TOPOLOGICAL FIELD THEORY ON THE HOCHSCHILD HOMOLOGY OF A CALABI-YAU ALGEBRA

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PLAN: I. HOCHSCHILD (CO)HOMOLOGY AND STRING TOPOLOGY

II. BABY AND BLUE THEOREMS (BABY = BV PART
BLUE = AS IN TITLE)

III PROOF OF THE BABY THEOREM

IV EXAMPLE OF STRING TOPOLOGY

V HYPOTHESIS OF BABY/BLUE THEOREM, IN LURIE'S LANGUAGE, MEANS A IS A CALABI-YAU ALGEBRA.

$\mathbb{F}$ FIELD (IN PARTICULAR OF CHARACT $p \neq 0$ — COULD DO SOMETHING EASIER IN CHARACT 0, VIA RATIONAL HOMOTOPY THEORY
OR COSTELLO)

I. LET A BE A dga, M AN (A,A)-BIMODULE
(EVERYTHING DIFF GRADED HERE)

Def: HOCHSCHILD COHOMOLOGY

$$HH^*(A,M) = \mathrm{Ext}_{A\otimes A}(A,M)$$

EXPLICITLY, LET $\varepsilon_A : B(A,A,A) \xrightarrow{\simeq} A$ QUASI-ISO OF A-BIMOD

$$C^*(A,M) = \mathrm{Hom}_{A\text{-BIMOD}}(B(A,A,A),M) \xrightarrow{H_*} HH^*(A,M)$$

$$C_*(A,M) = M \otimes_{A\otimes A^{op}} B(A,A,A) \longrightarrow HH_*(A,M)$$

LET $B: HH_*(A,A) \longrightarrow HH_{*+1}(A,A)$ THE CONNES BOUNDARY MAP.

$B^2 = 0$

<u>NOTE</u>: $HH^*(A, A^\vee) = HH_*(A,A)^\vee$ ← DUAL VECTOR SPACE

$\rightsquigarrow B^\vee : HH^*(A,A^\vee) \longrightarrow HH^{*-1}(A,A^\vee)$

<u>THM</u> (GERSTENHABER) $HH^*(A,A)$ IS A GERSTENHABER ALG,

i.e. IT'S A GRADED COMMUTATIVE ALG EQUIPPED WITH A LIE BRACKET OF DEGREE $\left[\begin{smallmatrix}-1\\+1\end{smallmatrix}\right.$ $\{,\}$ SATISFYING $\{a,bc\} = \{a,b\}c \pm b\{a,c\}$

CONVENTION : $A_n = A^{-n}$

GERST. STRUCT ON A_*, $\{,\}$ OF DEG ± 1

A^* — -1

<u>DEF</u>: A BV-ALG IS A GERSTENHABER ALG EQUIPPED WITH ~~AN OPERATOR~~ $\Delta : A_i \longrightarrow A_{i+1}$ s.t. $\Delta^2 = 0$, AND SUCH THAT THE LIE BRACKET IS GIVEN BY

$$\{a,b\} = \pm\Delta(ab) \mp (\Delta a)b \mp a(\Delta b)$$

<u>EX</u>. M^d d-DIMENSIONAL, SIMPLY CONNECTED, COMPACT, CLOSED MFD.

<u>THM</u> (CHAS-SULLIVAN) $H_{*+d}(LM)$ IS A BV-ALG.

$= \mathbb{H}_*(LM)$

<u>CONJECTURE</u>: ~~THERE~~ IS AN ISO OF GERSTENHABER ALGEBRAS

$$HH^*(S^*M, S^*M) \cong \mathbb{H}_*(LM)$$

SINGULAR COCHAINS OF M

PROVED OVER $\mathbb{Q}$ BY FELIX-THOMAS THEN !!

NOTE: THERE IS A FUNCTORIALITY ON ~~BOTH~~ SIDES
W.R.T. QUASI-ISOS / HOMOTOPY ~~EQUIV~~.

F.M. CONJECTURE PROVED BY MALM FOR $C_*(\Omega M)$
INSTEAD OF $S^*(M)$ — DOESN'T NEED M SIMPLY-CONN.
WHICH IMPLIES THE OTHER CASE BY MENICHI-THOMAS.
($S^*(M)$)

STRATEGIE

① DEFINE A BV-ALGEBRA STRUCTURE ON $HH^*(S^*(M), S^*(M))$
② GIVE AN ISO OF ALGEBRAS COMMUTING WITH THE
Δ-OPERATORS.

II BABY THEOREM [M] (MIXTURE OF FELIX-THOMAS-VIG. + GINZBURG)

LET A ~~BE~~ A DIFF GRADED ALGEBRA. LET [m] BE
A CLASS IN $HH^{-d}(A, A^\vee)$. LET $\eta: \mathbb{F} \to A$ UNIT MAP

$$\eta^* = HH^*(\eta, A) : HH^*(A, A^\vee) \longrightarrow HH^*(\mathbb{F}, A^\vee) = H(A^\vee)$$
$$[m] \longmapsto \eta^*[m]$$

SUPPOSE THAT

a) THE MORPHISM OF LEFT $H(A)$-MODULES (USING $A^\vee$ IS AN A-BIMODULE)

$$H(A) \longrightarrow H(A^\vee)$$
$$a \longmapsto a \cdot \eta^*[m]$$

IS AN ISOMORPHISM

THEN

1) THERE IS AN ISO OF VECTOR SPACES

$$HH^*(A, A) \xrightarrow{\mathcal{D}} HH^{*-d}(A, A^\vee)$$
$$a \longmapsto a \cap [m]$$

ASSUMING ALSO

b) $B^\vee([m]) = 0$

THEN 2) THE GERSTENHABER ALG $HH^*(A,A)$ EQUIPPED WITH $\Delta := \mathcal{D} \circ B^\vee \circ \mathcal{D}^{-1}$ IS A BV-ALGEBRA.

<u>BLUE THEOREM</u> [LURIE EX 4.2.8]

LET S BE A GOOD SYMMETRIC MONOIDAL $(\infty,1)$-CAT

(FOR LUC, S = CATEGORY OF CHAIN COMPLEXES)

LET A BE AN (A_∞)-ALGEBRA IN S EQUIPPED WITH A MAP OF COMPLEXES OF DEGREE d

$$A \otimes_{A\otimes A^{op}} B(A,A,A) \xrightarrow{tr} \mathbb{F} \qquad (tr = m \text{ ABOVE})$$

SUCH THAT $B^\vee(tr) = tr \circ B$ IS CHAIN HOMOTOPIC TO THE ZERO MAP (LURIE "SO(2)-EQUIVARIANT") ($\Leftrightarrow$ CONDITION b) ABOVE)

AND SUCH THAT THE ADJOINT OF THE COMPOSITE

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{C_*(\eta, A)} C_*(A,A) \xrightarrow{tr} \mathbb{F}$$

IS THE ADJOINT OF A QUASI-ISOMORPHISM $A \xrightarrow{\simeq} A^\vee$

($\Leftrightarrow$ CONDITION a), NON-DEGENERATE PAIRING IN H_*)

$$C_*(A,A) = A \otimes^R_{A\otimes A^{op}} A$$

(NOTE: $A \otimes A \xrightarrow{G} A \otimes^R_{A\otimes A^{op}} A$, $\quad a\otimes b \mapsto a \otimes^R_{A\otimes A^{op}} b = a\otimes b1 = ab \otimes 1$)

THEN BY THE COBORDISM HYPOTHESIS, | NON-COMPACT POSITIVE-BORDISM SYMMETRIC MONOIDAL VERSION 4.2.11, THERE IS A

$Z : \mathrm{Bord}_2 \longrightarrow \mathrm{Alg}(\mathcal{C})$ (ALGEBRAS IN $\mathcal{C}$.

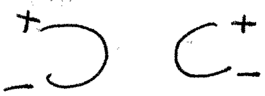
SUCH THAT $Z(+) = A$.

IN PARTICULAR, $Z(S^1) = A \underset{A \otimes A^{op}}{\otimes} B(A, A, A)$ IS A NON-COMPACT CLOSED TOPOLOGICAL CONFORMAL FIELD THEORY.

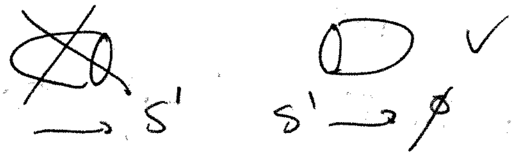
(RED)

DEFINITION 4.2.10 $\mathrm{Bord}_2^{or, nc}$ IS A 2-CATEGORY

OBJ = ORIENTED 0-MFDS $\pm$

1-MORPH = 1-dim ORIENTED COBORDISMS 

2-MORPH = 2-dim ORIENTED COBORDISM SUCH THAT EVERY CONNECTED COMPONENT HAS AN INCOMING BDRY PART. IN PARTICULAR



$\to HH_*(A, A)$ IS A NON-UNITAL BV-ALG AND

$HH^*(A, A^\vee)$ IS A UNITAL BV-ALG.

THEREFORE, THE BLUE THEOREM IMPLIES PART 2) OF THE BABY THEOREM, RESTRICTING TO GENUS 0 AND TAKING HOMOLOGY. MORE GENERALLY, $HH_*(A, A)$ IS A HOMOLOGICAL CFT, i.e. AN ALGEBRA OVER THE PROP $\{H_*(\mathcal{M}_{g, p+q})\}_{p \geq 1}$

(TAKING DUALS GET AN ACTION OF $\{H_*(\mathcal{M}_{g,p+q})\}_{q \geq 1}$.)

(HERE WE USE $H_*(fD_2) \underset{\text{OPERAD}}{\cong} \{H_*(\mathcal{M}_{0,*,1})\}$.)

<u>CONJECTURE</u> : $C^*(A, A^\vee)$ IS AN ALGEBRA OVER $S_*(fD_2)$ (DERIVED CYCLIC DELIGNE CONJECTURE).

(NOT TO TALK ABOUT $S_*(M_{g,p+q})$ ...)

<u>SKETCH OF PROOF OF BABY THEOREM</u> :

LET $B(A,A,A) \xrightarrow{\simeq} A$ (DOUBLE) BAR RESOLUTION.

HAVE $[m] \in HH^{-d}(A, A^\vee)$
$= H(\mathrm{Hom}_{A\text{-Bimod}}(B(AAA), A^\vee))$

$[m]$ IS REPRESENTED BY A MORPHISM OF A-BIMODULES OF DEGREE $-d$ COMMUTING WITH DIFFERENTIALS

$m : B(A,A,A) \longrightarrow A^\vee$

a) MEANS EXACTLY THAT m IS A QUASI-ISOMORPHISM.

WE HAVE A SEQUENCE OF QUASI-ISO OF A-BIMODULES

$$A \xleftarrow[\varepsilon_A]{\simeq} B(A,A,A) \xrightarrow[m]{\simeq} A^\vee$$

THIS MEANS THAT $A \cong A^\vee$ IS THE DERIVED CATEGORY OF (A,A)-BIMODULES.

BY APPLYING $HH^*(A,-)$, OBTAIN AN ISO

$$HH^*(A,A) \cong HH^{*-d}(A,A^\vee) \quad \text{AS GRADED VECTOR SPACES}$$

$$\xleftarrow{D}$$

ONE CAN SHOW THAT THERE IS A BV-STRUCTURE EXTENDING THE EXISTING PARTIAL STRUCTURES ON BOTH SIDES IFF $B^\vee([m]) = 0$

[GINZBURG: $HH^*(A,A) \xrightarrow{\cong} HH_*(A,A)$ ...
ASSUMES ISO — CLOSELY RELATED]

THE CONDITION COMES FROM

$$
\begin{tikzcd}
1 \in HH^*(A,A) \arrow[r, mapsto] \arrow[d, bend right, "\Delta(1)=0"'] & {[M]} \arrow[d, "B^\vee"] \\
0 \arrow[r] & 0
\end{tikzcd}
$$

IV EXAMPLE OF STRING TOPOLOGY [LURIE/MENICHI]

WE WANT TO TAKE $A = S^*(M)$ IN THE BABY/BLUE THM

LET $ev : LM \longrightarrow M$
$\qquad l \longmapsto l(0)$

JONES ISO: $H_*(LM) \xrightarrow[\cong]{\text{JONES}} HH^*(S^*(M), S^*(M)^\vee)$ $\left(\overset{\shortparallel}{S_*(M)}\right)$

$$
\begin{tikzcd}
H_*(LM) \arrow[rr, "\cong"] \arrow[dr, "H_*(ev)"] & & HH^*(S^*(M), S^*(M)^\vee) \arrow[dl, "HH^*(\eta, S^*(M)^\vee)"] \\
& H^*(M) \arrow[ul, bend left, "H_*(\sigma) \text{ (SECTION)}"] &
\end{tikzcd}
$$

$H_*(\sigma)[M] \longmapsto [M]$

$[M] \longmapsto H_*(\sigma)[M]$

CLAIM: $[M]$ SATISFIES HYPOTHESIS OF THM

PF: $HH^*(\eta, S^*(M)^\vee)([M]) = [M]$ BECAUSE DIAGRAM COMMUTES

AND $[M]$ GIVES APPROPRIATE ISO BY PD ..

$\rightarrow$ SO a) ✓

FOR b), $B^{\vee}(m) = B^{\vee} \circ \mathrm{JONES} \circ H_*(\lambda)[M]$

$$= \mathrm{JONES} \circ \underbrace{\Delta \circ H_*(\lambda)[M]}_{S^1\text{-ACTION ON CONSTANT LOOPS}} = 0$$

<u>DEFINITION</u> 4.2.6 : LET $\mathcal{C}$ BE A SYMM. MONOIDAL $(\infty,2)$-CATEGORY. [THINK $\mathcal{C} = \mathrm{Alg}(\mathrm{COMP.}) = \begin{cases} \text{ALG} \\ \text{BIMODULES} \\ \text{MORPH OF BIMODULES (CHAIN CPX OF)} \end{cases}$]

A CALABI-YAU OBJECT IN $\mathcal{C}$ IS THE DATA OF

(1) A DUALIZABLE OBJECT $X \in \mathcal{C}$

(2) A 2-MORPHISM $\mathrm{tr}: \mathrm{ev}_X \circ \mathrm{coev}_X \longrightarrow 1$

WHICH IS $SO(2)$-EQUIVARIANT AND IS THE COUNIT FOR AN ADJUNCTION BETWEEN ev_X AND coev_X

DEFINITION OF $\mathrm{Alg}(\mathrm{COMPLEXES})$: (AS $(\infty,2)$-CATEGORY)

OBJ = dg ALGEBRAS A

1-Mor(A,B) = dg (A,B)-BIMODULES

COMPOSITION : $A \xrightarrow{P} B \xrightarrow{Q} C \overset{\mathrm{DEF}}{=} P \underset{B}{\overset{R}{\otimes}} Q$

2-MORPH$({}_A P_B, {}_A P'_B)$ = CHAIN CPX OF MORPH OF (A,B)-BIMODULES

MONOIDAL STRUCTURE = TENSOR PRODUCT $A \underset{F}{\otimes} A'$ OF ALG OF BIMODULES OF 2-MORPH

EVERY ALGEBRA A HAS A^{op} AS DUAL :

$ev_A : A \otimes A^{op} \xrightarrow{A} \mathbb{F}$

A as $(A \otimes A^{op}, \mathbb{F})$-BIMODULE

$coev_A : \mathbb{F} \xrightarrow{A} A \otimes A^{op}$

A as $(\mathbb{F}, A \otimes A^{op})$-BIMOD.

$$\xrightarrow{coev} \xrightarrow{ev} = ev_A \circ coev_A = {}_{\mathbb{F}}A \overset{R}{\underset{A\otimes A^{op}}{\otimes}} A_{\mathbb{F}} = C_*(A,A).$$

CONDITION (2) OF CALABI-YAU:

$$\begin{tikzcd}
A\otimes A^{op} \arrow[r, "A"] \arrow[rr, bend left=40] \arrow[rrr, bend left=60, "A"] \arrow[rrr, bend right=60, "A"'] & \mathbb{F} \arrow[r, "A"] \arrow[rr, bend right=40] \arrow[r, phantom, bend left=30, "\Downarrow\mu"] & A\otimes A^{op} \arrow[r, "A"] \arrow[d, phantom, "\Downarrow tr"] & \mathbb{F} \\
 & & &
\end{tikzcd}
\cong
\begin{tikzcd}
A\otimes A^{op} \arrow[r, bend left=50, "A"] \arrow[r, bend right=50, "A"'] \arrow[r, phantom, "\Downarrow id_A"] & \mathbb{F}
\end{tikzcd}$$

AS MORPHISM OF $(A \otimes A^{op}, \mathbb{F})$-BIMOD.

$\rightsquigarrow$ HYPOTHESIS OF BLUE THM

(THE OTHER WAY AROUND, DOESN'T KNOW HOW TO DEFINE μ.)