TOPOLOGICAL | CHIRAL HOMOLOGY (~~[illegible]~~)
DERIVED

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GOAL: DEFINE TOPOLOGICAL HOMOLOGY (TCH)
DERIVED (i.e. CHAIN CPX)

IN 2 $\neq$ WAYS:
i) ONE USING THE BORDISM HYPOTHESIS ($\to$ PROPERTIES FOR FREE)
ii) BUILD TCH WITHOUT USING THE BORDISM HYPOTHESIS
($\to$ THEN CAN PROVE THAT ONE GET TFT IN THAT SETTING EASIER ...)

I) THE TENSOR $\otimes$-$(\infty,1)$-CATEGORY $Alg_n^{(0)} =: E_{\leq n} Alg$

WE WORK WITH THE dg CATEGORY OF CHAIN CPXS OVER A FIELD k: $Ch(k) = \begin{cases} Obj = \text{chain cpx} \\ Hom(C_\bullet, D_\bullet) = \{f \in Hom_k(C_\bullet, D_\bullet),\ d(f) = f(d_C) \pm d_D \circ f\} \end{cases}$

chain cpx

$Alg_0 := (\infty,1)$-CATEGORY OF CHAIN COMPLEXES

INGREDIENTS FOR TCH

$\boxed{I.1}$ A $\otimes$-$(\infty,1)$-CATEGORY MODELING $Ch(k)$

"MORALLY" THE OBJECTS SHOULD STILL BE CHAIN CPXS,
1-Mor = MORPHISMS OF CHAIN COMPLEXES
2-Mor = HOMOTOPIES BETWEEN MORPH
$\vdots \qquad \vdots$

THINKING OF $(\infty,1)$-CAT AS ω-CAT WITH EVERYTHING INVERTIBLE FROM 2 ONWARDS.

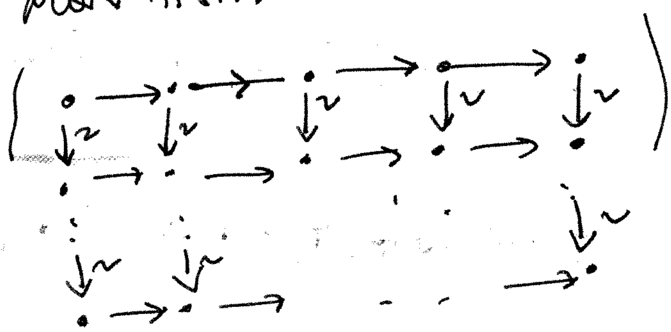
PROP (TOEN AND MANY MORE "OVERVIEW ON HIGHER STACKS")

LET $\mathcal{M}$ BE ANY MODEL CATEGORY
$\rightsquigarrow$ AN $(\infty, 1)$-CATEGORY BY TAKING THE DWYER-KAN LOCALIZATION $\underbrace{L(\mathcal{M}, W)}$ ← WEAK-EQUIV. OF $\mathcal{M}$

~~S~~ CATEGORY ENRICHED OVER SIMPLICIAL SETS
→ TAKE THE FAT NERVE FROM ALEXANDER'S TALK.

SHOULD ~~BE~~ EQUIVALENT TO TAKING THE FAT NERVE WITH QUASI-ISO'S ~~...~~ TAKING THE ROLE OF THE ISO'S AS VERTICAL MORPHISMS:

FAT NERVE $(W(\mathcal{M})) =$



$= N_\bullet(W(\mathcal{M})[\bullet])$

THAT WAS STEP 0 ....

STEP 1: Alg_1 (WE FOLLOW LUC'S TALK)

Alg_1 IS AN ~~⊗~~$(\infty, 2)$-CATEGORY.

$Obj(Alg_1) =$ dg $\begin{cases} \text{ASSOCIATIVE ALGEBRAS} \\ A_\infty \end{cases}$

$1\text{-}Mor(A, B) = Alg_0(A \otimes B^{op}\text{-MODULES})$

COMPOSITION: $A \xrightarrow{M} B \xrightarrow{N} C := {}_A M_B \overset{\mathbb{L}}{\underset{B}{\otimes}} {}_B N_C$

PROBLEM: CHOICE INVOLVED, NO REASON TO BE ASSOCIATIVE

THOUGH NOT REALLY A PROBLEM BECAUSE WE DON'T ASSUME HAVING $\begin{cases} \text{WELL-DEFINED} \\ \text{UNIQUELY DEFINED} \end{cases}$ ~~...~~ COMPOSITION IN AN $(\infty, 2)$-CATEGORY.

$\boxed{\text{I } 2.}$ NEED A WAY TO MAKE $\underset{B}{\otimes}$ INTO AN HONEST COMPOSITION OF 1-MORPHISMS, TO REALLY GET AN $(\infty,2)$-CATEGORY OUT OF THE COLLECTION OF $(\infty,1)$-CATEGORIES $(A\otimes B^{op})$-MODULES.

Alg_2 IS MONOIDAL WRT $\otimes$.

STEP $(n-1)\to n$:

Alg_n IS A $\otimes$-$(\infty, n+1)$-CATEGORY.

$Obj(Alg_n) = E_n$-ALGEBRAS [OVER A CHOSEN MODEL OF E_n-OPERAD]

(NOTE: E_n IS LIKE $E_1 \wr \ldots \wr E_1 \ldots$)

$1\text{-MORPH}(A,B) = Alg_{n-1}(A\otimes B^{op}\text{-MODULES})$

OR (A,B)-BIMODULES

E_n-BIMODULES IN THE OPERADIC SENSE

"LITTLE ISSUE:" NEED A MODEL OF E_n-ALGEBRAS SO THAT $A_1\otimes A_2$ IS AN E_n-ALGEBRA FOR A_1, A_2 E_n-ALGEBRAS.

$\boxed{\text{I } 3.}$ NEED A $\otimes$ OF E_n-ALGEBRAS.

i.e. WANT A DIAGONAL $E_n(r) \xrightarrow{\Delta} E_n(r)\otimes E_n(r)$

(E_n IS A "HOPF OPERAD").

— THERE ARE MODELS LIKE THAT, FOR EXAMPLE USING THE BARRAT-ECCLES OPERAD AND ITS FILTRATIONS.

$\boxed{\text{I.4}}$ $E_1 \hookrightarrow E_2 \hookrightarrow \cdots E_{n-1} \hookrightarrow E_n$

NEED TO CHOOSE SUCH A SEQUENCE OF RESTRICTIONS TO BE ABLE TO DEFINE THE INDUCTION STEP, COMPATIBLE WITH $\otimes$-PRODUCT.

(ANALOGOUS TO, IN THE DEFINITION OF Bord_n, RESTRICTING TO SOME DIRECTIONS GAVE PREVIOUS Bord_i's — AXIOM (3).)

— SHOULD BE ABLE TO DO THIS WITH THE BARRATT-ECCLES OPERAD, SEE [FRESSE] —

$$\begin{array}{c} E_{n-1}\text{-ALG} \\ \downarrow \\ {}_A N_B \underset{B}{\overset{\text{ll}}{\otimes}} {}_B P_C \end{array} \xleftarrow{\sim} {}_A N_B \underset{B}{\otimes} B(B,B,B) \underset{B}{\otimes} {}_B P_C$$

NEED A BAR CONSTRUCTION OF E_n-ALGEBRAS s.t. $B(A,A,A)$ IS AN E_k-ALGEBRA IF A IS A E_{k+1}-ALGEBRA.

[THERE ARE SUCH CONSTRUCTIONS: BRUN-FIEDOROWICZ-VOGT, FRESSE, MAY]

... AND A COMPATIBLE CHOICE FOR THE INDUCTION ...

$\rightsquigarrow$ Alg_n A $\otimes(\infty, n+1)$ - CATEGORY (OF $\underbrace{E_{\leq n}}$-ALG).

$\left\{\begin{array}{l} \text{OBJ} = E_n\text{-ALG} \\ \text{1-MOR} = E_{n-1}\text{-ALG} \\ \vdots \end{array}\right.$

__DEF__: $E_{\leq n}\text{-Alg} := \mathrm{Alg}_n^{(0)}$ = "THROW AWAY ALL NON-INVERTIBLE $(n+1)$-MORPHISMS IN Alg_n"

= $\otimes(\infty,n)$-CAT OBTAINED BY RIGHT ADJOINT OF THE INCLUSION $\otimes(\infty,n) \to \otimes(\infty,n+1)$-CAT.

LEMMA: $E_{\leq n}$-ALG HAS DUALS !

REASON: EVERYTHING UP TO LEVEL $n-1$ ARE ALGEBRAS AND THE DUAL OF A IS A^{op} (A^{op} = PULLBACK OF A OVER THE ANTI-PODAL $E_n \xrightarrow{inv} E_n$)

$\to$ PERFECT TARGET FOR A FIELD THEORY ...

BORDISM HYPOTHESIS IN $E_{\leq n}$-ALGEBRAS:

$$\text{ANY} \otimes\text{-}\mathrm{Hom}_{(\infty,n)\text{-CAT}}\left(\mathrm{Lan}\,\mathrm{Bord}_n^{fr}, E_{\leq n}\text{-ALG}\right) \xrightarrow{\sim} \iota\left(E_{\leq n}\text{-}Alg^{\sim}\right)$$

$$Z \longmapsto Z(\bullet^+)$$

THUS "ANY" E_n-ALGEBRA A GIVES RISE TO A "UNIQUE" TFT. Z_A (GIVEN A, CHOOSE A PATH TO SOMETHING IN THE IMAGE, THIS PATH IS UNIQUE UP TO HOMOTOPY...)

DEF: TOPOLOGICAL CHIRAL HOMOLOGY OF M (FRAMED n-dim MFD) WITH VALUE IN A (AN E_n-ALG) IS

$$\int_M A := Z_A(M)$$

IT IS A CHAIN CPX

IF N k-DIM, s.t. $N^k \times D^{n-k}$ IS FRAMED, DEFINE

$$\int_N A := Z_A(N)$$

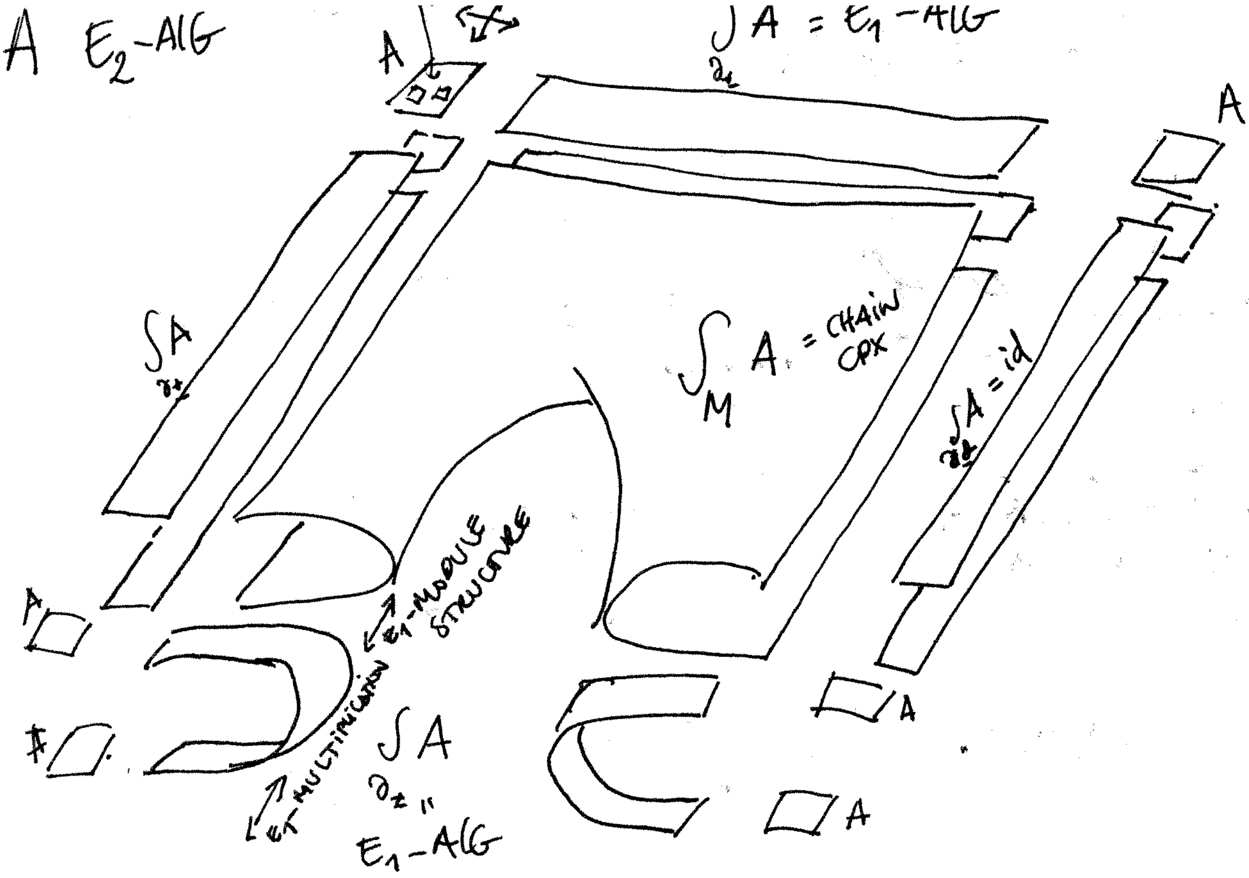
, WHICH IS AN E_{n-k}-(dg) ALGEBRA.

IN PARTICULAR, $\int_{M^n} A$ IS A MODULE OVER THE E_1-ALG

$$\underbrace{\int_{\partial M}}_{\text{HAS SEVERAL COMPONENTS}} A .$$

$\to$ MODULE OVER EACH OF THEM (OR OVER THE TENSOR PRODUCT)

A E_2-ALG



PROPERTIES OF TCH

① $\int_{M^k \cup_S N^k} A \xrightarrow{\sim} \int_M A \overset{\mathbb{L}}{\underset{\int_S A}{\otimes}} \int_N A$

$(E_{n-k})\text{-ALG} \underset{E_{n-k+1}\text{-ALG}}{\otimes} (E_{n-k})\text{-ALG} \rightsquigarrow E_{n-k}\text{-ALG}$

② $\int_{D^k} A \xrightarrow{\sim} A$ AS E_{n-k}-ALGEBRA.

$\forall\, 0 \leq k \leq n$

(HENCE A DETERMINES THE WHOLE THING)

③ $\int_{S^1} A = \int_{\circlearrowright} A \simeq \int A \underset{\int A}{\otimes} \int A \simeq A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} A$

$\longrightarrow$ HOCHSCHILD CPX

$$\int_{\bullet\!-\!-\!\bullet} A \simeq {}_A A_A$$

$$\wr|$$

$$\int_{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet} A \longleftarrow \int_{\bullet\,\bullet\,\bullet\,\cdots\,\bullet} A \simeq A^{\otimes k}$$

$$\wr|$$

$$A \overset{\mathbb{L}}{\underset{A}{\otimes}} A \overset{\mathbb{L}}{\underset{A}{\otimes}} \cdots \overset{\mathbb{L}}{\underset{A}{\otimes}} A \qquad\qquad A \underset{A}{\otimes} A \xrightarrow{\simeq} A$$

$$(a,b) \longrightarrow ab$$

$$\rightsquigarrow \quad B(A,A,A) \xrightarrow{\simeq} \int_{\bullet\!-\!\bullet} A \simeq A$$

$$\int_{S^2} A \simeq \int_{D^2} A \underset{\int_{S^1} A}{\otimes} \int_{D^2} A \simeq A \underset{(C_*(A,A),\,B)}{\overset{\mathbb{L}}{\otimes}} A$$

$M_\bullet$ SIMPLICIAL SET MODEL OF M. ASSUME A IS COMM.

$$\int_M A := A^{\otimes M_\bullet} = CH^{M_\bullet}(A,A) \quad \text{(PIRASHVILI'S NOTATION)}$$

GENERALIZATION OF HOCHSCHILD CPX

[A COMMUTATIVE $\rightsquigarrow$ CAN FORGET THE MANIFOLD STRUCTURE, JUST GOING TO SIMPLICIAL SETS
ANALOGOUS TO STABILIZING IN WALDHAUSEN THEORY ...]

FIX A AN E_n-ALGEBRA.

M FRAMED MANIFOLD OF dim n.

$\underline{TCH_A(M)}$ DEFINED BY

TAKE THE LITTLE n-CUBE MODEL FOR E_n-OPERAD:

$$\mathcal{E}_n(r) = Emb\left(\coprod_r \square^n, \square^n\right)$$

[SEE PASCAL'S TALK...]

$\underline{M}$: MODULE OVER $\mathcal{E}_n$ $\quad \underline{M}[r] = Emb^{fr}\left(\coprod_r D^n, M\right)$

$$TCH_A(M) := A \underset{\mathcal{E}_n}{\otimes} \underline{M}$$

is THE CHAIN COMPLEX

$$TCH_A(M) \simeq \bigoplus_{n_0 \to n_1 \to \dots \to n_k} A^{\otimes n_0} \otimes \mathcal{E}_n(n_0 \to n_1) \otimes \dots \otimes \mathcal{E}_n(n_{k-1} \to n_k) \otimes \underline{M}(n_k)$$

PROP ASSOCIATED TO $\mathcal{E}_n$

$\underline{\text{THM}}$: $TCH_A(M) \xrightarrow{\sim} \int_M A$

[LURIE]