Lecture IV (Christian)

Step 2: Comparison of Functor Homology

We saw that

\[ H_*(G_{\infty_0}(R), \mathbb{F}) \cong H_*(G_{\infty_0}(R) \times S(R), \mathbb{F}) \]
\[ H_*(O_{\infty}, \mathbb{F}) \cong H_*(O_{\infty} \times \mathbb{G}, \mathbb{F}) \]
\[ H_*(G_{\infty_1}, \mathbb{F}) \cong H_*(G_{\infty} \times \mathbb{G}, \mathbb{F}) \]

\[ G_n = \text{Aut} (\mathbb{Z}^*) \]

Difficulty: The functor homology in \( S(R) \), \( \mathbb{G} \), \( \mathbb{G} \) are inaccessible by direct computation.

AIM: Find a category \( D \) where functor homology is accessible, related to \( C \) (one of the previous categories) by a functor \( \psi : C \to D \) s.t.

\[ H_*(C, \psi^*F) \cong H_*(D, F) \quad \text{for } F \text{ polynomial.} \]

3. General method to prove that \( H_*(C, \psi^*F) \)

- LEFT KAN EXTENSION FOR \( \psi : C \to D \)

\( \psi^* : D \text{-Mod} \to C \text{-Mod} \) (by precomposition)

Usual left KAN extension is the left adjoint of \( \psi^* \).
\[ \text{Hom}_D(\mu, L, K) \cong \text{Hom}_E(L, \mu^*K) \]

We have a similar definition of left Kan extension replacing \( \text{Hom} \) by \( \otimes \):

\[ \psi_! : \text{Mod}_C \longrightarrow \text{Mod}_D \]

s.t. \( \psi_!(F) \otimes K = F \otimes \psi^*(K) \)

We can define \( \psi_! \) by \( \psi_!(F)(\alpha) = F \otimes \psi^*(P^\alpha) \)

**Grothendieck Spectral Sequence**

\[ \text{Mod}_C \xrightarrow{\psi_!} \text{Mod}_D \longrightarrow K \otimes K = \text{Mod}_K \]

We consider the Grothendieck S.S. associated to this composition:

\[ E_2^{p,q} = \text{Tor}_p^C((\mu^* \psi!), F)(K) = \text{Tor}_p^C(F, \psi^*(K)) \]

where the left derived functors \( L^j \psi_! \) are given by

\[ (L^j \psi_!(F))(\alpha) = \text{Tor}_\alpha^C(F, \psi_!(P^\alpha)) \]

Let \( L^\psi_!(G) = \ker \left( (L^\psi_!(G))/\psi^*(G) \rightarrow G \right) \)

Unit of the adjunction

**Prop:** For \( G : \mathcal{D}^{op} \rightarrow \text{Mod}_K \) and \( K : \mathcal{D} \rightarrow \text{Mod}_K \)

s.t. \( \text{Tor}_p^D(L^\psi_!(G), K) = 0 \neq \psi_!(G) \) we have
\[ \text{Tor}_r^C (\psi^*(G), \psi^*(K)) \cong \text{Tor}_r^D (G, K). \]

In particular, for \( f = \mathbb{1} \) (constant functor), the S.s. becomes

\[ E^2_{pq} = \text{Tor}_p^C (L_q \psi^!(1K), K) \Rightarrow H_{p+q} (C, \psi^*K) \]

\[ (L_q \psi^!(1K))(d) = H_0 (C/d, K) \]

where \( C/d \) is the category with objects \( C \in C \) equipped with a morphism \( d : C \to \psi(C) \) in \( D \), and maps \( (C, f) \to (C', f') \) making the obvious diagram commute.

\[ L_q (1K) = H_0 (C/d, K) \]

The previous proposition says: if \( K : D \to \mathbf{K} \)-mod is s.t. \( \text{Tor}_r^D (L_q (1K), K) = 0 \), then

\[ H_r (C, \psi^*K) \cong H_r (D, K) \]

\[ \text{\textit{Example}} \]

\[ \text{\textit{Orthogonal Group}} \]

\[ \mathbf{E}_q (1K) \xrightarrow{\psi} \mathbf{E}^\text{deg} (1K) = \{ \text{Obj: finite dim. quad spaces} \}
\]

\[ \text{Mor: injective maps of quad spaces} \]

Prop (Cancellation Result)

For \( K : \mathbf{E}^\text{deg} (1K) \to \mathbf{K} \)-mod polynomial, have

\[ \text{Tor} \mathbf{E}^\text{deg} (1K) (L_q (1K), K) = 0 \]
By previous proposition, we obtain

\[ H_*(E^*_QL, \gamma_*(K)) \cong H_*(E^*_{deg}, K) \]

A quadratic form on \( V \) is a homogeneous polynomial of degree 2 on \( V \), so it is an element of \( S^2(V^*) \)

\[ E^*_{deg}(1K) = (M(K))/(S^2)^* \]

\[ \text{obj} : (V, \alpha), V \in M(K), \alpha \in S^2(V^*) \]

\[ \Rightarrow H_*(E^*_{deg}, F) \cong \text{Tor}^*_*(V \mapsto K[S^2(V^*)], F) \]

By a result of Suslin, for \( F \) polynomial,

\[ \text{Tor}^*_*(K[\mathbb{A}^1], A, F) \cong \text{Tor}^*_{p(K)}(A, F) \]

So \( H_*(E^*_{QL}, \gamma_*(K)) \cong \text{Tor}^*_{p(K)}(V \mapsto K[S^2(V^*)], F) \) for polynomial \( F \)

\[ \Rightarrow \text{THM} : \text{for } F \text{ polynomial,} \]

\[ H_*(O_{\infty}, F_{\infty}) \cong \text{Tor}^*_{p(K)}(V \mapsto K[S^2(V^*)], F) \]

Sketch of proof of the cancellation result for

\[ K \text{ a finite field [D-V]} \]

\[ L_q(K) \text{ transforms inclusions } V \rightarrow V \downarrow H \text{ in an isomorphism.} \]

\[ \Rightarrow \text{by this remark, we can define } L_q^\ast(K) \text{ over } \mathbb{A} \]
CATEGORIES OF FRACTIONS WHERE WE INVERT $V \rightarrow V \cup H$

BY A PREVIOUS CONCESSION RESULT OBTAINED BY DIAMOND, WE OBTAIN THE RESULT.

NO MORE TRUE FOR A CONNUTATIVE RING! — BUT THERE IS AN ALTERNATIVE PROOF [DIAMOND]

- AUTOMORPHISMS OF FREE GROUPS $\text{Aut}(F_n)$

$$G \xrightarrow{\varphi} G'$$

FIN. GEN FREE GROUPS

$A \xrightarrow{u} B$, $H \subset B$ S.T. $B = u(A) * H$.

CLAIM (CONCESSION RESULT) FOR $K: G' \rightarrow K\text{-Mod}$ POLYNOMIAL,

$$\text{Tor}_1(L_\varphi(K), K) = 0$$

BY THE PREVIOUS PROPOSITION,

$$H_*(G, \varphi^*(K)) \cong H_*(G', K) = K\langle 0 \rangle$$

$0$ IS THE TERMINAL OBJECT

BY COMBINING STEPS 1 & 2,

THAT: FOR $K: G' \rightarrow K\text{-Mod}$ POLYNOMIAL S.T. $K(0) = 0$,

$$H_*(G_{\varphi^*}, F_{\varphi^*}) = 0$$
1. Explicit projective resolution of $\mathcal{O}_X$ given by the bar construction (already known to Ji Blažek – Pirashvili)

$$\cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow \mathcal{O}_X \quad P_i = \mathcal{O}_X^i$$

2. Take the tensor product of this resolution with $P_i \otimes \mathcal{O}_X$ to obtain a complex $\mathcal{C}$. The homology of this complex is isomorphic to

$$\text{Tor}^\mathcal{O}_X (X, \mathcal{O}_X \otimes P_i)$$

3. Vanishing criteria: is. conditions on $X$

$$s_1. (\ast) = 0$$

4. $X = L_1 (\mathcal{O}_X)$ satisfies these conditions

5. If $\text{Tor}^\mathcal{O}_X (X, \mathcal{O}_X \otimes P_i) = 0$, then $\text{Tor}^\mathcal{O}_X (X, F \otimes G) = 0$

($\ast$) is a consequence of a general result on the structure of polynomial functors on $\mathcal{O}_X$

Contravariant case:

$$\begin{array}{cccc}
G & \overset{y}{\longrightarrow} & \text{gr}^{op} \\
\downarrow & & \\
H_* (G, y^*(G)) & \overset{\text{not an iso}}{\longrightarrow} & H_* (\text{gr}^{op}, G) = 0 & \text{if } G(\delta) = 0
\end{array}$$