LECTURE II (CHRISTINE VESPA)

INGREDIENTS: Functor Toology and Polynomial Functors

(2) Functor Toology: (= Toology in Functor Categories)

C small category
IK a fixed commutative ring

Def: A left C-module is a functor \( F: C \to IK\{-}\)
\( \to C\{-}\) the category of left C-modules with
morphisms the natural transformations.

A right C-module is a functor \( G: C^{op} \to IK\{-}\)
\( \to \text{Mod-}C = \text{cat of right } C\{-}\)

Rem: A C-module determines not only a IK-module
\( F(C) \) for each obj \( C \), but a \( IK[\text{End}_C(C)] \)-module
(for \( C = FI \), \( \text{End}_{FI}(\mathbb{N}) = \mathbb{N}_0 \) ), and it is more
than a sequence of \( IK[\text{End}_C(C)] \)-modules
because of properties given by the functoriality.

Prop: C-modal and mod-C are Abelian categories
(because limits and colimits are carried in the
target categories which is Abelian)

Standard Projective Generators

Def: \( \forall C \in C, \quad P^C_c = IK[\text{Hom}_C(C, -)] \)
(corresponds to \( H(d) \) for \( C = FI \) in Church's talk)

Ex: If \( C \) had an initial object \( I \),
\( P^C_I = IK[\text{Hom}(I, -)] = IK \to \) the constant
Functor is projective

(In Church's lecture, \(m(0) = 2\) is the constant function)

\[ \text{Prop. Yoneda Lemma:} \]

1. \(\text{Hom}_{\text{C-mod}}(PC, F) \cong F(C)\) (Yoneda Lemma)

2. \(\forall C \in \text{C}, \text{PC is a projective object in C-mod}\)

3. Any projective object in \(\text{C-mod}\) is a direct summand in a coproduct of objects \(\text{PC}\)

4. For any object \(F \in \text{C-mod}\), there is an epimorphism \(P \to F\) (\(P\) is projective)

Thus, there is a set of projective generators in \(\text{C-mod}\), \(\{PC\}_{C \in \text{Obj} C}\)

Similarly, the functor \(P^\text{cop}_C = K[\text{Hom}_C(-, C)]\)

\(\Rightarrow\) projective generators of \(\text{Mod-C}\).

**Ex:** If \(T\) is the terminal object of \(C\),

\[ P^\text{cop}_T = K[\text{Hom}_C(-, T)] = K \text{ constant functor} \]

**Tensor Product**

**Def:** For \(F \in \text{C-mod}\) and \(G \in \text{C-mod}\), we define \(G \otimes F := \bigoplus_{C \in \text{Obj} C} G(C) \otimes F(C) \in K\text{-mod}\)

where \(\forall f : C \to C', \ x \in G(C'), \ y \in F(C), \ x \otimes F(f)(y) = G(f)(x) \otimes y\)
\textbf{Prop:} \( G \otimes_P^C G(C) \Rightarrow P \otimes^C F \Rightarrow F(C) \)

\textbf{Prop:} The \textit{bi-functor} \( - \otimes_C : \text{Mod} \times \text{C-Mod} \rightarrow \text{K-Mod} \)

is \textit{right exact in each variable}.

\((\Rightarrow \text{the standard projectives are flat})\)

\textbf{For functors}

\textbf{Def:} For \( F \in \text{C-Mod} \) and \( G \in \text{Mod-C} \), we define

\[ \text{Tor}^C_i (G, F) := \text{H}_i (P_\ast \otimes F) = \text{H}_i (G \otimes P_\ast) \]

where \( P_\ast : \rightarrow P_2 \rightarrow P_1 \rightarrow 0 \) is a projective resolution of \( G \) (or \( F \) a resolution of \( F \)).

\textbf{Homology of a category}

\textbf{Def:} For \( F \in \text{C-Mod} \), we define

\[ H_* (C ; F) := \text{Tor}^C_\ast (1K, F) \]

\textbf{Ex:} \( C = G \Rightarrow \text{Recover usual group homology.} \)

\textbf{Ex:} If \( C \) has an initial object \( I \), then

\[ H_* (C ; IK) = \begin{cases} 1K & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \]

\textbf{Ex:} If \( C \) has a terminal object \( T \), then

\[ H_* (C ; F) = F(+) \quad * = 0 \\ 0 \quad \text{if } i > 0 \]
**Question:** Is it easier to compute $H_\ast(E, F)$ than $\text{colim}_n (H_\ast(\text{Aut}_c(X^n), \mathbb{F}(X^n)))$?

Sometimes, yes.

**Ex:** $F : G \rightarrow \text{K-Mod}$ s.t. $F(0) = 0$

1. Twisted gen free group
2. $H_\ast(G, F) = 0$ (because $0$ is terminal and $F(0) = 0$)
3. $F$ polynomial ($D-V$)
4. $H_\ast(\text{Aut}_c, F_0) \quad A_n = \text{Aut}(\mathbb{Z}^n)$

**Def:** For $B \in (C^{op} \times C)^{-}\text{Mod}$, a bicategory, we define the Hochschild homology of $C$ with coefficients in $B$ by

$$HH_\ast(C; B) := \text{Tor}_{C^{op} \times C}^c (K[[\text{Hom}_C(-, -)]], B)$$

**Ex:** $HH_0(C, B) = \bigoplus_{C \in \text{Obj} C} B(C, C)$

**Exterior Tensor Product**

$F \in \text{C-Mod}, \ G \in \text{D-Mod}$

$F \otimes G \in (C \times D)^{-}\text{Mod}$,

$F \otimes G (c, d) := F(c) \otimes G(d)$. 
Prop: Let $C$-Mod $\Rightarrow F$ and $G \in$ Mod-$C$.
\[ H^0(C, GF) \cong G \otimes F \]

If $F$ or $G$ has values in projective $k$-modules, then
\[ H^\ast(C, GF) \cong \text{Tor}^C_{\ast}(G, F). \]

Rem: For $F \in$ $C$-Mod and $\Pi: C^{\text{op}} \times C \to C$
\[ H^\ast(C, \Pi^*F) \cong H^\ast(C, F) \]

2. Polynomial Functions

$(C, *, 0)$ is a monoidal category s.t. $\text{the unit } 0$

To the null object.

Ex: $(\mathbb{F}, +, 0)$, $(\mathbb{F}(R), +, 0)$, $(\text{gr}, *, 0)$

Finite pointed sets

$F: C \to A$, $A$ is an abelian category.

Def: Cross-effects of $F: C^\times \to A$

- $\text{cr}_1 F(X) = \ker (F(0): F(X) \to F(0))$
- $\text{cr}_2 F(X) = \ker (F(X_1 \times X_2) \to F(X_1 \oplus F(X_2))$

where $\tau_1: X_1 \times X_2 \xleftarrow{\text{id} \times 0} X_1 \times 0 \cong X_1$

$\tau_2: X_1 \times X_2 \xrightarrow{0 \times \text{id}} 0 \times X_2 \cong X_2$

$\text{cr}_n F(X_1, ..., X_n) = \text{cr}_2 (\text{cr}_{n-1} F(-) X_3, ..., X_n)) (X_1, X_2)$
For $F(0) = 0$, have
\[
F(X_1 \ast X_2) = F(X_1) \oplus F(X_2) \oplus cr_2 F(X_1, X_2)
\]
\[
F(x_1 \ast x_2 \ast x_3) = F(x_1) \oplus F(x_2) \oplus F(x_3) \oplus cr_2 F(x_1, x_2) \oplus cr_2^2 F(x_1, x_3) \oplus cr_2 F(x_2, x_3) \oplus cr_3 F(x_1, x_2, x_3)
\]

Prop: $F(0) = 0$, $F(x_1 \ast \ldots \ast x_n) = \bigoplus_{\varepsilon = 1}^{n} \bigoplus_{i_1 < i_2 < \ldots < i_{\varepsilon} < n} cr_2^\varepsilon F(x_{i_1}, \ldots, x_{i_{\varepsilon}})$

Def: A function $F : C \to A$ is said to be polynomial of degree $\leq n$ if $cr_{n+1} F = 0$.

Ex: 1) $F = (-)^{ab} : gr \to Ab$
\[
F(G \ast H) = (G \ast H)^{ab} = G^{ab} \oplus H^{ab} = F(G) \oplus F(H)
\]
So $F$ is polynomial of degree $1$.

2) $C = P(R)$, $M \in P(R)$ fixed; $R$ commutative
\[
F = - \otimes M : P(R) \to P(R)
\]
\[
F(N_1 \otimes N_2) = F(N_1) \otimes F(N_2) \to \text{poly of degree } 1
\]
\[
T^2 : P(R) \to P(R)
\]
\[
M \to M \otimes M
\]
\[
T^2(M \otimes N) = T^2(M) \otimes T^2(N) \oplus M \otimes N \oplus N \otimes M
\]
\[
\text{As } cr_3 T^2 = cr_2 (cr_2 T^2) = 0
\]
\[
T^2 \text{ is polynomial of deg } 2 \leq
\]
\[
\text{As } cr_3 T^2 = cr_2 (cr_2 T^2) = 0
\]

\[
\text{Polyomial is polynomial of degree } 1\text{ by previous example}
\]
\[ T^n(M) = M^2 \]  
\[ \Gamma^n(M) = (M^2)^3 \]  
\[ S^n(M) = (M^2)^4 \]  
\[ \Lambda^n(M) = M^2 \]  
\[ \implies \text{ARE POLYNOMIAL OF DEGREE n} \]

\text{EXTENDED DEFINITION (D-V) OF POLYNOMIAL FUNCTORS}  
\[ F: C \to Ab \]  
\text{WITH C SYM MON WITH INITIAL OBJECT (NOT NECESSARILY TERMINAL) THE UNIT}  
\text{SEE FRIDAY AFTERNOON}  
\[ \text{EX: } \emptyset = FI, \text{ SCR} \]

\text{More Symmetric Definition of } \alpha_n F:  
\[ \alpha_n F(X_1, \ldots, X_n) = \ker(F(X_1, \ldots, X_n) \to \bigoplus_{i=1}^n F(X_1, \ldots, X_i, \ldots, X_n)) \]