HIGHER STRING TOPOLOGY OPERATIONS

/ VERONIQUE GODIN '07

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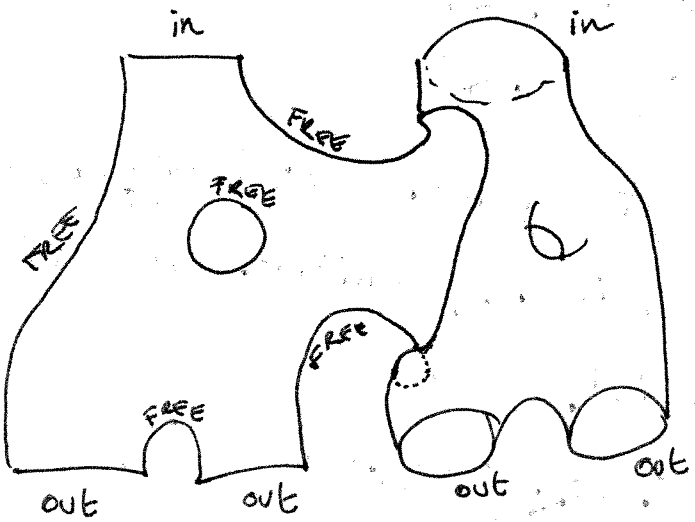
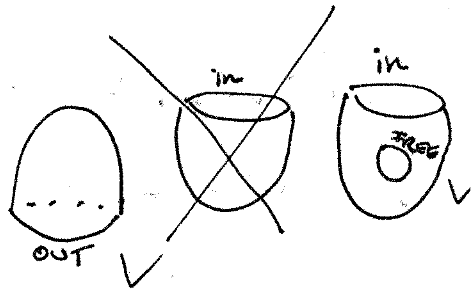
<u>IDEA</u>: EXTEND STRING TOPOLOGY ON $H_* LM$ TO OPERATIONS COMING FROM $H_*(B\mathrm{Diff}(S))$, S APPROPRIATE SURFACE

REPLACE SURFACES WITH FAT GRAPHS, LIKE REPLACING

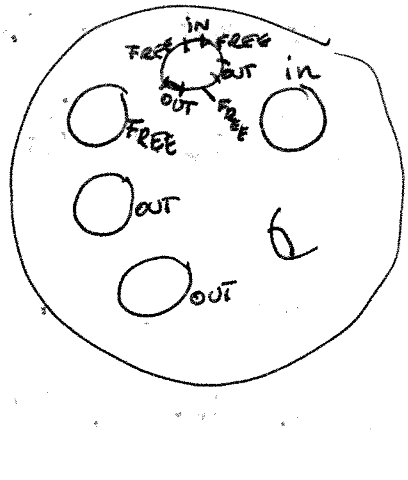
WITH

AN <u>OPEN-CLOSED COBORDISM S WITH POSITIVE BOUNDARY</u> IS A COBORDISM OF 1-MFDS WITH BDRY (THE IN<u>COMING</u> AND <u>OUT</u>GOING BOUNDARIES), THE REST BEING FREE BOUNDARIES, SUCH THAT NOT ALL OF THE BOUNDARY OF EACH COMPONENT IS INCOMING.





$\cong$



WE WILL CONSIDER $\mathrm{Diff}(S, \partial_{io})$, $\partial_{io} = \partial_{in} \cup \partial_{out}$

WHEN $\partial_{in} S_2 = \partial_{out} S_1$, WE CAN FORM $S_2 \# S_1 := S_2 \cup_{\partial'} S_1$,
 $\overset{\shortparallel}{\partial'}$

AND THERE IS A CORRESPONDING MAP

$$B\mathrm{Diff}(S_2, \partial_{io}) \times B\mathrm{Diff}(S_1, \partial_{io}) \to B\mathrm{Diff}(S_2 \# S_1, \partial_{io})$$

EULER CHARACTERISTICS

A_* BOUNDED GRADED VECTOR SPACE / $\mathbb{Z}$-MODULE

$$\mathrm{euler}(A_*) = \sum_i (-1)^i \dim A_i$$

$$\det(A_*) = \bigotimes_{i \text{ even}} \Lambda^{\dim A_i} A_i \otimes \bigotimes_{i \text{ ODD}} \Lambda^{\dim A_i} A_i^*$$

THEN, GIVEN A L.E.S. $\cdots \to A_i \to B_i \to C_i \to A_{i-1} \to \cdots$

$$\mathrm{euler}(B_*) = \mathrm{euler}(A_*) + \mathrm{euler}(C_*)$$

AND $\det(B_*) = \det(A_*) \otimes \det(C_*)$

<u>DEF</u>: χ_S IS THE COEFFICIENT SYSTEM ON $B\mathrm{Diff}(S, \partial_{io})$ WITH FIBER $\det(H_*(S, \partial_{in} S))$, SHIFTED TO LIE IN DEGREE $-\mathrm{euler}(H_*(S, \partial_{in}))$, SO

$$H_*(B\mathrm{Diff}(S), \chi_S) \cong H_{*-\mathrm{euler}}(B\mathrm{Diff}(S), \det)$$

GIVEN S_2, S_1 AS BEFORE, THERE IS A L.E.S.

$$\cdots \to H_*(S_1, \partial_{in} S_1) \to H_*(S_2 \# S_1, \partial_{in} S_1) \to \cdots$$

$$\to \cancel{H_*(S_2 \# S_1, S_1)} \to \cdots$$

$$\overset{\shortparallel\!\wr}{H_*(S_2, \partial_{in} S_2)}$$

$$\gamma_{S_2,S_1} : H_*(BDiff(S_2, \partial_{in}); \chi_{S_2}^{\otimes d}) \otimes H_*(BDiff(S_1, \partial_{in}); \chi_{S_1}^{\otimes d})$$

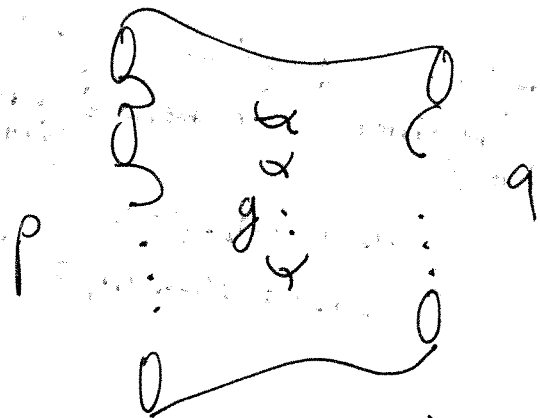
$$\longrightarrow H_*(BDiff(S_2 \# S_1; \partial_{in}); \chi_{S_2 \# S_2}^{\otimes d})$$

- DEGREE SHIFT MATCH ✓
- $H_*(\ , det_2) \otimes H_*(- , det_1) \longrightarrow H_*(- , det_2 \otimes det_1)$

(d IS THE DIMENSION OF THE MANIFOLD WE WORK WITH.

(IF THERE IS AT LEAST ONE WINDOW, THE LINE BDL IS TRIVIAL. BUT NOTE THAT WE STILL HAVE TO CHOOSE AN ORIENTATION.)

<u>EX</u>:



$$\text{Euler}(H_*(S, \partial_{in})) = 2 - 2g - p - 1$$

<u>Def</u>: $(\mathcal{O}_*, \mathcal{C}_*)$ PAIR OF GRADED VECTOR SPACES.
AN <u>OPEN-CLOSED HOMOLOGICAL CONFORMAL FIELD THEORY</u> <u>WITH POSITIVE BOUNDARY</u> OF DEGREE d (D+HCFT) CONSISTS OF, FOR EACH S,

WITH p_o / p_c INCOMING, q_o / q_c OUTGOING OPEN/CLOSED BOUNDARY COMPONENTS,

A MAP $H_*(BDiff(S, \partial_{in}); \chi_S^{\otimes d}) \otimes \mathcal{O}_*^{\otimes p_o} \otimes \mathcal{C}_*^{\otimes p_c} \longrightarrow \mathcal{O}_*^{\otimes q_o} \otimes \mathcal{C}_*^{\otimes q_c}$

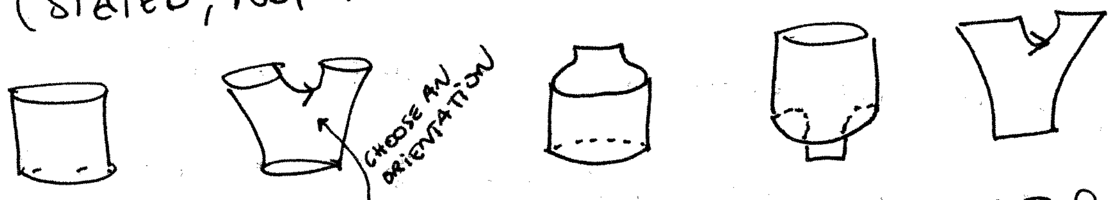
AND THOSE MAPS ARE ASSOCIATIVE WRT GLUING OF SURFACES

[MONOIDAL ??]

NOTE: THESE OPERATIONS ... EULER CHARACTERISTIC, I.E. LOWER DEGREE IN MOST CASES.

THM $(H_*(M), H_*(LM))$ ADMITS THE STRUCTURE OF A DEGREE $\dim M$ $\mathbb{Q}_+$HCFT.

EXAMPLES: (STATED, NOT PROVED - IN THE PAPER)



S (in / out)					
$H_*(S,\partial_{in})$	0	$0\ \mathbb{Z}\ 0\ldots$	$0\ \mathbb{Z}\ 0\ldots$	$0,0,\ldots$	$0,\mathbb{Z},0\ldots$
Euler	0	-1	-1	0	-1
$H_*(BDiff)$	$\mathbb{Z}\ \mathbb{Z}\ 0\ldots$	$\mathbb{Z}\ \underbrace{\mathbb{Z}^3\ \mathbb{Z}^3}_{\text{ITS COMPOSITIONS WITH } \Delta}\ \mathbb{Z}\ 0\ldots$	$\mathbb{Z}\ \mathbb{Z}\ 0\ldots$	$\mathbb{Z}\ \mathbb{Z}\ 0\ldots$	$\mathbb{Z}\ 0\ 0\ldots$
OPERATION	id, Δ	$\pm$ CHAS-SULLIVAN PRODUCT	$H_*(M)\to H_{*-d}(LM)$; $[M]\mapsto \mathrm{euler}(M).pt$	0 BECAUSE COMPOSITION WITH Δ; $H_*(LM)\to H_*(M)$ PROJECTION	INTERSECT. PRODUCT ON $H_*(M)$

Δ: $H_*(LM)\to H_{*+1}(LM)$ FROM S^1-ACTION

NOTE:

$$\left(H_*(BDiff(S,\partial_{in}));\ \underbrace{\det H_*(S,\partial_{in})^{\otimes d}}_{\cong\ \mathbb{Z}}\right)$$

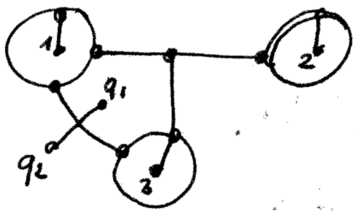
BUT NOT CANONICALLY

$\mathcal{F}at^a$ = CATEGORY OF ADMISSIBLE FAT GRAPHS.

$$= \coprod_{[S]} \mathcal{F}at^a_S \quad , \text{ WHERE IF } S \text{ HAS } \begin{cases} p \text{ CLOSED INCOMING} \\ q \text{ CLOSED OUTGOING} \\ \text{NO OPEN} \end{cases}$$

THEN $\mathcal{F}at^a_S = \mathcal{F}at^{ap}_{g,p+q}$ (IN YESTERDAY'S NOTATION).

EX:



$p=3$ $\quad \# eE = 7$
$q=2$ $\quad \# eV = 4$
$g=0$ $\quad \# eH = 14$

Γ FAT GRAPH. eE, eV, eH = EXTRA EDGES, EXTRA VERTICES, EXTRA HALF-EDGES, i.e. THOSE THAT DO NOT APPEAR IN AN INCOMING BOUNDARY.

FIX AN EMBEDDING OF $M \hookrightarrow W$, SOME AMBIENT EUCLIDEAN SPACE, AND LET ν BE ITS NORMAL BDL.

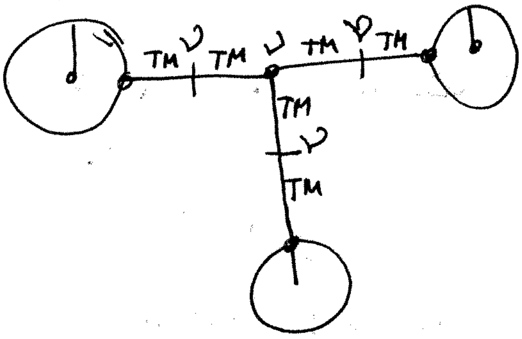
DEF: $\cdot$

$$\begin{array}{ccc} K_\Gamma & \longrightarrow & TM^{eH} \times \nu^{eV} \times \nu^{eE} \\ \downarrow & \lrcorner & \downarrow \\ M^\Gamma & \longrightarrow & M^{eH} \times M^{eV} \times M^{eE} \end{array}$$

$$\Gamma \xleftarrow{\text{SOURCE}} eH$$
$$\Gamma \longleftrightarrow eV$$
$$\Gamma \xleftarrow[\text{MIDPOINT}]{} eE$$

$\cdot \; W_\Gamma = W^{eE \sqcup eV}$

NOTE: $\dim K_\Gamma - \dim W_\Gamma = -\mathrm{EULER}(H_*(\Gamma, \partial_{in} \dots$
(EXCERCICE...)



WE WILL CREATE OPERATIONS

$$\Sigma^\infty M^{\partial_{in}\Gamma} \longrightarrow \mathrm{Thom}(K_\Gamma - M^\Gamma \times W_\Gamma)$$

BY DESUSPENDING THE RESULT OF:

PROP: THERE IS A CONTRACTIBLE SPACE $T(\Gamma)$ AND COLLAPSE MAPS

$$T(\Gamma) \times M^{\partial_{in}\Gamma} \times W_\Gamma \longrightarrow \mathrm{Thom}(K_\Gamma)$$

IDEA OF PROOF:

$$M^\Gamma \xrightarrow[①]{} M^{\partial_{in}} \times PM^{eE} \times M^{eV} \xleftarrow[②]{CST} M^{\partial_{in}} \times M^{eE} \times M^{eV} \xrightarrow[③]{} M^{\partial_{in}} \times W_\Gamma$$

③ IS AN EMBEDDING WITH NORMAL BDL $0 \times \nu^{eE} \times \nu^{eV}$, SO WE CAN CHOOSE $M^{\partial_{in}} \times W_\Gamma \longrightarrow \mathrm{Thom}(0 \times \nu^{eE} \times \nu^{eV})$
(FINITE DIM TUBULAR NBHD)

② $0 \times \nu^{eE} \times \nu^{eV} = cst^*(0 \times \nu_{1/2}^{eE} \times \nu^{eV})$ WHERE ($\begin{array}{ccc} \nu_{1/2} & \to & \nu \\ \downarrow & & \downarrow \\ PM & \xrightarrow[\frac{1}{2}]{} & M \end{array}$ (MID-POINT))

SO THERE IS A MAP (NO CHOICE!)

$$\mathrm{Thom}(0 \times \nu^{eE} \times \nu^{eV}) \to \mathrm{Thom}(0 \times \nu_{\frac{1}{2}}^{eE} \times \nu^{eV})$$

$$
\begin{tikzcd}
M^{\Gamma} \arrow[r] \arrow[d] & M^{\partial_{in}} \times PM^{eE} \times M^{eV} \arrow[d] \\
M^{V} \arrow[r, hook, "\sigma"'] & M^{V_{in}} \times M^{eH} \times M^{eV}
\end{tikzcd}
$$

σ HAS NORMAL BDL TM^{eH}, SO WE CAN CHOOSE COLLAPSE

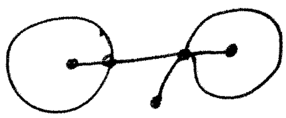
$$M^{\partial_{in}} \times PM^{eE} \times M^{eV} \longrightarrow \mathrm{Thom}(TM^{eH})$$

OR

$$\mathrm{Thom}(0 \times \nu_{1/2}^{eE} \times \nu^{eV}) \longrightarrow \mathrm{Thom}(\underbrace{TM^{eH} \oplus \text{PULLBACK}(0 \times \nu_{1/2}^{eE} \times \nu^{eV})}_{K_\Gamma})$$

COMBINING ①, ②, ③ AND FORMING A "SPACE OF CHOICES" $T(\Gamma)$, WE GET THE RESULT.

NOTE: IF YOU WANT TO DO THE PAIR OF PANTS, YOU HAVE TO WORK WITH SAY



$K_\Gamma : \underline{TM,\nu TM}$

$W_\Gamma : \overline{\quad W \quad}$

... SOME CANCELATIONS TO MAKE TO GET THE USUAL OPERATION.

EXTENDING TO FAMILIES (30 PAGES...)

1) OBTAIN THE RELEVANT MAPPING SPACES. WANT TO MODEL $E\mathrm{Diff}(S) \times_{\mathrm{Diff}(S)} M^S = (*)$ ← OR $M^{\partial_{in}}, M^{\partial_{out}}$ USING $\mathcal{F}at^a$.

2) UNDERSTAND HOW (K_Γ, W_Γ) VARIES WITH Γ, AND CREATE A VIRTUAL BDL K RELATED TO $\det(H_*(\Gamma, \partial_{in}))$, AS BUNDLE OVER (1).

3) EXTEND Thom COLLAPSE TO SIMPLICES .VI

FOR 1) (NOT TREATED IN THE PAPER FOR M^S)

"TRIVIAL" CLAIM: $\operatorname{hocolim}\left(\mathcal{F}at^q_S \to Top\right) \simeq (*)$

$$\Gamma \mapsto M^\Gamma$$

THEOREM (GODIN): $\operatorname{hocolim}\left(\mathcal{F}at^q_S \xrightarrow[\text{CONSTANT}]{} Top\right) \simeq BDiff(S, \partial_{in})$

(3) IS THE HARD PART, AND IS ~~[illegible]~~ THE BULK OF THE 30 PAGES.