**H_\bullet - Stability and Moduli Spaces**

**Part 1: Søren Galatius**

**Def:** Fix $P^m \subset C \times R^\infty$ closed smooth (possibly $\emptyset$)

$$N(P) = \bigcup_{W \subset (-\infty, 0]} \{ W \text{ compact and smooth} \}$$

$$\Theta W = \{ \Theta \in \mathbb{R}^p \mid \Theta \in (-\varepsilon, 0] \times P \subset W \}$$

$$N(P) = \frac{1}{W} \text{ BDiff}(W, \Theta W)$$

$$H^0(N(P)) = \text{ invariants of natural}$$

$$H^i(N(P)) = \text{ characteristic classes of BGLS}$$

**Tool:** Homological stability

**Given a bordism** $K: P \to Q$, get a cont. map

$$N(P) \to N(Q)$$

$$W \to (W \cup K) - e_1$$

**Theorem (Harer):** $m = 2$

**BDef(W, \Theta) \to BDef(W', \Theta)$$

$W, W'$ connected

**Harer:** B0 for $H_\bullet^*, \quad * << g$

**Massey-Weiss:** $H^*(\{0\}; \mathbb{Q}) \cong \mathbb{Q}[K_1, K_2, \ldots]$
\[ M = 2^n \]
\[
\text{Def: } W_{g,1} = \# S^n \times S^n \setminus \text{int}(D^{2n})
\]

For \( W \) connected, define its \textit{genus}:
\[
g(W) = \max \{ g | \exists W_i \hookrightarrow W \}
\]

\textbf{Note:}
\[
g(W \# (S^n \times S^n)) \geq g(W) + 1 \quad (\text{doesn't need to have equality here})
\]

\[
\bar{g}(W) = \max \{ g(W \# k(S^n \times S^n)) - 2 | k \}
\]

\textit{"stable genus"}

Then \( \bar{g}(W \# (S^n \times S^n)) = \bar{g}(W) + 1 \)

\underline{RESULTS:}

\textit{Description of } \{\text{BO}^n(W, \mathbb{C})\} \text{ for } n \geq 2, \pi_1 W = 0

For \( k \leq \frac{\bar{g}(W) - 1}{2} \)

\underline{FALSE:} \quad \text{BO}^n(W, \mathbb{C}) \rightarrow \text{BO}^n(W, \mathbb{C}) \text{ isom in } H_4 \text{ in a range}

\textit{— AT LEAST NOT ALWAYS!}

\underline{Tangential } n\text{-TYPE OF } W^{2n}

\[ \begin{array}{c}
X \\
\downarrow \phi \\\n\text{BO}(2n)
\end{array} \]

\( \phi \) \text{ fibre fibration}

\textit{n-coconnected} (i.e. \( \pi_3 \phi (\text{fiber}) = 0 \))

If \( \exists W \xrightarrow{\phi} \text{BO}(2n) \) s.t. \( \phi \) is n-connected

\textit{classifying map of } TW \rightarrow W
**Classical Fact (Moore–Pettitkov Factorisation)**

\[ TN \to BO(2n), \exists \text{ such an } (X, \theta, \xi). \]

Instead of classifying all manifolds, fix \( X \to BO(2n) \), and classify pairs \((W, e)\).

\[ N(P, e) := \{ (W, \xi) \mid \text{W at before, } \xi \text{ n-conn.}, \xi|_W = e \} \]

Fix \( E^* \otimes TP \to \theta^* \)

\[ \downarrow \quad \downarrow \]

\[ P \to X \]

**Lift of** \( P \to BO(2n) \)
**Classifying** \( E^* \otimes TP \)

\( j \to BO(2n) \) **Canonical BOC**

**Given a bordism** \( P \to \emptyset \), together with

\[ TK \xrightarrow{\lambda^*_k} \theta^* \]

\[ \downarrow \quad \downarrow \]

\[ K \to X \]

\[ N^0(P) \xrightarrow{UK} N^0(\emptyset) \]

Well-defined if \( P \to K \) is \((n-1)\)-connected.

This is the right kind of stabilization map.

(Assume \( X \) connected)

\[ N^0(P) = \bigoplus_{g > 0} N^0, g(P) \]

**Elementary Bordisms**: \( K : P \to \emptyset \) of index \( k \) is elementary if it admits a Morse function \( f \) such that

\[ f : K \to [0, 1], \quad f^{-1}(0) = P, \quad f^{-1}(1) = \emptyset, \quad \text{exactly one critical point of index } k. \]
It suffices to consider elementary bordisms of index $k \in \{n, 2n\}$.

If $k > n$, $N^g, g(P) \xrightarrow{K} N^g, g(\emptyset)$

If $n > 2$, $\pi_1 X = 0$.

**Thm.** Both maps induce isomorphisms in $H_\ast$, $* < g - \frac{4}{2}$.

**Special Case:**

**Thm A.** $T = ([0,1] \times P) \# S^g \times S^n$, then

$N^g, g(P) \rightarrow N^g, g(\emptyset)$ induces an isomorphism in $H_\ast$, $* < g - \frac{4}{2}$

and

$\pi_1$, $* < g - \frac{4}{2} + 1$

then enough to prove "stable stability" for other bordisms

$T^g \circ T^g \simeq T^g \circ T^g$

for $k \in \{n, 2n-1\}$

$T^g \circ k \simeq k \circ T^g$
\[
\begin{align*}
N^0(P) & \xrightarrow{T_p} N^0(Q) \rightarrow \cdots \\
\downarrow & \quad \downarrow \\
N^0(Q) & \xrightarrow{T} N^0(Q) \rightarrow \cdots 
\end{align*}
\]

Given Thm A, it suffices for \( k \in [n, 2n-1] \) to prove

**Thm B:** \( H_*(N^0(P)) \xrightarrow{k^*} H_*(N^0(Q)) \) is \( \mathbb{Z}[T] \)-linear.

It is an isomorphism after inverting \( T \) (i.e., after \( \otimes \mathbb{Z}[T, T^{-1}] \)).

(This last Thm B is also true when \( n=2 \), \( \pi_1 X \neq 0 \)).

\[ \text{Thm} \]
\[ N^0(P) \xrightarrow{G} Z^\infty X^\infty \]

\[ \text{ISO in } H_* \text{ in } \text{stable range} \]

\[ H_*(BO(\pi)) \cong H_*(Z^\infty X^\infty) \] 

\[ \text{above} \]

\[ \text{stable range} \]
PART II: OSCAR RAMONÓ-Williams

Slight variant in the model for $N^0(P)$:

$PC\mathbb{R}^\infty, \text{choose } S^{2n-1} \subset \mathbb{R}^{2n} \subset \mathbb{R}^{2n+1}, L = D^{2n-1}_- \times S^{2n-1}$

$N^0(P_{/p}) = [\text{some diagram}]$

$l^\#: TW \to \Theta^3$

which is standard on $P_{/p} \times [0,1] \times L$, $S^{2n-1}$

$P_{/L}$

Composition on the right by $T$ gives an endomorphism

$- o T : N^0(P_{/P}) \longrightarrow N^0(P_{/P})$

Let $N^0(P_{/P}) [T^{-1}]$ be the mapping telescope of this map.

Let $C^{-1}$ be the 'category' with

- objects: $(P_{/P})$, $PC\mathbb{R}^\infty$ and containing $L$

- morphisms $(k_{1,2} : (P_{/P}) \longrightarrow (Q_{/Q})$ inside $[0,5]\times \mathbb{R}^\infty$ that contain $[0,5] \times L$ which are $(n-1)$-connected rel $Q$.

Can topologize...
$N^0(-)$ is a functor: $C^{n-1} \rightarrow \text{Top}$

$(p, \ell_p) \mapsto N^0(p, \ell_p)$

$(k, \ell_k) \mapsto (k\ell) + e_1$

**Note:** We now give $T$ on a different side than $k$, these operations strictly commute.

$(- \circ T)$ gives a self net tranof. of $N^0(-)$, so get a functor $N^0(-)[T^{-1}]: C^{n-1} \rightarrow \text{Top}$

We have (stable stabilizers) if $2n \geq 4$. This sends every morphism in $C^{n-1}$ to a $H_k$-equiv.

Observation: $T_P = ([0,1] \times P) \# (S^n \times S^n)$, $T_P: N^0(p, \ell_p)[T^{-1}] \rightarrow$ Tp produces an iso on $H_*$ of $N^0(p)[T^{-1}]$.

But slide it through a standard color

$\rightarrow T_P$ induces an iso on $H_*$ of $N^0(p)[T^{-1}]$.

**The cobordism** $T_P$ has the form $\text{from where } M: P \rightarrow \partial$ consists of a single $n$-handle, attached previously.

Ex: The boundary of the core bounds a disc in $\varnothing + \text{normal bol}$ (compatible framing)
**Non-trivial Attachments:**

- No disc with bullet
- The bullet of the core

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If $M:T \rightarrow \varnothing$ is a corobin consisting of a single transient attacker handle, then $M_{oM} \cong T_p$ so $M_{oM}$ induces a $H_{\ast+1} (\mathbb{Z}_p)$. 

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**Idea:** Modify the MFD to make the attachment trivial.

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**EX:**

- Find a 1-handle and cut it out

$\Rightarrow$ Now the attachment is trivial

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Again

$\Rightarrow$ Trivial after cutting out.
Given $M : p \rightarrow q$, consider $\overline{\text{Mom}} : N^0(p) \rightarrow L$. We want to consider a semi-simplicial resolution:

$$K^p(p) \rightarrow N^0(p) \text{ s.t.}$$

1. $K^p(p) \simeq N^0(R)$ for some $R$

2. $(\overline{\text{Mom}})$ lifts to a semi-simplicial map, which in deg $p$ is given by $\overline{\text{Mom}}'$, where $M'$ consists of a trivial $n$-handle.

3. The map $|e| : |K^p(p)| \rightarrow N^0(p)$

$K^p(p)$ is the set of choices of $p+1$ handles making the attachment trivial (corresponds to, in the non-orientable surface case, to connections of 1-sided arcs with non-orientable, connected complement).

**Note:**

![Diagram](attachment:diagram.png)

The first one induces an injective map in $L^0$, and the second in $L^1$. More generally,

If $M : p \rightarrow q$ consists of a trivial $(k-1)$-handle, and $N : R \rightarrow p$ is a canceling $k$-handle, then $\text{Mom} : R \rightarrow q$, is diffeomorphic to a cylinder.

Stability for $(k-1)$-handle => Stab for canceling $k$-handle.
Make a resolution $L_0(P) \to N^\infty(P, l_0)$

where $0$-simplices are pairs $(w, l_w)$ with embedding of

a $(2n-k)$-handle with bdry

a meridian of the handle

attached via $m$