon \mathcal{E}\to\mathrm{Ch}$.

(2) So we obtain $\mathcal{C}^{m_1}(\mathcal{E}(p,-))\colon \mathcal{E}\to \mathrm{Ch}$. Making p variable we obtain
$$\mathcal{C}^{m_1}\mathcal{E}\colon \mathcal{E}^{\mathrm{op}}\otimes\mathcal{E}\to \mathrm{Ch}$$
and similarly
$$\mathcal{D}^{m_2}\mathcal{C}^{m_1}\mathcal{E}\colon \mathcal{E}^{\mathrm{op}}\otimes\mathcal{E}\to \mathrm{Ch}.$$

Theorem. *Take $i\colon A_\infty\to\mathcal{E}$. Then*
$$\mathrm{Nat}_{\mathcal{E}}((m_1,n_1),(m_2,n_2)) = (\mathcal{D}^{m_1}\mathcal{C}^{m_2}\mathcal{E})(n_1,n_2).$$
Using the explicit formulas from earlier, this becomes:
$$\prod\bigoplus \mathcal{E}\left(n_1+\sum j, n_2+\sum k\right)\left[\sum k-\sum j+n_1-n_2\right].$$

Now we know how to compute formal operations on all functors. What is much more interesting is the natural operations, which are defined for symmetric monoidal functors. It turns out in many cases that this map from formal to natural operations is an isomorphism. **Question:** When is $\mathrm{Nat}_{\mathcal{E}}\to\mathrm{Nat}^{\otimes}_{\mathcal{E}}$ an isomorphism? Or surjective, injective, etc?

Example. $m_1=m_2=0$, then $\mathrm{Nat}_{\mathcal{E}}((0,n_1),(0,n_2))=\mathcal{E}(n_1,n_2)$. And
$$\mathrm{Nat}^{\otimes}_{\mathcal{E}}((0,n_1),(0,n_2)) = \mathrm{Hom}(U^{\otimes n_1},U^{\otimes n_2})$$
where $U\colon \mathrm{Fun}^{\otimes}(\mathcal{E},\mathrm{Ch})\to\mathrm{Ch}$ is the functor that evaluates at 1. The functor $\mathrm{Nat}_{\mathcal{E}}\to\mathrm{Nat}^{\otimes}_{\mathcal{E}}$ is then of the form
$$\mathrm{Nat}^{\otimes}_{\mathcal{E}}((0,n_1),(0,n_2)) \longrightarrow \mathrm{Hom}(U^{\otimes n_1},U^{\otimes n_2}).$$
It is given by the structure of $\mathcal{E}$-algebra.

Definition. For all $\mathcal{E}$ there is a functor of dg props
$$\rho\colon \mathcal{E}\longrightarrow\widehat{\mathcal{E}}$$
where
$$\widehat{\mathcal{E}}(n_1,n_2)=\mathrm{Hom}(U^{\otimes n_1},U^{\otimes n_2})$$
is called the *completion of* $\mathcal{E}$. If this is an isomorphism then we say that $\mathcal{E}$ is *complete*.

Remark.

(1) $\mathrm{Fun}^{\otimes}(\widehat{\mathcal{E}},\mathrm{Ch})\xrightarrow{\cong}\mathrm{Fun}^{\otimes}(\mathcal{E},\mathrm{Ch})$
(2) $\widehat{\mathcal{E}}$ is complete.
(3) The completion $\widehat{\mathcal{E}}$ again has the structure of A_∞-prop *via* $\mathcal{E}\to\widehat{\mathcal{E}}$.
(4) Write $r\colon \mathrm{Nat}_{\mathcal{E}}\to\mathrm{Nat}^{\otimes}_{\mathcal{E}}$.
   (a) r is injective on morphisms / faithful if and only if $\rho\colon\mathcal{E}\to\widehat{\mathcal{E}}$ is faithful.
   (b) r is surjective on morphisms / full if and only if ρ is full.
(5) $\mathrm{Nat}^{\otimes}_{\mathcal{E}}\to\mathrm{Nat}_{\widehat{\mathcal{E}}}$ is an isomorphism.
(6) If $\mathcal{E}$ comes from a dg operad, then $\mathcal{E}\to\widehat{\mathcal{E}}$ is faithful.

Example (Cap product)**.** $A\colon$ Assoc $\to$ Ch. End(A) the dg prop with $\mathrm{End}(A)(p,q)=\mathrm{Hom}(A(p),A(q))$. The cap product
$$\cap\colon C_p(A,A)\otimes C^q(A,A)\longrightarrow C_{p-q}(A,A)$$
is given on $a\otimes D$, where $a=a_0\otimes\cdots\otimes a_p$ and $D\colon A^{\otimes}q\to A$, by $a\cap D=\pm a_0D(a_1,\dots,a_q)\otimes\cdots$.

Proposition. *There is*

$$F\colon C^*(A,A) \to \mathrm{Nat}_{\mathrm{End}(A)}((1,0),(1,0))$$

given by cap product. It is injective.

9. Anssi Lahtinen — String topology of classifying spaces & HHGFTs

Joint with Richard Hepworth. $\mathbb{Z}/2$ coefficients.

I. Background.
Godin '07. Suppose M is a closed oriented manifold. Then there is a degree $\dim(M)$ HCFT with $S^1 \mapsto H_*(LM)$.

Chataur-Menichi '07. Suppose G is a compact Lie group. Then there is a degree $-\dim(G)$ HCFT with $S^1 \mapsto H_*(LBG)$.

Rough definition. An HCFT $\mathcal{F}$ of degree d is an assignment

$$\big(1-\text{mfld}\, X\big) \longmapsto \big(\mathcal{F}_*(X) \text{ graded vector space}\big)$$

and

$$\big(\text{cobordism}\, \Sigma\colon X \to Y\big) \longmapsto \big(H_{*-d\cdot\chi(\Sigma,X)}(B\mathrm{Diff}(\Sigma)) \otimes \mathcal{F}_*(X) \to \mathcal{F}_*(Y)\big)$$

where $\mathrm{Diff}(\Sigma)$ is the group of diffeomorphisms of Σ fixing X and Y pointwise. Compatible with disjoint union, composition and diffeomorphisms of cobordisms.

The way I have stated these two results makes it look like the two results are exactly equivalent. But in fact the HCFT constructed by Godin is of a stronger type than the one constructed by Chataur-Menichi. For example, in Godin's theory the 1-manifolds may have boundary, while for Chataur-Menichi they must be closed. Also Godin's theory comes with a unit for the value on a circle and a unit and counit for the value on an interval; but Chataur-Menichi have neither a unit nor a counit. The main goal of the project was to extend Chataur-Menichi's result to one more closely analogous to Godin's. But in fact we ended up describing something much more complicated.

II. Homological h-graph field theories (HHGFTs). The key idea in this extension is that instead of 1-manifolds, sufaces and diffeomorphisms we can equally well work with spaces that have the homotopy type of a finite graph and homotopy equivalences.

Definition. An *h-graph* is a space homotopy equivalent to a finite graph.

Example. pt, I, S^1, $S^1 \vee S^1$, any compact connected surface Σ with non-empty boundary.

Definition. An *h-graph cobordism* $S\colon X \to Y$ is a diagram of the form

$$X \xrightarrow{\;i\;} S \xleftarrow{\;j\;} Y$$

of h-graphs such that

- $X \sqcup Y \xrightarrow{i,j} S$ is a closed cofibration.
- $i\colon X \to S$ is surjective on π_0.
- There should exist a homotopy cocartesian square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow j \\ B & \longrightarrow & X \end{array}$$

where A has the homotopy type of a finite set and B is an h-graph. What this condition says is that up to homotopy S is obtained from Y by attaching a finite graph along a finite number of points.

(One thing which this last requirement, which may seem a little mysterious, is that it allows us to form compositions of h-graph cobordisms.)

Example. Any open-closed cobordism $\Sigma\colon \partial_0 \to \partial_1$ between 1-manifolds such that $\partial_0 \to \Sigma$ is surjective on π_0. So the cobordisms we already had are still there.

In addition you can imagine any finite graph, and turn it into an h-graph cobordism by deciding that part of it is the incoming boundary and part of it is the outgoing boundary. Now we want families of such things.

Definition. Suppose that X and Y are h-graphs and that B is a space. A *family of h-graph cobordisms* $S/B\colon X \to Y$ consists of:

- A fibration $S \to B$.
- Maps $X \times B \to S \leftarrow Y \times B$ over B.

such that:

- The map $(X \sqcup Y) \times B \to S$ is a closed fibrewise cofibration.
- The diagram $X \to S_b \leftarrow Y$ is an h-graph cobordism for each $b \in B$. (The diagram is obtained by restricting to fibres.)

Example. Suppose $\Sigma\colon \partial_0 \to \partial_1$ is an ordinary open-closed cobordism, and that $\partial_0 \to \Sigma$ is surjective on π_0. Let $D = \mathrm{Diff}(\Sigma)$. Then

$$BD \times \partial_0 \longrightarrow ED \times_D \Sigma \longleftarrow BD \times \partial_1$$

(all over BD). Write $UD = ED \times_D \Sigma$. Then we have $UD/BD\colon \partial_0 \to \partial_1$.

Example. Suppose $S\colon X \to Y$ any h-graph cobordism. Have a universal family of h-graph cobordisms

$$U\mathrm{hAut}(S)/B\mathrm{hAut}(S)\colon X \longrightarrow Y$$

where $\mathrm{hAut}(S) = \{f\colon S \to S \mid f|(X \sqcup Y) = id\}$ is the topological monoid of self-homotopy equivalences.

Definition. An *HHGFT* Φ *of degree* d consists of:

- A strong symmetric monoidal functor

  $$\text{(h graphs and homotopy equivalences)} \xrightarrow{\Phi_*} \text{(graded vector spaces)}$$

- For each family of h-graph cobordisms $S/B\colon X \to Y$, a map

  $$H_{*-d\chi(S,X)}(B) \otimes \Phi_*(X) \xrightarrow{\Phi(S/B)} \Phi_*(Y)$$

Satisfying 5 axioms:

(1) Base change (and homotopy invariance). Given

$$\begin{array}{ccc} X & \xrightarrow{S/B} & Y \\ \simeq\big\downarrow \phi_X & & \phi_Y \big\downarrow\simeq \\ X' & \xrightarrow[S'/B']{} & Y' \end{array}$$

and $\phi_B\colon B \to B'$ and $\phi_S\colon S \to S'$ (over B and under $(X \sqcup Y) \times B$, such that φ_S is a homotopy equivalence in every fibre. The diagram

$$\begin{array}{ccc} H_{*-d\chi(S,X)}(B) \otimes \Phi_*(X) & \xrightarrow{\Phi(S/B)} & \Phi_*(Y) \\ \downarrow & & \downarrow \\ H_{*-d\chi(S',X')}(B') \otimes \Phi_*(X') & \xrightarrow[\Phi(S'/B')]{} & \Phi_*(Y') \end{array}$$

(2) Gluing. Given $X \xrightarrow{S/B} Y \xrightarrow{T/C}$, compare $\Phi(S/B)$, $\Phi(T/C)$, $\Phi((T \circ S)/(C \times B))$. Here $(T \circ S)/(C \times B)$ is the family whose fibres are the composites of the fibres of T with the fibres of S.
(3) Identity: $\Phi(X \times I/\mathrm{pt}\colon X \to X)$ acts as identity.
(4) Monoidality: Given $S_i/B_i\colon X_i \to Y_i$, $i = 1,2$, compare $\Phi(S_1/B_1)$, $\Phi(S_2/B_2)$ and $\Phi((S_1 \sqcup S_2)/(B_1 \times B_2))$.
(5) Unit. $\Phi(\emptyset/\mathrm{pt})$ acts as identity.

An HHGFT induces an HCFT. I get it by applying the HHGFT to the universal families $UD(\Sigma)/BD(\Sigma)$.

Theorem (H-L)**.** *Chataur-Menichi's HCFT extends to an HHGFT.*

The HCFT here is open-closed, and also has a counit. So we do get an extension of the HCFT to one comparable to Godin.

(Some) benefits.

(1) New cobordisms, and so new operations. For example

1

from S^1 to I. This is not an ordinary cobordism. The resulting operation $\Phi_*(S^1) \to \Phi_*(I)$ is a retraction of coalgebras.
(2) New factorisations.

2

(3) Connection to automorphism groups of free groups (with boundary). (These groups are homotopy equivalent to $\mathrm{hAut}(S)$ for some suitable h-graph cobordism S.)

10. Daniel Berwick-Evans — Two-dimensional Yang-Mills theory and string topology of classifying spaces

Joint with Dmitry Pavlov.

There are some structural aspects of Yang-Mills theory that look a lot like string topology.

I. Yang-Mills theory.

Warm-up. G a finite group. Consider the group ring $\mathbb{C}[G]$. This is a symmetric semisimple Frobenius algebra.

Theorem (Schommer-Pries)**.** *There is an equivalence*

$$\{\text{local } 2-\text{d topological field theories}\} \longleftrightarrow \{\text{semisimple symmetric Frobenius algebras}\}$$

The objects on the left are symmetric monoidal functors $\mathrm{Fun}^{\otimes}(2-\mathrm{Bord}, \mathrm{ALG})$*. This theory sends a point to an algebra A, an interval to an A-bimodule M, and bordisms to morphisms of bimodules.*

For $A = \mathbb{C}[G]$ we get a 2-TFT called Dijkgraaf-Witten theory.

$$\mathrm{pt} \mapsto \mathbb{C}[G], \quad S^1 \mapsto Z(A) = \mathbb{C}[G]^G, \quad \text{pants} \mapsto (m\colon Z(A) \otimes Z(A) \to Z(A)).$$

2-d Yang-Mills theory continues this story for G a compact Lie group with a biinvariant metric. (The theory will not be topological.) I'll tell you what a physicist would write down if you asked what is Yang-Mills theory.

Classical YM theory. Consider the stack of maps $\Sigma^2 \mapsto \mathrm{pt}/\!/G^\nabla$. There is a function on $\mathrm{Map}(\Sigma, \mathrm{pt}/\!/G)$ called the *Yang-Mills action*. For every map from a manifold S into here, I will give you a map on S. Given $\Sigma \to \mathrm{pt}/\!/G^\nabla$, corresponding to a principal G-bundle $P \to \Sigma$ with connection ∇, take

$$S_{YM} = \int_\Sigma tr(F \wedge *F)$$

where $F = \mathrm{curv}(\nabla)$. Want to form

$$\int_{\mathrm{Map}(\Sigma, \mathrm{pt}/\!/G^\nabla)} e^{-S} D\phi.$$

This will define quantum Yang-Mills. When G is finite then $\mathrm{Map}(\Sigma, \mathrm{pt}/\!/G^\nabla)$ is finite and we (or in fact Dan Freed) can compute this. When G is not finite then all hell breaks loose.

Observations:

- $L(\mathrm{pt}/\!/G^\nabla) = \mathrm{Map}(S^1, \mathrm{pt}/\!/G^\nabla) \cong G^{ad}/\!/G$ which is finite-dimensional.
- $\mathrm{Map}(S^1 \times I, \mathrm{pt}/\!/G^\nabla) \cong \mathrm{Map}(I, G/\!/G)$. On the left here is a gauge theory problem, and on the right it's a classical mechanics problem; we can quantise on the right (it's quantum mechanics) and this will (by magic) coincide with the quantisation on the left.
- The isomorphism $C^\infty(\mathrm{Map}(S^1 \times I, \mathrm{pt}/\!/G^\nabla)) \cong C^\infty(\mathrm{Map}(I, G/\!/G))$ sends S_{YM} to S_{mech}, where $S_{mech}(\gamma) = \int_I |\dot{\gamma}|^2 dt$. (A map $I \to G/\!/G$ is $I \leftarrow P \to G$ with P a principal G-bundle with connection.)

So quantizing Yang-Mills theory is quantising classical mechanics on $G//G$. Audience question: What is quantisation? We quantise and get $C^\infty(G)$ as space of states, together with the action by the Laplacian Δ which is the Hamiltonian / time-evolution operator.

Theorem (BE-Pavlov)**.** *Let G be a Lie group with bi-invariant metric. There is a local 2-d field theory,*

$$YM_G\colon 2-V\mathrm{Bord} \longrightarrow \mathrm{ALG}$$

generalizing Dijkgraaf-Witten theory. (Here again ALG *is the 2-category of algebras, bimodules and intertwiners.)*

$2-V$Bord:

- objects are 0-manifolds;
- 1-morphisms are 1-manifolds;
- 2-morphisms are 2-manifolds with volume form.

Construction of YM_G:

$$\begin{aligned}
\mathrm{pt} &\longmapsto C^\infty(G) \text{ with convolution product}\\
I &\longmapsto {}_AA_A\\
\cap &\longmapsto {}_{A^{op}\otimes A}A_{\mathbb{C}}\\
S^1 &\longmapsto Z(A) = C^\infty(G)^G \cong C^\infty(G//G)\\
1 &\longmapsto {}_AA_A \otimes {}_AA_A \xrightarrow{m} {}_AA_A\\
2 &\longmapsto e^{-V\Delta}\colon {}_AA_A \to {}_AA_A\\
\text{cyclinder volume } V &\longmapsto e^{-V\Delta}\colon Z(A) \to Z(A)\\
\text{pants made of cylinders volume } V_1, V_2, V_3 &\longmapsto e^{-V_3A} \circ m \circ (e^{-V_1\Delta} \otimes e^{-V_2\Delta})\\
\text{cup volume } 0 &\longmapsto ev_e\colon Z(A) \to \mathbb{C}, \quad f \mapsto f(e).\\
\text{cap volume } V > 0 &\longmapsto \mathbb{C} \to Z(A), \quad 1 \mapsto e^{-V\Delta}\delta_e
\end{aligned}$$

Taking the zero-volume limit gives a 2-TFT without unit. We can try to solve this using differential forms.

YM via push-pull constructions. Corresponding to the volume $V = 0$ pair of pants.

$$\begin{array}{ccccc}
 & & \mathrm{Map}(\infty, \mathrm{pt}//G^\nabla) & & \\
 & \swarrow^{L\times R} & \| & \searrow^{c=m} & \\
L(\mathrm{pt}//G^\nabla) \times L(\mathrm{pt}//G^\nabla) & & (G\times G)//G & & L(\mathrm{pt}//G^\nabla)\\
\| & & & & \|\\
G//G \times G//G & & & & G//G \times G//G
\end{array}$$

Then we take a push-pull:

$$\begin{array}{ccccc}
 & & C^\infty(G\times G)^G & & \\
 & \swarrow^{(L\times R)^*} & & \searrow^{m^!} & \\
C^\infty(G)^G \otimes C^\infty(G)^G & & & & C^\infty(G)^G
\end{array}$$

Let G be connected. Models for equivariant de Rham of $G//G$.

(1) $S^\bullet(\mathfrak{g}^*)^G \otimes \Lambda^\bullet(\mathfrak{g}^*)^G$. Cartan.
(2) $S^\bullet(\mathfrak{g}^*)^G \otimes \Omega^\bullet(G)^G$. Weil.
(3) $S^\bullet(\mathfrak{g}^*)^G \otimes \Lambda^\bullet(\mathfrak{g}^*)^G \otimes^G \Omega^\bullet(G)$. BRST.

The result:

(1) Get product, coproduct, a unit which is the harmonic volume form on G. This is a TFT without counit.
(2) We can get a VFT where the cap is determined by

$$\mathbb{C} \to \Phi(G), \qquad 1 \mapsto e^{-V\Delta}(\mathrm{dvol}_G \cdot \delta_e).$$

As $V \to \infty$, this becomes the previous theory.

Questions:

- Where is the counit?
- What does zero (or maybe infinite) volume limit mean precisely?

11. Sander Kupers — Higher operations using radial slit configurations II

Plan:

(1) Radial slits revisited.
(2) Global construction of operations.
(3) HCFTs and checking the axioms.
(4) (Partial) compactification and fun.

Last time:

(1) We defined radial slit configurations.
(2) For each radial slit configuration we defined a surface and some graphs.
(3) For generic ones we defined a string operation.

Conventions:

- M is a compact closed oriented manifold of dimension d.
- Homology is with field coefficients, for argument $\mathbb{Q}$.

1. Radial slits again. Σ a 2d-cobordism such that each connected component has non-empty incoming and outgoing boundary. One obtains

$$\mathrm{Rad}_\Sigma \cong B\mathrm{Diff}^+(\Sigma, \partial\Sigma).$$

We started by defining possibly-degenerate preconfigurations

$$\mathrm{PRad}^{\mathrm{deg}} \subset (1,\infty) \times \Big(\bigsqcup_{i=1}^{n} \mathbb{C}\Big)^{2h} \times S_{2h} \times L \times S_{2h} \times \{0,1\}^{2h} \times \Big(\bigsqcup_{i=1}^{n} \mathbb{C}\Big)^{m}.$$

An element L is written

$$L = (R, (\zeta_i), \lambda, \omega, \theta, P_i).$$

We then formed quotients (vertically)

$$\begin{array}{ccc}
\mathrm{PRad}^{\mathrm{deg}} & \longleftarrow & \mathrm{PRad} \\
\downarrow & & \downarrow \\
\mathrm{LRad}^{\mathrm{deg}} & \longleftarrow & \mathrm{LRad} \\
\downarrow & & \downarrow \\
\mathrm{Rad}^{\mathrm{deg}} & \longleftarrow & \mathrm{Rad}
\end{array}$$

where the first quotient identified diagrams related by slit jumps, and the second identified diagrams under relabellings (of the indices i).

Theorem. $\mathrm{Rad}_\Sigma \simeq B\mathit{Diff}^+(\Sigma, \partial\Sigma)$.

Building a surface from L:

$$image2$$

This gives a space

$$\mathrm{Sector}(L) \subset \Big(\bigsqcup_{j=1}^{2h} \Big(\bigsqcup_{i=1}^{n} \mathbb{C}\Big)\Big).$$

Allowing L to vary, we get

$$\text{Sector} \subset \text{PRad} \times \Big(\coprod_{j=1}^{2h}\Big(\coprod_{i=1}^{n}\mathbb{C}\Big)\Big)$$

and by taking identifications in each fibre (the identifications required to make the surface) we obtain a space

$$P\Sigma \longrightarrow \text{PRad}.$$

We want this to be the universal surface bundle.

(1) Want this to be compatible with slit jumps and parameterising point jumps. In other words, need $L \sim L'$ to give a canonical isomorphism $\Sigma(L) \cong \Sigma(L')$. This will produce a reduced bundle

$$L\Sigma \longrightarrow \text{LRad}.$$

(2) Want this to be compatible with relabelling. Then we obtain the reduced bundle

$$\Sigma \longrightarrow \text{Rad}.$$

Theorem (Bödigheimer)**.** $\Sigma \to \text{Rad}_\Sigma$ *is the universal surface bundle.*

This is in Bödigheimer (2006) and possibly also a paper of Ebert.

In $\Sigma(L)$ there are several types of graphs.

*Image*3

It gives us a diagram

$$\Gamma_{in} \longrightarrow \Gamma \longrightarrow \Sigma \longleftarrow \Gamma_{out}$$

over Rad. (All of these can be pulled back over LRad or PRad, and I will indicate that sort of thing using an L or a P.) Now

$$\Gamma_{in} \cong (S^1)^n \times \text{Rad}, \qquad \Gamma_{out} \cong (S^1)^m \times \text{Rad}$$

and there is also a section $r\colon \Sigma \to \Gamma$ which is a homotopy equivalence.

Some other graph spaces over PRad (*not* compatible with slit jumps). The following thing is actually not degenerate:

*image*4

We obtain $P\tilde{\Gamma}$ (don't glue vertices yet), $P\tilde{\Pi}$ (don't identify edges or vertices), $P\Pi$ (don't identify edges). For generic L we have $\tilde{\Gamma} = \tilde{\Pi}$ and $\Pi = \Gamma$. These guys are not compatible with slit jumps.

So now I have a very big diagram which tells me all the spaces I need for my construction.

$$\begin{array}{ccccccccc}
\Gamma_{in} & \longrightarrow & & \longrightarrow & \Gamma & \longleftarrow & \Sigma & \longleftarrow & \Gamma_{out} \\
\uparrow & & & & \uparrow & & & & \\
L\Gamma_{in} & & & & L\Gamma & & & & \\
\downarrow & & & & \downarrow & & & & \\
P\Gamma_{in} & \longleftarrow & P\tilde{\Gamma} & \longleftarrow P\tilde{\Pi} \longrightarrow P\Pi & \longrightarrow P\Gamma & & & &
\end{array}$$

Now we will need some local systems on these spaces. Think of local systems as bundles of abelian groups locally isomorpic to $\mathbb{Q}$ together with locally constant function to $\mathbb{Z}$.

Definition. Form LRad $\times\, \mathbb{Q}$. Now S_{2h} acts on LRad with quotient Rad, and it acts on $\mathbb{Q}$ by sign(σ). We define

$$\mathcal{L} = \{\mathrm{LRad} \times_{S_{2h}} \mathbb{Q}, -h\}.$$

Proposition. *$\mathcal{L}$ is compatible with disjoint union and composition.*

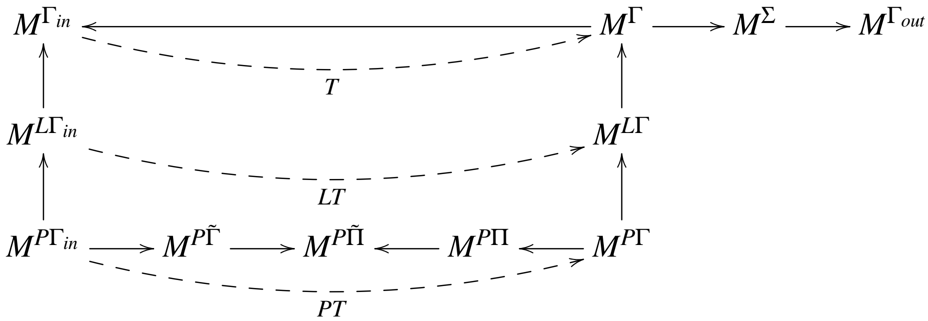
In this case $\mathcal{L}^{\otimes d}$ will be encoding orientations and degree-shifts of the Thom isomorphism.

2. Global construction.

2.1 outline
2.2 fibrewise mapping spaces and parameterised spectra
2.3 tubular neighbourhood and Pontrjagin-Thom map.

2.1 Outline:

i) Map the diagram fibrewise into M.



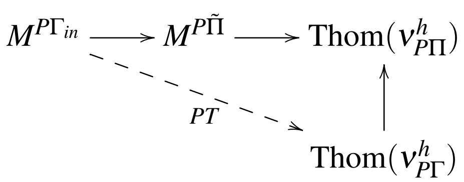
ii) Construct a fibrewise tubular neighbourhood

$$Pf\colon \nu^h_{P\Pi} \longrightarrow M^{P\tilde{\Gamma}} \qquad \text{for} \qquad M^{P\Pi} \to M^{P\tilde{\Pi}}.$$

This produces a fibrewise Pontrjagin-Thom collapse map

$$M^{P\tilde{\Pi}} \longrightarrow \mathrm{Thom}(\nu^h_{P\Pi}).$$

We check that the following composite factorises.



(This works because the tubular neighbourhood is chosen correctly.) Now this map PT has enough good properties that it induces the maps LT and T in the diagram above. So the output is a map

$$M^{\Gamma_{in}} \longrightarrow \mathrm{Thom}(\nu_\Gamma)$$

where ν_Γ is a vector bundle over M^Γ.

iii) Create parameterized spectra.

$$\Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma_{in}} \longrightarrow \mathrm{Thom}_{sp}(\nu_\Gamma)$$
$$\Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma} \longrightarrow \Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma_{out}}$$

(The middle terms are related by Thom isomorphism.)

$$\begin{array}{ccc} \Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma_{in}} \wedge_{\mathrm{Rad}} H\mathbb{Q} & \longrightarrow & \mathrm{Thom}_{sp}(\nu_\Gamma) \wedge_{\mathrm{Rad}} H\mathbb{Q} \\ & & \downarrow \\ & & \Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma} \wedge_{\mathrm{Rad}} \Sigma^{hd} H\mathbb{Q}_{\tilde{\mathcal{L}}^{\otimes d}} \longrightarrow \Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma_{out}} \wedge_{\mathrm{Rad}} \Sigma^{hd} H\mathbb{Q}_{\tilde{\mathcal{L}}^{\otimes d}} \end{array}$$

Moving the degree-shift gives

$$\Xi\colon \Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma_{in}} \wedge_{\mathrm{Rad}} H\mathbb{Q}_{\mathcal{L}^{\otimes d}} \longrightarrow \Sigma^\infty_{\mathrm{Rad}} M_+^{\Gamma_{in}} \wedge_{\mathrm{Rad}} H\mathbb{Q}.$$

Taking homology gives

$$H_*(\mathrm{Rad}, \mathcal{L}^{\otimes d}) \otimes H_*(LM;\mathbb{Q})^{\otimes in} \longrightarrow H_*(LM;\mathbb{Q})^{\otimes m}.$$

2.3 Tubular neighbourhood.

- Fix a normal bundle ν for $\nabla\colon M \to M^2$, and a propagating flow on ν.
- We want
$$Pf\colon \nu^h_{P\Pi}$$
where $\nu^h_{P\Pi}$ is the pullback of h copies of ν under the map $M^{P\Pi} \to M^h$.
- Fix $L \in \mathrm{PRad}$ and $(g,\nu) \in (\nu^h_{P\Pi})_L$ where $g\colon P\Pi_L \to M$ and $\nu \in \nu^{\oplus h}_{ev(g)}$. then
$$Pf(g,\nu)\colon y \mapsto Z_1(v_{1,\lambda(1)}, \eta_1(y))\left[Z_2(v_{2,\lambda(2)}, \eta_2(y))\left[\cdots\left[g(q(y))\right]\cdots\right]\right]$$
where $y \in P\tilde{\Pi}_L$ and $q\colon P\tilde{\Pi}_L \to P\Pi_L$, and the Z_i are the flows that come out of the propagating flow and the $\eta_i(y)$ are 'flow control functions' going from $P\tilde{\Pi}_L \to [0,1]$.

3. HCFT. Recall that $\mathcal{L}$ was compatible with disjoint union and composition. The spaces Rad are *almost* a prop in topological spaces.

Definition. $H_*(\mathrm{PRad};\mathcal{L}^{\otimes d})$ prop in graded vector spaces.

Definition. An *HCFT* (without units or counits) of degree d is an algebra over this prop.

Theorem. *$H_*(LM;\mathbb{Q})$ is an HCFT of dimension d.*

12. Nathalie Wahl — String topology *via* Hochschild homology

We saw this morning that topology is tricky, and I'm gonna show you that algebra is easy. Of course this is what I say, but then I was preparing my talk and I realised that there's all these little things that you need to know.

Theorem (Jones)**.** *Let M be 1-connected and work over a field. There is a quasi-isomorphism*

$$C_*(C^*(M), C^*(M)) \xrightarrow{\simeq} C^*(LM)$$

where $C^*(M)$ denotes the singular cochain algebra.

Now $H^*(M)$ =PD algebra=Frobenius algebra (meaning that it has a non-degenerate pairing). So $C^*(M)$ =some homotopy version of this. Why am I saying this? Because if you know that there is some algebraic structure on the algebra then you know there should be some algebraic structure on the Hochschild homology.

Theorem (Costello, Kontsevich-Soibelman)**.** *If A is an 'A_∞-Frobenius algebra', then $C_*(A,A)$ admits an action of the prop*

$$\bigsqcup_{g\geqslant 0} C_*(\mathcal{M}_{g,p+q}) = \bigsqcup_{g\geqslant 0} C_*(BDiff(S_{g,p+q}))$$

i.e. maps

$$C_*(A,A)^{\otimes p} \otimes C_*(\mathcal{M}_{g,p+q}) \longrightarrow C_*(A,A)^{\otimes q}$$

compatible under $\circ$ and $\sqcup$.

Theorem (Tradler-Zeinalian)**.** *If A is a symmetric dg-Frobenius algebra then $C_*(A,A)$ admits such an action by a prop of Sullivan diagrams.*

Here 'symmetric' means that the pairing is symmetric.
Plan:

- Relate these things to yesterday's talk.
- Give a proof of both at once.
- Use this setup to compute nontrivial higher string operations.

Recall: A prop is a symmetric monoidal category whose objects are $\mathbb{N}$.

Example. A_∞ is the prop with

$$A_\infty(m,n) = \mathbb{Z} - \text{free on disjoint unions of trees with } m+n \text{ leaves.}$$

Here degree $= \sum_v |v| - 3$ and diff $= \sum \text{blow} - \text{ups of vertices}$.

Example. $\mathcal{O}$ is the prop with

$$\mathcal{O}(m,n) = \mathbb{Z} - \text{free on fat graphs with } m+n \text{ leaves.}$$

(It is the same as A_∞ except that there the graphs were of a very special kind.)

Note. $i\colon A_\infty \to \mathcal{O}$ is an A_∞-prop.

Example. The prop $H_0(\mathcal{O})$ which has $H_0(\mathcal{O})(n,m) = H_0(\mathcal{O}(n,m))$ which is free on the topological typs of graphs, or equivalently on the topological types of open cobordisms $nI \to mI$. It is concentrated in degree 0. It is an A_∞-prop via $A_\infty \to \mathcal{O} \to H_0(\mathcal{O})$.

Proposition (Lauda-Pfeffer)**.**

$$H_0(\mathcal{O}) - \text{algebras} \cong \{\text{symmetric Frobenius algebras}\}$$

Theorem (W).

$$\begin{array}{ccccc} C_*(\mathrm{Rad}) & \xrightarrow{\simeq} & \coprod_{g\geqslant 0, r\geqslant 0} C_*(\mathcal{M}^r_{g,p+q}) & \xrightarrow{\simeq} & \mathrm{Nat}_{\mathcal{O}}(p,q) \\ \downarrow & & \downarrow & & \downarrow \\ C_*(\mathrm{Rad}^{\mathrm{deg}}) & \xrightarrow{\simeq} & \coprod_{g\geqslant 0, r\geqslant 0} SD^r_{g,p,q} & \xrightarrow[\simeq]{} & \mathrm{Nat}_{H_0(\mathcal{O})}(p,q) \end{array}$$

Constructing (formal) natural operations on $C_*(A,A)$. Fix a general A_∞-prop

$$i\colon A_\infty \to \mathcal{E}.$$

"Free $\mathcal{E}$-algebras"=representable functors $\mathcal{E}(p,-)\colon \mathcal{E} \to \mathrm{Ch}$, $n \mapsto \mathcal{E}(p,n)$.

Theorem (WW). *If a prop $\mathcal{D}$ acts naturally on the Hochschild complex of the representable functors $\mathcal{E}(p,-)$, i.e. if there is an action*

$$C^n(\mathcal{E}(p,-)(0) \otimes \mathcal{D}(n,m) \longrightarrow C^m(\mathcal{E}(p,-))(0)$$

natural in p, then $\mathcal{D}$ acts naturally on the Hochschild complex of $\mathcal{E}$-algebras, i.e. there are maps

$$C_*(A,A)^{\otimes n} \otimes \mathcal{D}(n,m) \longrightarrow C_*(A,A)^{\otimes m}$$

for any $A = \Phi(1)$ where $\Phi\colon \mathcal{E} \to \mathrm{Ch}$ strong symmetric monoidal. (Weaker action for weaker monoidality.)

Recall: If $\Phi\colon \mathcal{E} \to \mathrm{Ch}$ is an functor, its Hochschild $C(\Phi)\colon \mathcal{E} \to \mathrm{Ch}$ is given by

$$C(\Phi)(m) \;=\; \Phi(-+m) \otimes_{A_\infty} \mathcal{L} \;\cong\; \bigoplus_{n\geqslant 1} \Phi(n+m) \otimes L_n \;\cong\; \bigoplus_{n\geqslant 1} \Phi(n+m)[n-1]$$

Picture: $L_n = \langle l_n \rangle$

image

where l_n is a corolla at a white vertex with leaves labelled $1, \ldots, n$ and the leaf 1 marked. Then

$$C^n(\Phi)(m) \cong \bigoplus_{k_1,\ldots,k_n \geqslant 1} \Phi(k_1 + \cdots + k_n + m) \otimes_{A_\infty} (l_{k_1} \otimes \cdots \otimes l_{k_n}).$$

Double bar construction (symmetric version). Given $\Phi\colon \mathcal{E} \to \mathrm{Ch}$, we construct $B(\Phi, \mathcal{E}, \mathcal{E})\colon \mathcal{E} \to \mathrm{Ch}$, quasi-isomorphic (as functors) to Φ. It starts as a double complex with

$$B_p(\Phi, \mathcal{E}, \mathcal{E}) = \bigoplus_{k_0,\ldots,k_p} \Phi(k_0) \otimes_{\Sigma_{k_0}} \mathcal{E}(k_0,k_1) \otimes_{\Sigma_{k_1}} \cdots \otimes_{\Sigma_{k_{p-1}}} \mathcal{E}(k_{p-1},k_p) \otimes_{\Sigma_{k_p}} \mathcal{E}(k_p,-)$$

and then one takes the total complex.

Proof. $\Phi\colon \mathcal{E} \to \mathrm{Ch}$ strong symmetric monoidal, $A = \Phi(1)$.

$$\begin{array}{ccc}
C_*(A,A)^{\otimes n} \otimes \mathcal{D}(n,m) & \dashrightarrow & C_*(A,A)^{\otimes m} \\
\downarrow \cong & & \downarrow \cong \\
C^n(\Phi)(0) \otimes \mathcal{D}(n,m) & & C^m(\Phi)(0) \\
\sim \uparrow\downarrow \sim & & \uparrow \cong \\
C^n(B^{\Sigma}(\Phi,\mathcal{E},\mathcal{E}))(0) \otimes \mathcal{D}(n,m) & & C^m(B^{\Sigma}(\Phi,\mathcal{E},\mathcal{E}))(0) \\
\cong \uparrow & & \cong \uparrow \\
B^{\Sigma}(\Phi,\mathcal{E},C^n(\mathcal{E})(0)))(0) \otimes \mathcal{D}(n,m) & \longrightarrow & B^{\Sigma}(\Phi,\mathcal{E},C^m(\mathcal{E})(0)))(0)
\end{array}$$

We use the fact that $C^n(B_p(\Phi,\mathcal{E},\mathcal{E}))(0)$ is given by

$$\bigoplus_{\substack{k_0,\ldots,k_p \\ j_1,\ldots,j_n \geqslant 1}} \Phi(k_0) \otimes_{\Sigma_{k_0}} \mathcal{E}(k_0,k_1) \otimes_{\Sigma_{k_1}} \cdots \otimes_{\Sigma_{k_{p-1}}} \mathcal{E}(k_{p-1},k_p) \otimes_{\Sigma_{k_p}} \mathcal{E}(k_p, j_1+\cdots+j_n) \otimes_{A_\infty} (L_{j_1} \otimes \cdots \otimes L_{j_n}$$

□

Note. The action is explicit and easy to unravel. Take $m = n = 1$.

$$\begin{array}{c}
(a_1 \otimes \cdots \otimes a_p) \otimes d \\
\downarrow \\
((a_1 \otimes \cdots \otimes a_p) \otimes (id_p \otimes l_p) + \cdots) \otimes d \\
\downarrow \\
((a_1 \otimes \cdots \otimes a_p) \otimes (\sum e_i \otimes l_{q_i}) + \cdots \\
\downarrow \\
(b_1 \otimes \cdots \otimes b_{q_i}) \otimes l_{q_i} + 0
\end{array}$$

Where the map

$$C(\mathcal{E}(p,-))(0) \otimes \mathcal{D}(1,1) \longrightarrow C(\mathcal{E}(p,-))(0)$$

sends $id_p \otimes l_p \otimes d$ to $\sum e_i \otimes l_{q_i}$.

I should say where this came from ... it started at a workshop here many years ago called ‘Strings in Copenhagen’.

Take $\mathcal{E} = \mathcal{O}$. Want some $\mathcal{D}$ acting on $C^n(\mathcal{O}(p,-))(0)$.

$$C^n(\mathcal{O}(d,-))(0) = \bigoplus_{k_1,\ldots,k_n \geqslant 1} \mathcal{O}(p, k_1+\ldots+k_n) \otimes_{A_\infty} (L_{k_1} \otimes \cdots \otimes L_{k_n})$$

An element here is a fat graph with p inputs and $k_1 + \cdots + k_n$ outputs, together with the corollas of k_i leaves around white vertices with marked edge. There is nothing to stop me from attaching these graphs together, and that’s what I do. The vertices from the first part are ordinary, called ‘black’. The vertices from the second are ‘white’

The result is exactly the model Costello and Kontsevich-Soibelman use for moduli spaces of cobordisms from p open to n closed. Now we take

$$C^n(\mathcal{O}(p,-)(0)\otimes\mathcal{D}(n,m)\longrightarrow C^m(\mathcal{O}(p,-))(-)$$

where $\mathcal{D}(n,m)$ is a chain complex of bordisms from n closed to m closed.

Sullivan diagrams and examples of non-trivial operations.

Proposition. *$C^n(H_0(\mathcal{O})(p,-))(0)$ is isomorphic to a chain complex of 'Sullivan diagrams' with p 'open boundaries'.*

Here a Sullivan diagram on n circles is an equivalence class of fat graphs with n embedded boundary components.

image 1

Such a graph has edges divided into 'circle edges' (around the outside) and 'chords' (the rest). The equivalence relation is given by sliding chords along each other. For example:

image 2

'Anything outside the circles is allowed to do all sorts of things.' 'You are inverting the edge collapses of chords.' Such a thing has a topological type as an open-closed cobordism.

- Outgoing closed = the n circles.
- Incoming closed = leaf alone in its boundary circle.
- Incoming/outgoing open = leaves .
- Degree = number of circle edges minus n.
- Differential = sum of collapse of circle edges.

Example.

$Image3.$

- Gluing = sum over all meaningful things.

Example.

$image4$

Action on $C_*(H^*(S^n),H^*(S^n))\simeq C^*(LS^n)$, nontrivial on $HH_*(H^*(S^n),H^*(S^n))$. (We won't have to worry about signs.)

$$H^*(S^n)=\langle 1,x\rangle$$

where $x^2=0$, 1 unit, $\Delta(x)=x\otimes x$, $\Delta(1)=(1\otimes x)\pm(x\otimes 1)$. We will consider the reduced Hochschild complex.

$$C_*(H^*(S^n),H^*(S^n))=\mathbb{Z}\langle 1,\ x,\ 1\otimes x,\ x\otimes x,\ 1\otimes x\otimes x,\ x\otimes x\otimes x,\ \ldots\rangle$$

with generators in degree 0, n, $n-1$, $2n-1$, $2n-2$, $3n-2$ and so on. There is only one generator in each degree. And most differentials are zero. (Most of them multiply x with itself, which produces zero.)

Example.

image 5

image 6

image 7

Corollary. *The classes* $[t_1], [t_3], \ldots$ *are nontrivial in* $H_*(SD)$.