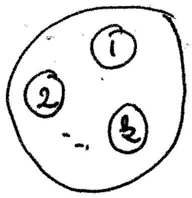
CHIRAL HOMOLOGY AND CONFIGURATION SPACES

PASCAL LAMBRECHTS

(AS EXPLAINED BY DAVID ...)
+ SALVATORE, "CONFIGURATION SPACES WITH SUMMABLE LABELS"

$\mathcal{D}_d$ = LITTLE DISC OPERAD (TOPOLOGICAL CATEGORY)

$\mathcal{D}_d(k) = \Big\{$  $\Big\} = \mathrm{Emb}^{st}\left(\coprod_k \overset{\circ}{D}{}^d, \overset{\circ}{D}{}^d\right)$

└ STANDARD ~~EMBEDDING~~ – BORDERS ARE ALLOWED TO MEET
= BY TRANSLATION AND DILATION

ASSOCIATED CATEGORY $\underline{\underline{\mathcal{D}}}_d$ WITH

$$\begin{cases} \mathrm{OBJ} = \mathbb{N} = \{0, 1, \dots\} & \text{THINKING OF } k \text{ AS } \coprod_k D^d \\ \mathrm{Hom}_{\underline{\underline{\mathcal{D}}}_d}(k,l) = \mathrm{Emb}^{st}\left(\coprod_k D^d, \coprod_l D^d\right) \end{cases}$$

(SYMMETRIC MONOIDAL CATEGORY)

A RIGHT $\mathcal{D}_d$-MODULE IS THE DATA OF SPACES $N(k)$, $k \geq 0$
WITH MAPS $N(k) \otimes \mathcal{D}_d(m_1) \otimes \dots \otimes \mathcal{D}_d(m_k) \longrightarrow N(m_1 + \dots + m_k)$ (*)

A RIGHT MODULE IS THE SAME AS A FUNCTOR

$$N : \underline{\underline{\mathcal{D}}}_d^{op} \longrightarrow \mathrm{Top}$$
$$k \longmapsto N(k)$$

$$(*) \equiv \bigotimes_{i=1}^{k} \mathcal{D}_d(m_i) \longrightarrow \mathrm{maps}(N(k), N(l)) \qquad l = \sum m_i$$

$$\subset \underline{\underline{\mathcal{D}}}_d(k,l) = \bigoplus_{\substack{\{[n_1] \dots [n_k]\} \text{ PARTITION OF } l \\ \sum n_i = l}} \otimes \mathcal{D}_d(n_i)$$

M is a framed d-manifold,

we associate a right $\mathcal{D}_d$-module to M:

$$f_M : \underline{\mathcal{D}}_d^{op} \longrightarrow Top$$

$$k \longmapsto f_M(k) = \mathrm{Emb}^{fr}\left(\coprod_k D^d, M\right)$$

framed embeddings

$= \{ j : \coprod_k D^d \hookrightarrow M$ + path from the pull-backed framing at the center of D^d to the standard framing, $\forall k \}$?

(could also consider the induced map on tangent bdl an require it to be standard (?)).

$$k \longmapsto f_M(k)$$

$$\downarrow \qquad\qquad \uparrow \text{restriction of embedding}$$

$$l \longmapsto f_M(l)$$

Let A be a (topological) $\mathcal{D}_d$-algebra

(ex: $A = \Omega^d X$).

Define $A^{\otimes -} : \underline{\mathcal{D}}_d \longrightarrow Top$

$$k \longmapsto A^{\otimes k}$$

$$\mathcal{D}_d^{(k)} \ni f \downarrow \qquad f_* \downarrow \text{— operad action}$$

$$1 \longmapsto A$$

(symmetric monoidal functor)

$$\underline{\underline{\mathcal{D}}}_d \underset{A^{\otimes -} \text{ (COVARIANT)}}{\overset{f_M \text{ (CONTRAVARIANT)}}{\rightrightarrows}} \mathrm{Top}$$

CAN DEFINE A "COEND" / HOCOEND: $f_M \overset{ho}{\underset{\underline{\underline{\mathcal{D}}}_d}{\otimes}} A^{\otimes -} \in \mathrm{Top}$

$$f_M \underset{\underline{\underline{\mathcal{D}}}_d}{\otimes} A^{\otimes -} = \coprod_{k \in Ob(\underline{\underline{\mathcal{D}}}_d)} f_M(k) \otimes A^{\otimes k} \Big/ \sim$$

[QUESTION: WHEN IS THIS PARTICULAR COEND EQUIVALENT TO THE HOMOTOPY COEND?]

WHERE $\sim$ IS DEFINED BY

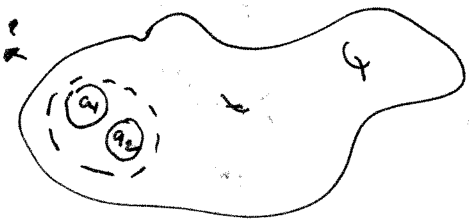
GIVEN $k \xrightarrow[\text{IN } \underline{\underline{\mathcal{D}}}_d]{\alpha} k'$

$$f_M(k) \xleftarrow{\alpha^*} f_M(k') \ni j$$

$$\underset{\overset{\cup}{a}}{A^{\otimes k}} \longrightarrow A^{\otimes k'}$$

THEN $(\alpha^*(j), a) \sim (j, \alpha_*(a))$

ELEMENT OF $f_M \underset{\underline{\underline{\mathcal{D}}}_d}{\otimes} A^{\otimes -}$ IS k DISKS LABELED BY k ELEMENTS OF THE ALGEBRA. RELATION: IF THE DISKS COME FROM BIGGER DISCS, THEN WE CAN MULTIPLY IN THE ALGEBRA:



THE FULTON-MACPHERSON

$\mathcal{F}_d$ IS AN OPERAD HOMOTOPY EQUIV TO $\mathcal{D}_d$

$$j \in \mathcal{D}_d(3) \longmapsto^{\simeq} \mathrm{Conf}(3, \mathring{D}^d) \quad (\mathring{D}^d = \mathbb{R}^d)$$

THIS IS NOT AN OPERAD BECAUSE WE CANNOT COMPOSE

FULTON-MACPHERSON BUILD AN OPERAD OUT OF Conf( ) BY COMPACTIFYING THE SPACES. NOTE THAT IT IS A LOT SMALLER THEN $\mathcal{D}_d$, AND OF THE "RIGHT" DIMENSION.

CONSIDER $\mathrm{Conf}(k, \mathbb{R}^d) \,/\, \text{TRANSLATION - DILATION}$ ↶ TRANSLATION ($\mathbb{R}^d$), DILATION ($]0, +\infty[$)

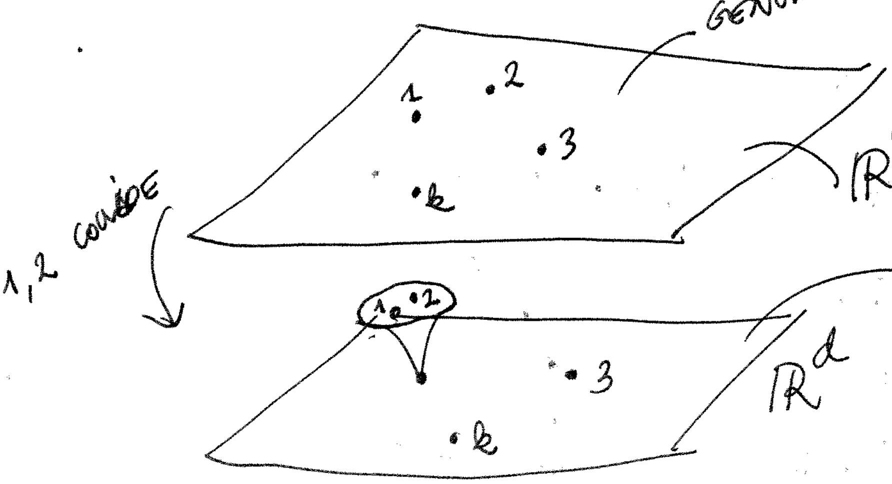
(DIMENSION $= kd - d - 1$)

COMPACTIFICATION IN A CLEVER WAY SO THAT

- IT BECOMES A MANIFOLD WITH CORNERS
- IT IS AN OPERAD

THAT COMPACTIFICATION IS $\mathcal{F}_d(k)$

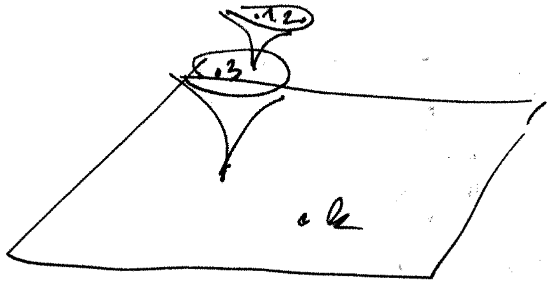
WHAT IS A POINT IN $\mathcal{F}_d(k)$?

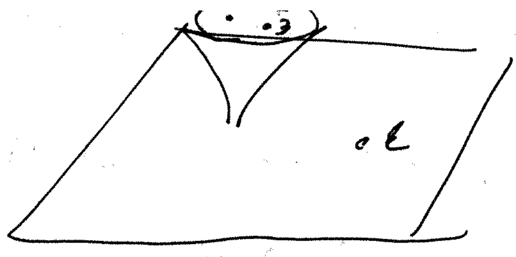


GENUINE CONFIGURATION (UP TO TRANSLATION AND SCALING) → POINT IN THE INTERIOR OF $\mathcal{F}_d(k)$

CONFIGURATION OF $k-1$ POINTS + CONFIGURATION OF 2 POINTS IN THE TANGENT SPACE — IN THE BOUNDARY

LOWER STRATA IN THE BULK





$\mathcal{F}_d(k)$ IS A STRATIFIED MANIFOLD OF DIMENSION $kd-d-1$.
THE STRATA ARE INDEXED BY ROOTED TREES (IN $\mathbb{R}^{d+1}$ ...) WITH LEAVES LABELED BY $1, \dots, k$.

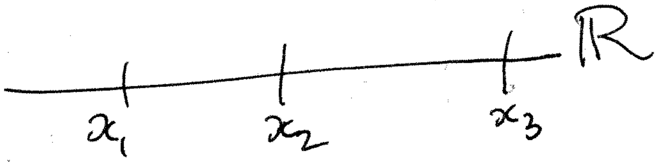
EXAMPLE:

$k=2$: $\mathrm{Conf}(2, \mathbb{R}^d) / \text{TRANSL-DILATION} \cong S^{d-1}$ (ANY d)

$\shortparallel$

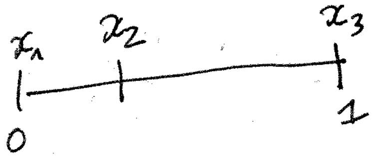
$\mathcal{F}_d(2)$ (THE POINTS CANNOT APPROACH EACH OTHER ALL AT ONCE BECAUSE OF SCALING!)

$k=3$, $d=1$



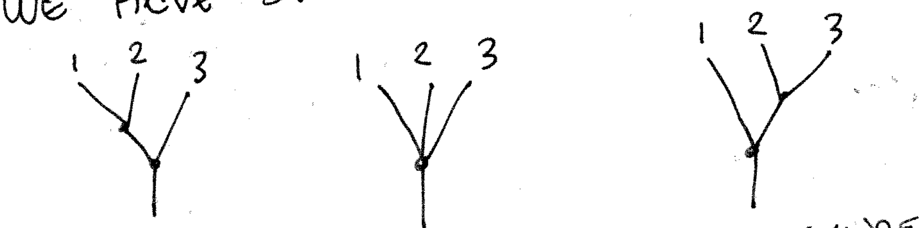
$\mathrm{Conf}(3, \mathbb{R}^k) / \text{tr, dil} \cong \Sigma_3 \times \mathrm{Conf}^{incr}(3, \mathbb{R}) / \sim \;\cong \Sigma_3 \times (0,1)$

CAN ASSUME $x_1 = 0$, $x_3 = 1$



$\mathcal{F}_1(3) = [0,1] \times \Sigma_3$

WE HAVE STRATA OF $\mathcal{F}_1(3)$ INDEXED

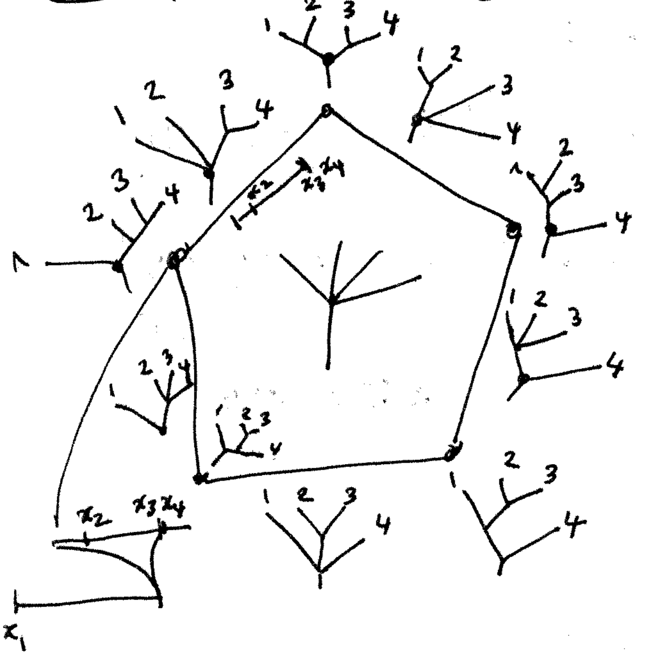


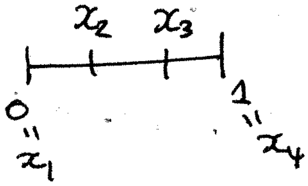
1,2 COLLIDE — NO COLLIDE — 2,3 COLLIDE

CODIMENSION OF A STRATUM IS ...

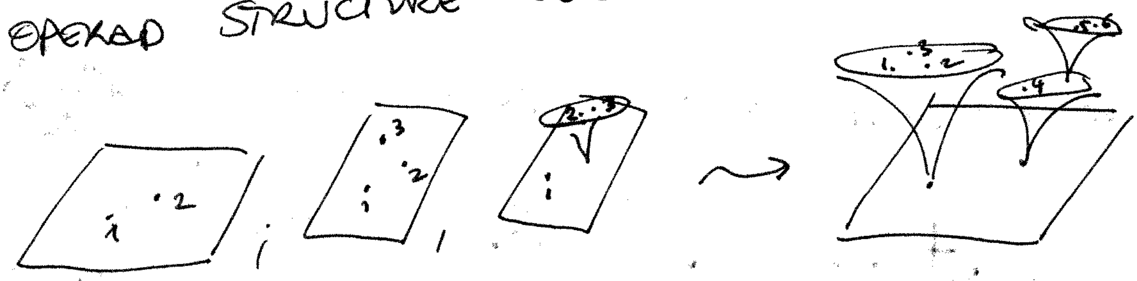
$k=4$, $d=1$: $\mathcal{F}_1(4) = \Sigma_4 \times K_2$

K_2: 2-dim'l STASHEFF ASSOCIAHEDRON





OPERAD STRUCTURE CORRESPONDS TO STACKING TREES:

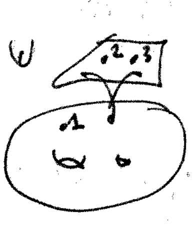


BACK TO THE MANIFOLD M.

$\mathrm{Conf}(k, M) = \{(x_1 \ldots x_k) \in M^k \mid x_i \neq x_j \text{ for } i \neq j\}$

WE DON'T HAVE TRANSLATION AN SCALING GLOBALLY IN M.

$\mathrm{Conf}(k, M) \subset \mathrm{Conf}[k, M]$ COMPACTIFIED



~ FULTON-MACPHERSON IN THE TANGENT SPACE WHEN POINTS APPROACH EACH OTHER / COLLIDE.

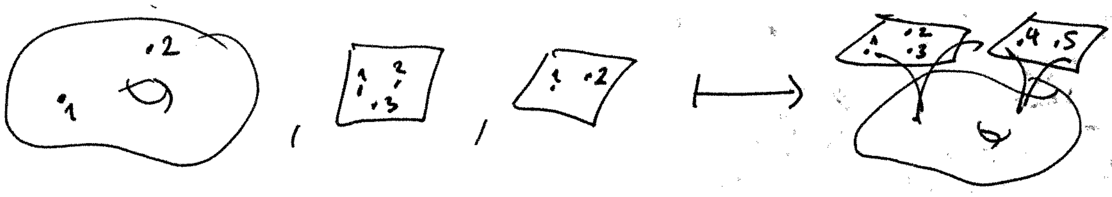
A POINT OF $\mathrm{Conf}[k, M]$ IS A CONFIGURATION OF $j \leq k$ POINTS IN M, $x_1 \ldots x_j$, AND t_i POINTS IN $T_{x_i}M$ (LIVES IN $\mathcal{F}_d(t_i)$) $\forall i$ WITH $\sum_{i=1}^{j} t_i = k$.

HAD $f_M : \underline{\underline{\mathcal{D}}}_d \longrightarrow \mathrm{Top}$

REPLACE IT BY COMPACTIFIED CONFIGURATIONS, WHICH IS SMALLER ~~AND~~ IS A MANIFOLD:

<u>FACT</u>: $\mathrm{Conf}[\bullet, M]$ IS A RIGHT MODULE OVER $\mathcal{F}_d$

i.e. $\exists$ maps $\mathrm{Conf}[k,M] \times \mathcal{F}_d(n_1) \times \dots \times \mathcal{F}_d(n_k) \to \mathrm{Conf}[n, M]$, $n = \sum n_i$



<u>HOPE</u>: $\mathrm{Conf}[\bullet, M]$ IS COFIBRANT IN SOME SENSE
— FREE AS $\mathcal{F}_d$-MODULE

($\mathcal{F}_d$ IS COFIBRANT IF WE DON'T TAKE ARITY 0.)

THERE SHOULD BE A MAP $\mathcal{F}_d \to \mathcal{D}_d$ ...

LET A BE A $\underline{\underline{\mathcal{D}}}_d$-ALGEBRA IN Top, eg $A = \Omega^d X$ OR COMM. MONOID

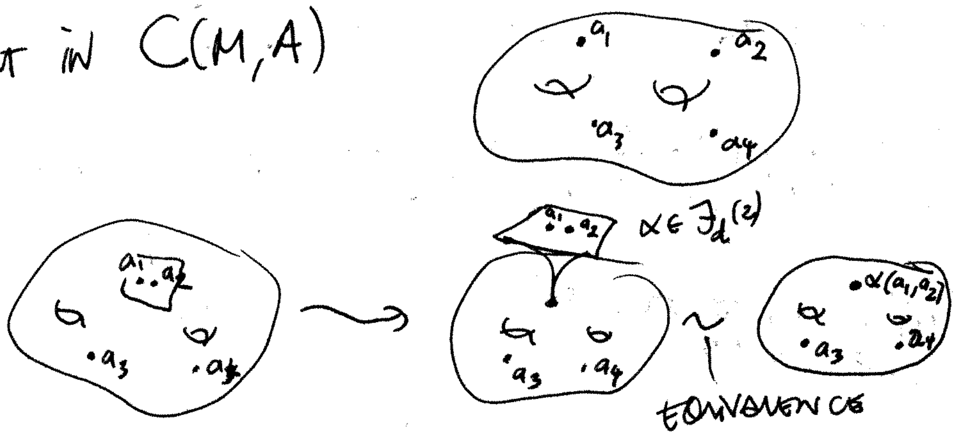
$$TCH_A(M) = \int_M A \simeq \mathrm{Conf}[\bullet, M] \underset{\mathcal{F}_d}{\otimes} A^{\otimes \bullet}$$

(THIS IS THE HOMOTOPY COEND IF $\mathrm{Conf}[\,,M]$ IS COFIBRANT)

$$= \coprod_{k \geq 0} \mathrm{Conf}[k, M] \times A^k \Big/ \text{LEFT-RIGHT MODULE STRUCTURE} := C(M, A)$$

<u>NOTE</u>: CAN DO THIS FOR ANY $\mathcal{F}_d$-MODULE. QUESTION: HOW DO WE RECOGNIZE MANIFOLDS AS GIVING PARTICULAR TYPES OF $\mathcal{F}_d$-MODULES VIA $\mathrm{Conf}[\;]$ — $\exists$ WAY TO SEE POINCARÉ DUALITY [?]

ELEMENT IN $C(M,A)$



[QUESTION: ARE THE OPERADS $\mathcal{F}_d$ HOPF OPERADS, i.e. IS THE TENSOR PRODUCT OF TWO $\mathcal{F}_d$-ALGEBRA AGAIN AN $\mathcal{F}_d$-ALG.]

EXAMPLE: A ABELIAN

$$C(M,A) = \coprod_{k \geq 0} \{ (x_i \in M,\ a_i \in A) \mid x_i \neq x_j \} \Big/ \text{ADDING IN } A \text{ WHEN POINTS COLLIDE}$$

EXAMPLE: THIS WORKS ALSO IF A IS A PARTIAL $\mathcal{F}_d$-ALGEBRA

CLASSICAL EXAMPLE: TAKE A = BASED SPACE $a_0 \in A$

→ A PARTIAL ~~ABELIAN~~ MONOID BY $a + a_0 = a = a_0 + a \quad \forall a \in A$

⇝ IN $C(M,A)$, POINTS CAN ONLY COLLIDE IF ONE OF THEM IS LABELED WITH a_0 (OR RATHER, THEN CAN COLLIDE, BUT WE REMEMBER HOW AND THERE IS NO IDENTIFICATION)