Lecture III (Tom Church)

First Goal: Prove Thm (C-E) that
\[ \deg H_i^{\text{FI}}(W) \leq \deg H_0^{\text{FI}}(W) + \deg H_i^{\text{FI}}(W) + i - 1. \]

Precursors: Eisenbud-Fishstao-Wyman (over C)
Rutman (char \to 0)

Rem: Bound not preserved by "shifting" the \text{FI}-module, where "shift" means
\[ 0 \to V \to M \to W \to 0 \]

But the Thm holds \textup{\textit{to the bound}}.

\[ 0 \to V \to M \to W \to 0 \]

\[ \forall c \in M(d) \text{ is inside a free module} \]
\[ \Rightarrow V \text{ is torsion-free, i.e. } \forall f : [m] \to [n], \forall v \in V, f(v) = 0 \Rightarrow v = 0. \]

Because \( M(d) \) is torsion free.

"Down"

“IDEA:" SHifting W twice will make it nice:
\[ 0 \to \ker L \to M' \to V \to 0 \]
Free

What does \( L \) have that \( V \) doesn't?

Answer: \( L \) is "\textit{saturated}"
$V \in M(1)$ is not saturated: $2e_1 \in V_3$ but $2e_1 \notin V_2$ (from the example)

**Def:** $U \subset M(A)$ is saturated if for any $m \in M_n$, for any $f: [n] \to [N]$, $f_*(m) \in U$ implies $m \in U$.

**Intrinsic Version:** A torsion-free FI-module $U$ is saturated if for all $U \subset U'$, s.t. $\exists f_1, f_2: T \to R$ with $f_1 \leq f_2$, then $U \leq \text{Im}(U_0 \to U')$.

Consequently, for $k$, $H_k(S^2 \vee V)_2 = 0$

**Prop:** $0 \to U \to M \to V \to 0$ is free torsion-free $\Rightarrow U$ saturated.

**If:** $m \in M_n$, then $f: [n] \to [N]$ s.t. $f_*(m) \in U$,

$m \in V_n$, $f_*(m) \in U_n$,

$m = 0 \Rightarrow f_*(m) = 0 \Rightarrow m = 0 \Rightarrow m \in U_n$.

**Rec:** saturated powers of $U$ say precisely that you can recover the FI-module $U$ from $\bigcup_{m \in M_n} U_m = U$.

**Rec:** recover $U_T = \{ u \in U | S^{\text{int-T}} \text{ fixes } u \}$.
More technical condition: (saturation not quite enough)

If \( \mathcal{O} \to X \to M \to U \to \mathcal{O} \) then \( X \) is free saturated "2-saturated":

\[
\text{Def: } X \text{ is 2-saturated if it's saturated and for any } s, s', \text{ set } X + \mathcal{M}_s + \mathcal{M}_{s'} = X_s + X_{s'}, \text{ as submodules of } M^x_{\mathcal{O}}
\]

Example: \( U \subseteq \mathcal{M}(\{a, b, c\}) \)

\[
\begin{array}{c}
2 \\
{} + \quad 3 \\
\quad {} + \\
1 \quad 4
\end{array}
\]

\( T = \{1, 2, 3, 4\} \)

\( S = \{1, 2, 3\} \)

\( S' = \{1, 2, 4\} \)

\[
u = r + \lambda a + \lambda b + \lambda c
\]

\( u \in M_{S} + M_{S'} \)

But \( u \notin U_{S} + U_{S'} \).

Prop: If \( \mathcal{O} \to X \to M \to U \to \mathcal{O} \) and \( U \) is saturated, then \( X \) is 2-saturated.

(Alternative characterization: \( \forall \xi, H_1(\mathcal{M}(\{a, b, c\})) = 0 \))

\[
\text{Def: } Y \text{ is } k\text{-saturated if } (k-1)\text{-saturated and } \quad Y + \mathcal{N}(\mathcal{M}_{S_1} + ... + \mathcal{M}_{S_k}) = Y_{S_1} + ... + Y_{S_k}
\]

\[
\text{Def: We say } U \text{ is } k\text{-saturated above } N \text{ if the above holds when } |S_1| \geq N
\]

Prop B: If \( X \subseteq M(d) \) is \((k+1)\text{-saturated } \geq N\) then \( X \) is generated by \( X_N \) (or \( X_k \) for \( k \geq N \))
From $\mathbb{T} = [n]$, $S_1 = [n] - 1, \ldots, S_{d+1} = [n] - (d+1)$

Every injection $f: [d] \to [n]$ factors through some $S_i \to M_T = M_{S_1} + \cdots + M_{S_{d+1}}$.

$X_T = X_+ \cap (M_{S_1} + \cdots + M_{S_{d+1}}) = X_{S_1} + \cdots + X_{S_{d+1}}$

i.e. $X_n$ is spanned by monos from $X_{n-1} \to X_n$.

Prop C: If $0 \to V \to M \to W \to 0$, $W$ generated in degree $k$ and $0 \to U \to M' \to V \to 0$, $V$ generated in degree $0$, then $U$ is $k$-saturated for all $k$ with $N = D + k + 1$.

Intrinsic Definition of $k$-saturated:

$J_1 \in \text{ker } J_1 \cdots \text{ker } J_k = \text{ker } J_1 + \cdots + \text{ker } J_k$

$J_2 \in \mathbb{Z}_{S_1 \cup \ldots \cup S_{13}}$, $J_1 = \text{Id} - (1 - 1)$

For $u = 1^+ 2^- 3^+$, $J_1 u = u - 3^+$

Saturation $\Rightarrow \text{ker } J_1 = \text{im} (U_{q_1, \ldots, q_3} \to U_{q_1, \ldots, q_3})$ as subsets of $U_{q_1, \ldots, q_3}$.

$J_2 = (\text{Id}) - (2 - 2)$. $J_1, J_2$ commute.

$J_1 J_2$ annihilates anything not involving 2, also not involving 1.

$J_1 J_2 \to \text{ker } J_1 J_2 = M_{S_1} + M_{S_2}$