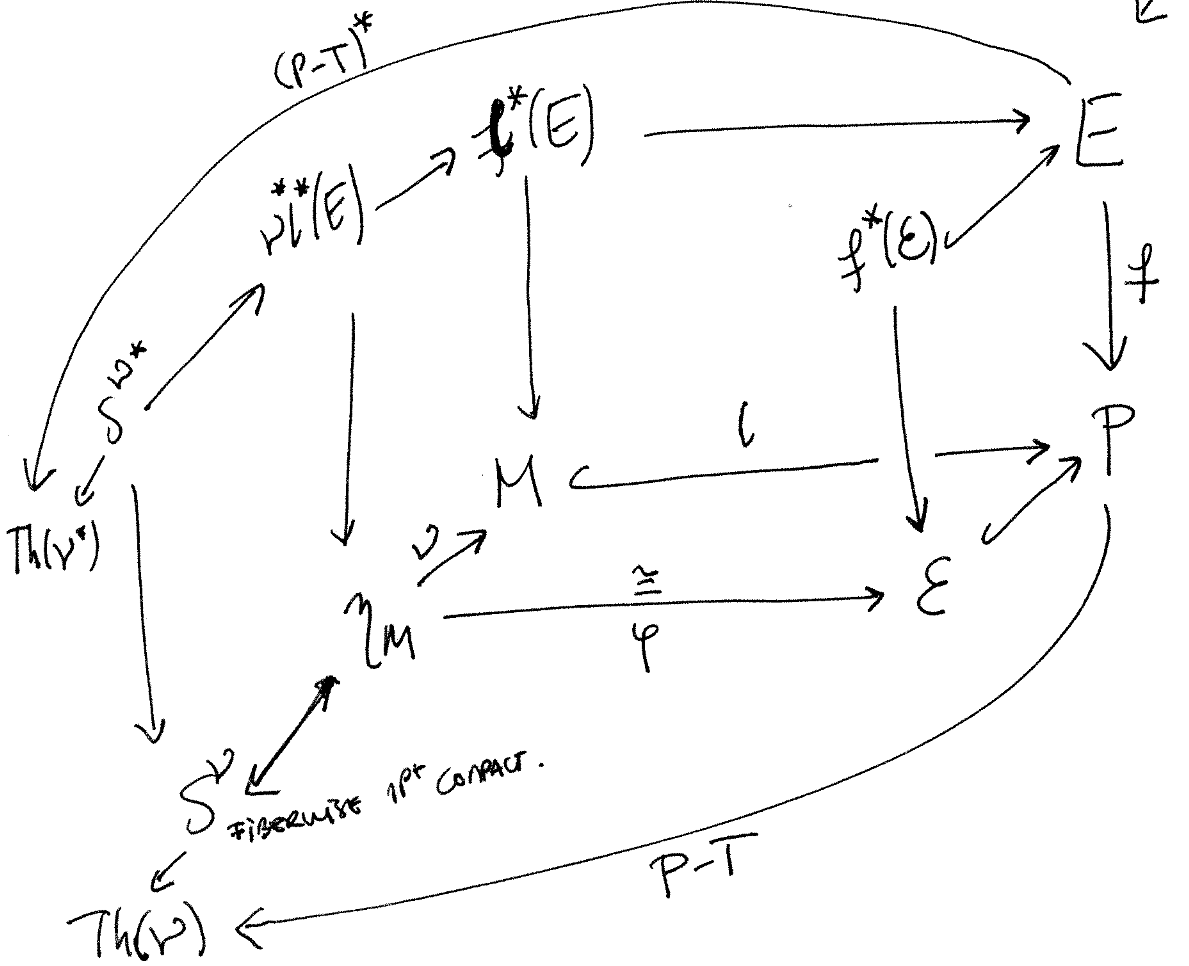
TARDE, UMKEHR MAPS

THE ENEMY!
(ENNEMI!)



M, P MANIFOLDS

η_M NORMAL BDL

φ TUBULAR NBHD

S^ν SPHERE BDL

f FIBRATION

THEN BO GIVES UMKEHR MAPS

$$u : H_*(P) \xrightarrow{P-T} H_*(Th(\nu)) \longrightarrow H_{*+k}(M)$$

$$u^* : H_*(E) \xrightarrow{(P-T)^*} H_*(Th(\nu^*)) \longrightarrow H_{*+k}(M)$$

$\rightsquigarrow$ COHEN-KLEIN — [illegible] ...

NOTE: ENTIRE DIAGRAM IS "OVER P".

DEF: LET X BE A TOPOLOGICAL SPACE. $\mathcal{T}_X$ IS THE OVER CATEGORY OF X:

$$\begin{cases} Obj(\mathcal{T}_X) = \left\{ \begin{array}{c} Y \\ \downarrow \\ X \end{array} \right\} \\ Mor = \left\{ \begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd} \right\} \end{cases}$$

DEF: $\mathcal{R}_X$ IS EXTENSION OF $\mathcal{T}_X$ TO RETRACTS: (RETRACTIVE SPACES)

$$Obj(\mathcal{R}_X) = \left\{ \begin{tikzcd} Y \arrow[d, shift right, "s"'] \\ X \arrow[u, shift right, "r"'] \end{tikzcd} \right\} \qquad s \circ r = 1\!\!1_X$$

$$Mor(\mathcal{R}_X) = \left\{ \begin{tikzcd} Y \arrow[rr] \arrow[dr, shift left] & & Z \arrow[dl, shift left] \\ & X \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd} \right\}$$

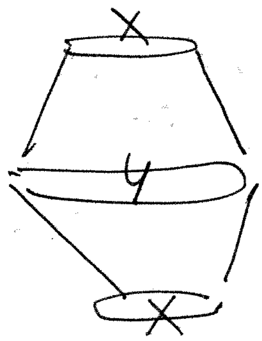
(POINTED VERSION OF $\mathcal{T}_X$ — IN EACH FIBER, WE HAVE A BASEPOINT)

DEF: LET $Y, Z \in \mathcal{R}_X$. DEFINE $Y \cup_X Z$ VIA A PUSHOUT:

$$\begin{tikzcd} y \arrow[rr, mapsto] & & y \\ Y \amalg Z \arrow[rr] \arrow[d] & & Y \arrow[d] \\ z \arrow[r, mapsto] & r_Y(s_Z(z)) & \\ Z - z \arrow[rr] & & Y \cup_X Z \arrow[ul, phantom, "\ulcorner"] \\ r_Z(s_Y(y)) & & \end{tikzcd}$$

DEF: THE SUSPENSION OF Y IS GIVEN BY

$$S_X Y = X \times \{0\} \cup_{s_Y} Y \times I \cup_{s_Y} X \times \{1\}$$



THE REDUCED SUSPENSION

$$\Sigma_X Y = S_X Y \big/ (r_Y S_Y(Y), t) \sim (r_Y S_Y(Y), s)$$

SMASH PRODUCT

$$\begin{tikzcd} & Y \cup_X Z \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \times_X Z \arrow[dl] \\ & Y \wedge_X Z & \end{tikzcd}$$

Def: BY A FIBERED SPECTRUM $\mathcal{E} \in \underline{\mathrm{Spec}(R_X)}$, WE UNDERSTAND

$$\{\underset{\cup}{\overset{R_X}{\mathcal{E}_j}}\}_{j \in \mathbb{N}} \quad \text{AND} \quad \sigma : \Sigma_X \mathcal{E}_j \longrightarrow \mathcal{E}_{j+1}$$

DEFINE Ω_X AS THE ADJOINT TO Σ_X.

$\mathcal{E}$ IS AN Ω-SPECTRUM (IS FIBRANT IS SOME APPROPRIATE MODEL CATEGORY) if $\mathcal{E}_j \to \Omega_X \mathcal{E}_{j+1}$ w.e. $\forall j$

INTERESTING SPECTRA:

* SUSPENSION SPECTRUM $(\Sigma_X^\infty Y)_j = \Sigma_X^j Y$
* IF $C \in \mathrm{Spec}(\mathrm{Top}_*)$, WE LET $C \times X \in \mathrm{Spec}(R_X)$ BE GIVEN BY $(C \times X)_j = C_j \times X$ AND FOR $E \to X$ FIBRATION, $C \otimes E$ IS GIVEN BY

$$\begin{tikzcd} E \times * \arrow[r, hook] \arrow[d] & E \times C_j \arrow[d] \\ X \arrow[r] & (E \otimes C)_j \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

LET $\nu \to X$ BE A VECT. BDL AND ...

THEN WE CAN ~~DEFINE~~ $(\nu_{\mathcal{E}})_j = S^\nu \wedge_X \mathcal{E}_j$

IF ν IS A VIRTUAL BDL, e.g. REPRESENTED BY A NORMAL BDL $n_M \to X \hookrightarrow \mathbb{R}^L$ $(\nu = -TX)$

$$(\nu_{\mathcal{E}})_{j+L} = S^{n_X} \wedge_X \mathcal{E}$$

(COULD DO THING MORE GENERALLY FOR ANY VIRTUAL BDL BY FINDING A LARGE ENOUGH BDL TRIVIALIZING IT)

COHOMOLOGY WITH COEFF IN $\mathcal{E} \in \mathrm{Spec}(\mathcal{R}_X)$ IS GIVEN BY $H^*(Y;\mathcal{E})_j = \hom_{\mathcal{J}_X}(Y, \mathcal{E}_j)$

FOR $Y \in \mathcal{J}_X$

$$H^*(-;\mathcal{E}) : \mathcal{J}_X \to \mathrm{Spec}(\mathrm{Top}_*)$$

j^{th} SPACE OF A SPECTRUM WITH STRUCTURE MAPS FROM $\mathcal{E}_j$ $\quad S^1_X Y \to S^1_X \mathcal{E}_j \to$

BASE POINT $Y \to X \to \mathcal{E}_j$

HOMOLOGY WITH COEFF IN $\mathcal{E} \in \mathrm{Spec}(\mathcal{R}_X)$ IS GIVEN BY $H_*(Y;\mathcal{E}) = (Y \times_X \mathcal{E}_j) \cup_Y CX$

THESE THEORIES (H^* AND H_*) ARE HOMOTOPY INVARIANTS IF $\mathcal{E}$ IS AN Ω-SPECTRUM.

<u>THM</u>: LET $\overset{\mathcal{E}}{P}{}_{\downarrow X}$ BE A MANIFOLD. THEN THERE IS A WEAK EQUIV. OF SPECTRA

$$H^*(P, \mathcal{E}) \simeq H_*(P, {}^{-T(P)}\mathcal{E})$$

WHERE $\quad {}^{-T(P)}\mathcal{E} = \mathcal{E} \wedge_X \underbrace{(S^{-TP} \amalg X)}_{\text{LEVELWISE}}$

DEFINED USING

$$\begin{tikzcd}
X \amalg S^{-TP} \arrow[dr] & -TP \arrow[l, hook'] \arrow[d] & \\
& P \arrow[d] & \mathcal{E} \arrow[dl, shift left] \\
& X \arrow[ur, shift left] \arrow[uul, bend left] &
\end{tikzcd}$$

SO $\quad H_\bullet(P, {}^{-T(P)}\mathcal{E}) = \left(P \wedge_X (\mathcal{E} \wedge_X (S^{-TP} \amalg X))\right) / X$

<u>EXAMPLE</u> (ATIYAH DUALITY)

$$M = X \qquad \mathcal{E} = \overset{S^0}{\Sigma^\infty_{fib}}(M \amalg M)$$
$$\downarrow$$
$$M$$

$$H^\bullet(M, \mathcal{E}) = \mathrm{Hom}_{Spec}(M_+, S^0)$$

$$H_\bullet(M, {}^{-TM}\mathcal{E}) = M^{-TM}$$

MORE ABOUT THIS ON NEXT PAGE

RECALL:

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \in \mathcal{T}_X \rightsquigarrow \begin{array}{c} Y \sqcup X \\ \downarrow \uparrow \\ X \end{array} \in \mathcal{R}_X$$

$$\begin{array}{c} \rightsquigarrow \\ \Sigma^\infty_{fib}(Y \sqcup X) \\ \downarrow \\ X \end{array}$$

EX: 1) $X = M = Y$ $\quad \begin{array}{c} M \\ \downarrow id \\ M \end{array} \rightsquigarrow \begin{array}{c} M \sqcup M \\ \downarrow \uparrow \\ M \end{array}$ TAKE $\mathcal{E} = \Sigma^\infty_{fib}(M \sqcup M) \downarrow M$, $\underline{S}^0 \downarrow$

EXERCICE: IN THIS CASE, THE THEOREM SAYS THAT

$$\mathrm{Hom}_{Spec}(M_+, \underline{S}^0) \simeq M^{-TM}$$ [ATIYAH DUALITY]

2) $H\mathbb{Z} \rightsquigarrow$ FIBERWISE VERSION $\begin{array}{c} H\mathbb{Z}(M) \\ \downarrow \\ M \end{array} = H\mathbb{Z}(M) \times M \in Spec(\mathcal{R}_M)$ ($H\mathbb{Z}$)

THEN THE THEOREM IMPLIES

$$\mathrm{Hom}_{Spec}(M, H\mathbb{Z}) \simeq M^{-TM} \wedge H\mathbb{Z}$$

$$\downarrow \pi_*$$

$$H^*(M) \cong H_*(M^{-TM}) \xrightarrow[\text{IF } \underline{M \text{ ORIENT}}]{\text{THOM ISO}} H_{*-n}(M)$$

$\longrightarrow$ POINCARÉ DUALITY.

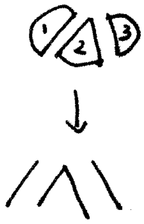
(NOTE: 2) $\cong$ 1) $\wedge_M H\mathbb{Z}$).

RECALL: $\mathrm{Fat}^{a_k}_{g,k+1}$ = FAT GRAPHS WITH k INCOMING ADMISSIBLE BOUNDARIES AND 1 OUTGOING.

$\mathrm{Fat}^{a_1}_{g,1+k}$ = ... $1 \to k$ ...

EX:

CONSIDER THE PULL-BACK



$$\begin{tikzcd} L_\Gamma M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \prod_{i=1}^{k} M^{S_i} \arrow[l, dashed, bend right, "\text{WANT AN UMKEHR}"'] \arrow[d] \\ M^{\pi_0(\Gamma\setminus S_{k+1})} \arrow[r] & \underbrace{\prod_{i=1}^{k} M^{\overline{S_i\cap(\Gamma\setminus S_{k+1})}^{(?)}}}_{=X} \end{tikzcd}$$

$\overline{S_i\cap(\Gamma\setminus S_{k+1})}$ = EXTRA EDGES OF S_i

DEF/THM:
AN UMKEHR MAP OF u WRT $\mathcal{E}$ IS $u^! = H^\bullet(u,\mathcal{E})$

FOR $u: Q \longrightarrow P$ AND $\mathcal{E} \in \mathrm{Spec}(R_X)$

$$\begin{tikzcd} Q \arrow[rr, "u"] \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$

TAKE $\mathcal{E} = \Sigma^\infty_{fib}\left(\prod_{i=1}^{k} M^{S_i}\right) = \left(\prod_{i=1}^{k} M^{S_i}\right) \otimes \underline{S}$

$$
\begin{tikzcd}
 & L_\Gamma M \arrow[r] \arrow[d, "r"] \arrow[dl, bend right, "ev_*"'] & \prod M^{S_i} \arrow[d, "res"] \arrow[ddr, bend left, "ev_*"] & \\
 & M^{\pi_0(\Gamma \setminus S_{\ell+1})} \arrow[r] \arrow[dl, bend left, "\text{pr (CHOICE HERE!!)}"] & X := \prod M^{S_i \cap (\Gamma \cap S_{\ell+1})} \arrow[dr, "ev_{v_\ell} = h"] & \prod_{\pi_0(S_i \cap \ldots)} M \arrow[l, "cst"'] \arrow[d, "pr"] \\
M \arrow[uur, bend left, "cst"] \arrow[ur, bend left, "\Delta"] \arrow[rrr, bend right, "\Delta^\ell"'] & & & M^\ell \arrow[u, bend right, "\Delta"']
\end{tikzcd}
$$

OR THERE

$$\hom_{\mathcal{T}_{M^\ell}}\Big(M^\ell, \prod_{i=1}^{\ell} M^{S_i} \underset{ev}{\otimes} X\Big)$$

$$\|$$

$$\hom_{J_{M^\ell}}\Big(M^\ell, \prod_{i=1}^{\ell} M^{S_i} \underset{\text{prohores}}{\otimes} X\Big)$$

$$\Big\downarrow \text{PUSH } \Delta$$

$$\hom_{J_{M^{\pi_0(S_i \cap (\Gamma \cap S_{\ell+1}))}}}\Big(M^\ell, \prod_{i=1}^{\ell} M^{S_i} \underset{\text{hores}}{\otimes} X\Big)$$

$$\Big\downarrow \text{PUSH } e$$

$$\hom_X\Big(M^k, \prod M^{S_i} \otimes X\Big)$$