Lévy Processes in Finance
Summer school Sandbjerg

Jan Kallsen
TU München/CAU Kiel

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Outline

1. Mathematical finance
2. Stochastic calculus
3. Applications to finance
1 Mathematical finance
2 Stochastic calculus
3 Applications to finance
Price processes and trading strategies

- **securities price process** \( S = (S_0(t), \ldots, S_d(t))_{t \in [0, T]} \)
  - \( \mathbb{R}^{d+1} \)-valued semimartingale on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)\)
  - \( S_i(t) \) is price of security \( i \) at time \( t \)
  - Example:
    - Bond: \( S_0(t) = e^{rt} \)
    - Stock: \( S_1(t) = S_1(0)e^{L(t)} \) with Lévy process \( L \)

- **trading strategy** or portfolio \( \varphi = (\varphi_0(t), \ldots, \varphi_d(t))_{t \in [0, T]} \)
  - \( \mathbb{R}^{d+1} \)-valued predictable process
  - \( \varphi_i(t) \) is number of shares of security \( i \) at time \( t \)
  - value process \( V_{\varphi}(t) := \varphi(t)^\top S(t) = \sum_{i=0}^{d} \varphi_i(t)^\top S_i(t) \)
  - \( \varphi \) is self financing if \( S \)-integrable and \( V_{\varphi} = V_{\varphi}(0) + \varphi \cdot S \)
Discounting
for simplified bookkeeping

- Idea: express prices in multiples of **numeraire** security $S_0$.
  - Assume $S_0, (S_0)_- > 0$.
- discounted price process $\hat{S} := \frac{1}{S_0} S = (1, \frac{S_1}{S_0}, \ldots, \frac{S_d}{S_0})$
- discounted value process $\hat{V}_\varphi := \frac{V_\varphi}{S_0} = \varphi^\top \hat{S}$
- Example: for $S_0(t) = e^{rt}$ we have
  $\hat{S}(t) = e^{-rt} S(t)$ and $\hat{V}_\varphi(t) = e^{-rt} V_\varphi(t)$.
- Lemma:
  $\varphi$ self financing $\iff \varphi$ $\hat{S}$-integrable and $\hat{V}_\varphi = \hat{V}_\varphi(0) + \varphi \cdot \hat{S}$
- Lemma: For $(\hat{S}_1, \ldots, \hat{S}_d)$-integrable $(\varphi_1, \ldots, \varphi_d)$ and $\nu \in \mathbb{R}$
  ex. unique $\varphi_0$ such that $\varphi = (\varphi_0, \ldots, \varphi_d)$ self financing with $V_\varphi(0) = \nu$.
- Consequence: do not worry about self financability and $\varphi_0$. 
Arbitrage and consequences

An admissible self financing strategy $\varphi$ is an arbitrage if $V_\varphi(0) = 0$, $V_\varphi(T) \geq 0$, $P(V_\varphi(T) > 0) > 0$.

- Without admissibility constraint Black-Scholes model is not arbitrage free.
- We take careless engineering point of view here. Precise statements are difficult.

**Rule (law of one price)**

Let $\varphi, \psi$ be admissible strategies in an arbitrage-free market. If $V_\varphi(T) = V_\psi(T)$, then $V_\varphi = V_\psi$.

**Rule (fundamental theorem of asset pricing, FTAP)**

The market is arbitrage free $\iff$ there exists some equivalent martingale measure (EMM) $Q$ (i.e. $Q \sim P$ is probability measure and $\hat{S}$ is $Q$-martingale).
Derivatives

- **European option**: $\mathcal{F}_T$-measurable random variable $X$
  - $X$ stands for payoff at time $T$.
  - Examples:
    - European call $X = (S_1(T) - K)^+$ or put $X = (K - S_1(T))^+$
    - Lookback call $X = (\sup_{t \in [0, T]} S_1(t) - K)^+$
  - Discounted payoff is $\hat{X} := \frac{X}{S_0(T)}$.

- Futures and American options are more involved.

- What is a reasonable price for the option at time $t < T$?

- We distinguish **liquidly traded** and **over-the-counter** options.
Consider underlyings $S_0, S_1$ (given processes) and derivative price process $S_2$ with $S_2(T) = X$ (yet to be determined).

The FTAP implies:

\[ \text{no arbitrage } \implies \hat{S}_2(t) = E_Q(\hat{X}|\mathcal{F}_t) \text{ for some EMM } Q \]

Rule

$S_2$ does not depend on the chosen EMM $Q$ $\iff$ $X$ is attainable (i.e. $X = V_{\varphi}(T)$ for some admissible strategy $\varphi$).

Corollary (second fundamental theorem of asset pricing)

An arbitrage free market is complete (i.e. any option is attainable) $\iff$ there exists only one EMM.

Most models involving non-Gaussian Lévy processes are incomplete.
**Problem of incompleteness**

for liquidly traded options

- Problem: range of possible prices typically too large
  - e.g. trivial arbitrage bounds for European call in geometric Lévy model for standard Lévy processes (VG, NIG, . . . )
  - Away from Black-Scholes case arbitrage theory does not yield useful information.

- Approach 1:
  - consider some derivatives as underlyings, e.g. vanilla options
    - get fewer EMM’s, hence tighter price bounds for other derivatives
    - Problem: how to get reasonable and arbitrage free model for new underlyings?
      - Approach seems rather cumbersome.

- Approach 2 (**martingale modelling**):
  - model prices directly under market’s pricing measure
    - used in practice
Martingale modelling
for liquidly traded options

- Choose parametric class of models for underlyings $S_0, S_1$ under market’s EMM $Q$.
  - e.g. $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with variance-gamma Lévy process $L$

- Consider martingale constraint, i.e. $\hat{S}^1$ $Q$-martingale.
  - here: restrict variance-gamma parameters such that $E(e^{L(1)-r}) = 1$

- Determine remaining parameters by calibration to observed prices of liquid options at $t = 0$.
  - choose parameters such that discounted observed price $= E_Q($discounted payoff$)$
    for all observed vanilla options
    - use e.g. least squares procedure if more options than parameters

- Use model e.g. for pricing of new non-liquidly traded derivatives.
Question: is the parametric class of models appropriate?
  ▶ Statistical tests are useless because they concern real world probabilities \( P \).

Answer: inconsistencies may indicate inappropriate class or model.
  ▶ e.g. no choice of parameters leads to reasonable fit of observed option prices
  ▶ or recalibration leads to heavily time-varying models

However: Entirely different classes may produce reasonable fit, but differ substantially for further e.g. path-dependent options (model risk).
Two different situations:

- only interested in valuation of new options
  - modelling under pricing measure $Q$ suffices, physical measure $P$ not needed
- real-world probabilities needed as well (e.g. for value at risk assessments etc.)
  - model dynamics of $S_0, S_1$ under both $P$ and $Q$
  - choose typically same parametric class under both $P$ and $Q$
  - e.g. $S_1(T) = S_1(0)e^{L(t)}$, $L$ VG Lévy process under $P$ and $Q$
  - determine $P$-parameters by estimation using stock price data, $Q$-parameters by calibration using option prices
  - OBS! condition $Q \sim P$ leads to constraints on possible parameters
Situation:

- Underlyings $S_0, S_1$ liquidly traded, but not the option
- Client wants to buy option $X$ from bank at time $t = 0$
- Reasonable prices $\pi$?

$\pi \leq \pi_{\text{high}} := \inf\{\pi \in \mathbb{R} : \exists \ \text{adm. } \varphi \text{ with } V_0(\varphi) = \pi \text{ and } V_\varphi(T) \geq X\}$

- Justification: otherwise client can do better by investing in such superhedge $\varphi$

$\pi \leq \pi_{\text{low}} := \sup\{\pi \in \mathbb{R} : \exists \ \text{adm. } \varphi \text{ with } V_0(\varphi) = \pi \text{ and } V_\varphi(T) \leq X\}$

- Justification: otherwise bank should sell such subhedge $\varphi$ rather than option

How to determine $\pi_{\text{high}}, \pi_{\text{low}}$?
Individual pricing
of over-the-counter derivatives (ct’d)

### Rule (superhedging)

\[
\hat{\pi}_{\text{high}} := \frac{\pi_{\text{high}}}{S_0(0)} = \sup \{ E_Q(\hat{X}) : Q \text{ EMM for } S_0, S_1 \}
\]

\[
\hat{\pi}_{\text{low}} := \frac{\pi_{\text{high}}}{S_0(0)} = \inf \{ E_Q(\hat{X}) : Q \text{ EMM for } S_0, S_1 \}
\]

- Price bounds coincide with price interval for liquid options.
- hence same problem as before: range of prices too large
- alternatives:
  - pricing based on quadratic hedging
  - utility indifference pricing
Quadratic hedging
and pricing over-the-counter derivatives

- **quadratic hedging of option $X$:**
  - minimize expected squared hedging error
  \[
  \varepsilon^2(\varphi) := E((\hat{V}_\varphi(T) - \hat{X})^2)
  \]
  over all admissible strategies $\varphi$
  - optimal strategy $\varphi^*$, minimal hedging error $\varepsilon^2 := \varepsilon^2(\varphi^*)$
  - $\hat{V}_{\varphi^*}(T)$ is orthogonal projection in $L^2$ of $\hat{X}$ on \{\hat{V}_\varphi(T) : \varphi \text{ admissible strategy}\}.$

- **application to over-the-counter trade**
  - charge e.g. $\pi = V_{\varphi^*}(0) + \lambda \varepsilon^2$ as option’s premium
  - use $V_{\varphi^*}(0)$ to buy hedging strategy $\varphi^*$.
  - take multiple of $\varepsilon^2$ as compensation for unhedgeable risk (parameter $\lambda$ stands for risk aversion)

- **How to determine $\varphi^*$, $V_{\varphi^*}(0)$, $\varepsilon^2$?**
Theorem (quadratic hedging)

Assumption: \( \hat{S} \) is martingale (general case is more involved).
Then

\[
V_{\varphi^*}(0) = E(\hat{X})
\]

\[
\varphi^*(t) = \frac{d\langle \hat{V}, \hat{S}_1 \rangle(t)}{d\langle \hat{S}_1, \hat{S}_1 \rangle(t)}
\]

\[
\varepsilon^2 = E\left(\langle \hat{V}, \hat{V} \rangle(T) - (\varphi^*)^2 \cdot \langle \hat{S}_1, \hat{S}_1 \rangle(T)\right)
\]

with \( \hat{V}(t) := E(\hat{X}|\mathcal{F}_t) \).

- How to determine angle brackets \( \langle \hat{V}, \hat{S}_1 \rangle \) etc.?
Utility indifference pricing of over-the-counter derivatives

- Problem of approach via quadratic hedging:
  gains are penalized, hence economically questionable

- Alternative:
  increasing utility functions and indifference pricing

- Idea:
  sell option only if this is profitable for the bank (i.e. utility rises)
Expected utility maximization

- fix increasing, strictly concave utility function \( u \).
- Examples:
  - \( u(x) = \frac{x^{1-p}}{1-p} \) for \( p \in (0, \infty) \setminus \{1\} \)
  - \( u(x) = \log(x) \) (behaves like \( p = 1 \) above)
  - \( u(x) = 1 - e^{-px} \) for \( p > 0 \)
- maximize expected utility \( E(u(\hat{V}_\varphi(T))) \) over all admissible strategies \( \varphi \) with fixed initial value \( V_\varphi(0) = v \)
- solution approaches:
  - dynamic programming
  - martingale methods (here)

**Rule (dual characterization)**

\( \varphi \) admissible strategy such that
\[
\frac{dQ}{dP} = \frac{u'(\hat{V}_\varphi(T))}{E(u'(\hat{V}_\varphi(T)))} \text{ density of EMM}
\]
\[\implies \varphi \text{ is optimal.}\]
Utility indifference pricing
of over-the-counter derivatives

Question:
- underlyings $S_0, S_1$ liquidly traded, but not the option
- client wants to buy option $X$ from bank at time $t = 0$
- reasonable prices $\pi$ for the bank?

Answer:
- utility of bank without option trade:
  $U_0 := \sup\{E(u(\hat{V}_\varphi(T))) : \varphi \text{ adm. with } V_0(\varphi) = v\}$
- utility of bank with option trade:
  $U_X(\pi) := \sup\{E(u(\hat{V}_\varphi(T) + \hat{\pi} - \hat{X})) : \varphi \text{ adm. with } V_0(\varphi) = v\}$
- require $U_X(\pi) > U_0$ (trade raises bank’s utility)
- utility indifference price: limiting price with $U_X(\pi) = U_0$
- reasonable prices are those above indifference price

Problem: utility indifference prices are hard to compute

Alternative:
compute approximate solution for small number of options
Approximate utility indifference pricing of over-the-counter derivatives

Notation:

- \( \pi(n) \): utility indifference price per unit of \( X \) for \( n \) options
- \( \varphi(n) \): optimal strategy for \( n \) sold options, i.e. for \( U_{nX}(n\pi(n)) \)

Goal: approximate \( \pi(n), \varphi(n) \) for small \( n \)

Expansion:

\[
\hat{\pi}(n) = \hat{\pi}(0) + n\delta + o(n)
\]
\[
\varphi(n) = \varphi^* + n\eta + o(n)
\]

Interpretation:

- \( \pi(0) \): limiting price for very small number of options
- \( \delta \): risk premium per option that is to be sold
- \( \varphi^* \): optimal strategy for pure investment problem without options
- \( \eta \): hedging strategy per option

How to determine \( \pi(0), \delta, \eta \)?
Approximate utility indifference pricing
for exponential utility

Rule

Consider \( u(x) = 1 - e^{-px} \).
Let \( \varphi^*, Q \) be optimal pair from dual characterisation.
Define \( \hat{V}(t) := E_Q(\hat{X} | \mathcal{F}_t) \). Then

\[
\hat{\pi}(0) = \hat{V}(0) = E_Q(\hat{X}),
\]

\[
\delta = \frac{p}{2} E_Q \left( \langle \hat{V}, \hat{V} \rangle^Q(T) - \eta^2 \cdot \langle \hat{S}_1, \hat{S}_1 \rangle^Q(T) \right),
\]

\[
\eta(t) = \frac{d \langle \hat{V}, \hat{S}_1 \rangle^Q(t)}{d \langle \hat{S}_1, \hat{S}_1 \rangle^Q(t)}.
\]

- \( Q \) is the minimal entropy martingale measure.
- \( \delta, \eta \) solve quadratic hedging problem for \( X \) under \( Q \).
- Slightly more involved results hold for power/logarithmic utility with additional numeraire change.
Integral transform methods
for pricing and hedging in general

Consider e.g. $X = (S_1(T) - K)^+$. We have encountered objects like:

- $\hat{V}(t) = E(\hat{X}|\mathcal{F}_t)$,
- $\varphi(t) = \frac{d\langle \hat{V}, \hat{S}_1 \rangle(t)}{d\langle \hat{S}_1, \hat{S}_1 \rangle(t)}$,
- $\varepsilon^2 = E\left(\langle \hat{V}, \hat{V} \rangle(T) - \varphi^2 \cdot \langle \hat{S}_1, \hat{S}_1 \rangle(T)\right)$.

How to compute them?

- Typically no closed-form solution available.
- Often available: closed-form solutions for $\hat{X} = \hat{S}_1(T)^z$, $z \in \mathbb{C}$.
- E.g. $E(\hat{X}) = E(\hat{S}_1(T)^z) = e^{-zrT}E(\exp(zL(T)))$, i.e. extended characteristic function of $L(T)$ for $S_1 = S_1e^L$.
Idea: use integral representation

\[ \hat{X} = \int_{R-i\infty}^{R+i\infty} \hat{S}_1(T)^z \ell(z) dz \]

with explicitly given \( \ell(z), R \).

Linearity yields

- \( \hat{V}(t) = \int_{R-i\infty}^{R+i\infty} \hat{V}_z(t) \ell(z) dz \) with \( \hat{V}_z(t) := E(\hat{S}_1(T)^z | \mathcal{F}_t) \),
- \( \varphi(t) = \int_{R-i\infty}^{R+i\infty} \varphi_z(t) \ell(z) dz \) with \( \varphi_z(t) := \frac{d \langle \hat{V}_z, \hat{S}_1 \rangle(t)}{d \langle \hat{S}_1, \hat{S}_1 \rangle(t)} \),
- \( \varepsilon^2 = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \varepsilon_{yz}^2 \ell(y) \ell(z) dy dz \) with
  \( \varepsilon_{yz}^2 = E \left( \langle \hat{V}_y, \hat{V}_z \rangle(T) - \varphi_y \varphi_z \cdot \langle \hat{S}_1, \hat{S}_1 \rangle(T) \right) \).

determine \( \ell(z) \) via inverse Laplace transform

- for \( \hat{X} = (\hat{S}_1(T) - K)^+ \) we have \( \ell(z) = \frac{K^{1-z}}{z(z-1)}, R > 1 \)
- for \( \hat{X} = (K - \hat{S}_1(T))^+ \) we have \( \ell(z) = \frac{K^{1-z}}{z(z-1)}, R < 0 \)
Outline

1. Mathematical finance
2. Stochastic calculus
3. Applications to finance
Basic concepts
of the general theory of stochastic processes

filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\)
process \(X = (X(t))_{t \in \mathbb{R}_+}\)
càdlàg: right-continuous with left-hand limits \(X_\cdot = (X(t^-))_{t \in \mathbb{R}_+}\)
of finite variation: difference of two increasing processes
adaptedness: \(X(t) \mathcal{F}_t\)-measurable for all \(t\)
predictability: slightly stronger than “\(X(t) \mathcal{F}_{t-}\)-measurable for all \(t\)”
stopping time: random time \(\tau\) with \(\{\tau \leq t\} \in \mathcal{F}_t\)
stopped process \(X^\tau = (X(\tau \wedge t))_{t \in \mathbb{R}_+}\) with stopping time \(\tau\)
martingale: \(E(X(t)|\mathcal{F}_s) = X(s)\) for \(s \leq t\)
submartingale: \(E(X(t)|\mathcal{F}_s) \geq X(s)\) for \(s \leq t\)
supermartingale: \(E(X(t)|\mathcal{F}_s) \leq X(s)\) for \(s \leq t\)
density process: \(Z(t) = E\left(\frac{dQ}{dP}|\mathcal{F}_t\right) = \frac{dQ}{dP}|\mathcal{F}_t\)
localized classes \(\mathcal{C}_{loc}\) (local martingales etc.): \(X^{\tau_n} \in \mathcal{C}\) for all \(n\)
special semimartingale: \(X = X(0) + M^X + A^X\) with predictable \(A^X\)
compensator: finite variation process \(A^X\) in above decomposition
semimartingale: \(X = X(0) + M + A\) with adapted \(A\)
Lévy processes

Definition

- processes of constant growth in stochastic sense
- random counterpart of linear functions
- continuous-time counterpart of random walks

**Definition (Lévy process)**

adapted càdlàg process $X$ with $X(0) = 0$, $X(t) - X(s)$ independent of $\mathcal{F}_s$ (independent increments), $\mathcal{L}(X(t) - X(s))$ depends only on $t - s$ (stationary increments)

**Examples:**

- linear function $X(t) = bt$
- Brownian motion with drift $X(t) = \mu t + \sigma W(t)$
- Poisson process
- compound Poisson process $X(t) = \sum_{i=1}^{N(t)} Y_i$
Theorem (Lévy-Khintchine formula)

Characteristic function is \( \varphi_{X(t)}(u) = E(e^{iu^T X(t)}) = \exp(t \psi(u)) \) with characteristic exponent

\[
\psi(u) = iu^T b^h - \frac{1}{2} u^T cu + \int (e^{iu^T x} - 1 - iu^T h(x))K(dx)
\]

and Lévy-Khintchine triplet \((b^h, c, K)\). 

\( h \) denotes some truncation function as e.g. \( h(x) = x1\{|x| \leq 1\} \).

Interpretation:

- Drift coefficient \( b \) stands for linear function.
- Diffusion coefficient \( c \) stands for Brownian motion.
- Lévy measure \( K \) stands for jumps.

\[
b^h = \tilde{b}^h + \int (\tilde{h}(x) - h(x))K(dx)
\]
Lévy processes
Moments

- $p$-th moments exist if $K|_{\{|x|>1\}}$ has $p$-th moments.
- $E(X(t)) = b^h t = -i \psi'(0) t$ for $h(x) = x$ (i.e. $h = \text{id}$)
- $\text{Var}(X(t)) = (c + \int x^2 K(dx)) t = -\psi''(0) t$ for dimension $d = 1$
- $p$-th exponential moments exist if
  $K|_{\{|x|>1\}}$ has $p$-th exponential moments.
- $E(e^{\rho^T X(t)}) = \exp(t(\rho^T b^h + \frac{1}{2} \rho^T c \rho + \int (e^{\rho^T x} - 1 - \rho^T h(x)) K(dx)))$, which equals $\exp(t \psi(-i \rho))$ if $\psi$ has analytic extension.
- $X$ is (local) martingale iff $E(X(1)) = 0$.
- $e^X$ is (local) martingale iff $E(e^{X(1)}) = 1$. 


Lévy processes
Path properties

- \( X \) has differentiable paths \( \iff c = 0, K = 0 \)
  \( \iff X \) is linear function.

- \( X \) has continuous paths \( \iff K = 0 \)
  \( \iff X \) is Brownian motion with drift.

- \( X \) is piecewise constant \( \iff c = 0, K(\mathbb{R}) < \infty, b^0 = 0 \)
  \( \iff X \) is compound Poisson process.

- \( X \) has finitely many jumps on \([0, t]\) \( \iff K(\mathbb{R}) < \infty \)

- \( X \) is of finite variation \( \iff c = 0, \int_{|x| \leq 1} |x|K(dx) < \infty \)
Theorem (Lévy-Itô decomposition)

Let $X$ have finite expectation and triplet $(b^{id}, c, K)$.

\[
X(t) = b^{id} t + \sqrt{c} W(t) + \lim_{\varepsilon \to 0} \left( \sum_{s \leq t} \Delta X(s) 1_{\{|\Delta X(s)| > \varepsilon\}} - \int x 1_{\{|\Delta X(s)| > \varepsilon\}} K(dx) s \right)
\]

\[
= b^{id} t + \sqrt{c} W(t) + \int_{[0,t] \times \mathbb{R}} x (\mu^X - \nu^X)(d(s, x))
\]

with random measure of jumps of $X$

\[
\mu^X([0, t] \times B) := \#\{(s, x) \in [0, t] \times B : \Delta X(s) = x \neq 0\}
\]

and compensated random measure of jumps of $X$

\[
\nu^X([0, t] \times B) := K(B)t.
\]
Lévy processes in finance
for modelling of stock prices etc.

- Brownian motions with drift
  - most important and the only continuous Lévy processes
  - density, characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 2 parameters
  - all exponential moments exist

- stable Lévy motions
  - the only self-similar Lévy processes
  - either Brownian motion or pure jump
  - characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 4 parameters
  - infinite variance (except for Brownian motion)
Lévy processes in finance
for modelling of stock prices etc. (ct’d)

- **Merton model**
  - Brownian motion + drift + Gaussian jumps
  - finitely many jumps on $[0, t]$
  - characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 5 parameters
  - all exponential moments exist

- **Kou model**
  - Brownian motion + drift + exponential jumps
  - finitely many jumps on $[0, t]$
  - characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 6 parameters
  - some exponential moments exist
Lévy processes in finance
for modelling of stock prices etc. (ct’d)

- variance gamma (VG) processes
  - pure jump Lévy process
  - infinitely many jumps on $[0, t]$ but of finite variation
  - density, characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 4 parameters
  - some exponential moments exist

- normal inverse Gaussian (NIG) processes
  - pure jump Lévy process
  - of infinite variation
  - density, characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 4 parameters
  - some exponential moments exist

- etc.
Quadratic variation

and friends

- **quadratic variation** of semimartingale $X$:

$$[X, X](t) = \lim_{\sup \, |t_i - t_{i-1}| \to 0} \sum_{i \geq 1} (X(t_i \wedge t) - X(t_{i-1} \wedge t))^2$$

- covariation of $X$ and $Y$:

$$[X, Y] = \frac{1}{4} ([X + Y, X + Y] - [X - Y, X - Y])$$

- Example: $X$ Lévy process with triplet $(b^h, c, K)$

  - $[X, X]$ is Lévy process.
  - Triplet of $[X, X]$ is $(c, 0, \tilde{K})$ relative to $\tilde{h} = 0$, where $\tilde{K}(B) = \int 1_B(x^2)K(dx)$.

- **predictable quadratic variation** of $X$:

$$\langle X, X \rangle = A^{[X, X]}$$ (compensator of $[X, X]$)

- predictable covariation of $X, Y$: $\langle X, Y \rangle = A^{[X, Y]}$

- Example: $X$ Lévy process with finite variance

  - $\langle X, X \rangle$ is linear function.
  - $\langle X, X \rangle(t) = \text{Var}(X(t))$
Continuous martingale part
of a semimartingale

- Local martingales $X, Y$ are orthogonal if $XY$ local martingale.
- Local martingale $X$ is purely discontinuous if orthogonal to all continuous local martingales.
- $X$ local martingale $\Rightarrow$
  there is unique decomposition $X = X(0) + X^c + X^d$ with
  $X^c$ continuous, $X^d$ purely discontinuous, $X^c(0) = 0 = X^d(0)$.
- $X = X(0) + M + A$ semimartingale $\Rightarrow$
  continuous martingale part $X^c := M^c$ does not depend on decomposition of $X$
- $[X, Y](t) = \langle X^c, Y^c \rangle(t) + \sum_{s \leq t} \Delta X_s \Delta Y_s$
- $X$ Lévy process with Lévy-Itô decomposition
  $X(t) = b^{id}t + \sqrt{c}W(t) + \int_{[0,t] \times \mathbb{R}} x(\mu^X - \nu^X)(d(s, x))$
  $\Rightarrow X^c = \sqrt{c}W$
Stochastic integral

Heuristics

- Informal “definition”:
  - \( \varphi \) predictable process
  - \( X \) semimartingale
  - \( \varphi \cdot X(t) \approx \lim_{\sup |t_i - t_{i-1}| \to 0} \sum_{i \geq 1} \varphi(t_{i-1})(X(t_i \land t) - X(t_{i-1} \land t)) \)
  - alternative notation: \( \int_0^t \varphi(s)dX(s) = \varphi \cdot X(t) \)
  - \( \varphi \cdot X(t) := \sum_{i=1}^{d} \varphi_i(t) \cdot X_i(t) \) for \( \mathbb{R}^d \)-valued \( \varphi, X \)

- Interpretation:
  - \( \varphi \) trading strategy
  - \( X \) stock price process
  - \( \varphi \cdot X(t) = \int_0^t \varphi(s)dX(s) \) gains due to price changes
**Stochastic integral**  
**Properties**

- $\varphi \cdot X$ is linear in $\varphi$ and $X$
- $\varphi \cdot (\psi \cdot X) = (\varphi \psi) \cdot X$
- $[\varphi \cdot X, Y] = \varphi \cdot [X, Y]$
- $\langle \varphi \cdot X, Y \rangle = \varphi \cdot \langle X, Y \rangle$
- $\Delta (\varphi \cdot X) = \varphi \cdot \Delta X$
- $(\varphi \cdot X)^T = \varphi \cdot X^T$
- $(\varphi \cdot X)^c = \varphi \cdot X^c$
- $X$ sigma-martingale $\Rightarrow \varphi \cdot X$ sigma-martingale  
  (slightly more general than local martingale)
- **Integration by parts:**  
  $XY = X(0)Y(0) + X_- \cdot Y + Y_- \cdot X + [X, Y]
Theorem (Itô’s formula)

Let $X$ semimartingale in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ of class $C^2$.

\[
f(X(t)) = f(X(0)) + Df(X_-) \cdot X(t) + \frac{1}{2} \sum_{i,j=1}^{d} D_{ij} f(X_-) \cdot \langle X^c_i, X^c_j \rangle(t) \\
+ \sum_{s \leq t} \left( f(X(s)) - f(X(s-)) - Df(X(s-))^\top \Delta X(s) \right)
\]

For $d = 1$:

\[
f(X(t)) = f(X(0)) + f'(X_-) \cdot X(t) + \frac{1}{2} f''(X_-) \cdot \langle X^c, X^c \rangle(t) \\
+ \sum_{s \leq t} \left( f(X(s)) - f(X(s-)) - f'(X(s-)) \Delta X(s) \right).
\]
Random measures
and their integration

- recall random measure of jumps of semimartingale $X$:
  $\mu^X([0, t] \times B) := \#\{ (s, x) \in [0, t] \times B : \Delta X(s) = x \neq 0 \}$
- random measure here means: measure on $\mathbb{R}_+ \times \mathbb{R}^d$ for fixed $\omega$
- integral process $\xi \ast \mu$
  - $\xi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ in some sense predictable function
  - $\mu$ random measure
  - $\xi \ast \mu(t) := \int_{[0,t] \times \mathbb{R}^d} \xi(s, x) \mu(d(s, x))$ pathwise integral
  - For $\mu = \mu^X$ we have $\xi \ast \mu^X(t) = \sum_{s \leq t} \xi(s, \Delta X(s)) 1_{\{\Delta X(s) \neq 0\}}$.
- compensator $\nu^X$ of $\mu^X$:
  the in some sense predictable random measure such that $\xi \ast \mu - \xi \ast \nu$ local martingale for most predictable $\xi$
- stochastic integral $\xi \ast (\mu^X - \nu^X)$:
  extension of $\xi \ast \mu^X - \xi \ast \nu^X$ to some $\xi$ where this is not defined
Random measures
Properties

- $X$ Lévy process with triplet $(b^h, c, K) \Rightarrow \nu^X(d(t, x)) = K(dx)dt$
- $\xi \ast \mu$ and $\xi \ast (\mu^X - \nu^X)$ are linear in $\xi$.
- $\varphi \cdot (\xi \ast \mu) = (\varphi \psi) \ast \mu$
- $\varphi \cdot (\xi \ast (\mu^X - \nu^X)) = (\varphi \psi) \ast (\mu^X - \nu^X)$
- $\xi \ast (\mu^X - \nu^X)$ is local martingale, typically (but not always) with jumps
  $\Delta(\xi \ast (\mu^X - \nu^X))(t) = \xi(t, \Delta X(t))1_{\{\Delta X(t)\neq 0\}}$
- $\langle \eta \ast (\mu^X - \nu^X), \xi \ast (\mu^X - \nu^X) \rangle = (\eta \xi) \ast \nu^X$
- Itô’s formula:

\[
f(X(t)) = f(X(0)) + f'(X_-) \cdot X(t) + \frac{1}{2} f''(X_-) \cdot \langle X^c, X^c \rangle(t) + (f(X_- + x) - f(X_-) - f'(X_-)x) \ast \mu^X(t)\]

- $X$ special semimartingale:

\[
X = X(0) + A^X + X^c + x \ast (\mu^X - \nu^X)\
\]

generalizes Lévy-Itô decomposition
The solution $\mathcal{E}(X)$ to $Z = 1 + Z_- \cdot X$ is called Doléans exponential or stochastic exponential of $X$.

Solution for $\Delta X > -1$:

$$\mathcal{E}(X) = \exp \left( X(t) - X(0) - \frac{1}{2} \langle X^c, X^c \rangle(t) \right. \left. + \sum_{s \leq t} \left( \log(1 + \Delta X(s)) - \Delta X(s) \right) \right)$$

Yor’s formula:

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y])$$
Idea: constant *relative* rather than absolute growth
  - more appropriate e.g. for stock prices

**geometric Lévy process:**
- $\mathbb{R} \setminus \{0\}$-valued semimartingale with $X(0) = 1$
- $\frac{X(t)}{X(s)}$ independent of $\mathcal{F}_s$
- $\mathcal{L}(\frac{X(t)}{X(s)})$ depends only on $t - s$

**equivalence for positive $X$:**
- $X$ is a geometric Lévy process.
- $X = e^L$ for some Lévy process $L$.
- $X = \mathcal{E}(\tilde{L})$ for some Lévy process $\tilde{L}$.

**moreover:**
- can relate triplets of $L$ and $\tilde{L}$
- multivariate analogue exists
Semimartingale characteristics as local Lévy-Khintchine triplets

- **Local dynamics of functions and processes**

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<th>Stochastic Processes</th>
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<td>Linear function $X(t) = bt$ [slope b]</td>
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<td>Function $X(t)$ [local slope $b(t) = \frac{dX(t)}{dt}$]</td>
<td>Semimartingale $X(t)$ [local triplet $(b^h, c, K)(\omega, t)$]</td>
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- **Goal:** Definition of local triplet $(b^h, c, K)(\omega, t)$

- **Interpretation:**
  - Semimartingale $X$ resembles locally around $t$
  - A Lévy process with triplet $(b^h, c, K)(\omega, t)$.  

```
Lemma: \( X \) Lévy process with triplet \((b^h, c, K)\) \(\Rightarrow\)

\[
M(t) := e^{iu^\top X(t)} - \int_0^t e^{iu^\top X(s^-)}\psi(u)\,ds
\]
is local martingale for any \( u \in \mathbb{R}^d \), where \( \psi(u) \) characteristic exponent of \((b^h, c, K)\).

**Definition (differential characteristics)**

A predictable triplet \((b^h, c, K)(\omega, t)\) is called **differential characteristics** or **local triplet** of semimartingale \( X \) if

\[
M(t) := e^{iu^\top X(t)} - \int_0^t e^{iu^\top X(s^-)}\psi(s, u)\,ds
\]
is a local martingale for any \( u \),

where \( \psi(t, u) := iu^\top b^h(t) - \frac{1}{2} u^\top c(t) u + \int (e^{iu^\top x} - 1 - iu^\top h(x))K(t, dx) \) denotes the characteristic exponent of \((b^h, c, K)(\omega, t)\).

**dependence on truncation function:**

\[
b^{\tilde{h}}(t) = b^h(t) + \int (\tilde{h}(x) - h(x))K(t, dx)
\]
Semimartingale characteristics

Alternative characterization

- $X$ semimartingale with local triplet $(b^h, c, K)$. Define
  - $X^h := X - (x - h(x)) \ast \mu^X$
  - $A^{X^h}$ compensator of $X^h$
    (i.e. $X^h - A^{X^h}$ local martingale)
  - $\nu^X$ compensator of measure of jumps of $X$

Then

\[
A^{X^h}(t) = \int_0^t b^h(s) ds,
\]
\[
\langle X_i^c, X_j^c \rangle = \int_0^t c_{ij}(s) ds,
\]
\[
\nu^X([0, t] \times dx) = \int_0^t K(s, dx) ds.
\]
Semimartingale characteristics
Rules

Lemma (Lévy processes)

\( X \) is a Lévy process \( \iff \) local triplet of \( X \) does not depend on \((\omega, t)\).

Lemma (stochastic integrals)

Let \((b^h, c, K)\) be the local triplet of \( X \). Then \( \varphi \cdot X \) has local triplet

\[
\begin{align*}
\tilde{b}^h(t) &= \varphi(t)^\top b^h(t) + \int \left( \tilde{h}(\varphi(t)x) - \varphi(t)h(x) \right) K(t, dx), \\
\tilde{c}(t) &= \varphi(t)c(t)\varphi(t)^\top, \\
\tilde{K}(t, B) &= \int 1_B(\varphi(t)x)K(t, dx).
\end{align*}
\]
Lemma ($C^2$-functions)

Let $(b^h, c, K)$ be the local triplet of $X$ and $f : \mathbb{R}^d \to \mathbb{R}^n$ of class $C^2$. Then $f(X)$ has local triplet

\[
\tilde{b}^h_i(t) = Df_i(X(t-))^\top b^h(t) + \frac{1}{2} \sum_{k,l=1}^{d} D_{kl} f_i(X(t-)) c_{kl}(t) \\
+ \int \left( \tilde{h}_i \left( f(X(t-)) + x \right) - f(X(t-)) \right) - Df_i(X(t-))^\top h(x) \right) F(t, dx),
\]

\[
\tilde{c}_{ij}(t) = \sum_{k,l=1}^{d} D_{k} f_i(X(t-)) c_{kl}(t) D_{l} f_j(X(t-)),
\]

\[
\tilde{K}_t(B) = \int 1_B(f(X(t-)) + x) - f(X(t-)) \right) K_t(dx).
\]
Lemma (change of measure)

Let \((b^h, c, K)\) be the local triplet of \(X\). Suppose that \(Q \sim P\) has density process \(E(\varphi \cdot X^c + \psi \ast (\mu^X - \nu^X))\). Then \(X\) has \(Q\)-local triplet

\[
\begin{align*}
\tilde{b}^h(t) &= b^h(t) + \varphi(t)^\top c(t) + \int h(x)\psi(t, x)K(t, dx), \\
\tilde{c}(t) &= c(t), \\
\tilde{K}(t)(B) &= \int 1_B(x)(1 + \psi(t, x))K(t)(dx).
\end{align*}
\]

Lemma (predictable covariation)

Let \((b^h, c, K)\) be the local triplet of \(X\). Then

\[
d\langle X_i, X_j \rangle(t) = (c_{ij}(t) + \int x_i x_j K(t, dx)) \, dt.
\]
Outline

1. Mathematical finance
2. Stochastic calculus
3. Applications to finance
Option pricing in geometric Lévy model
Integral transform method

- **Model:**
  - $S_0(t) = e^{rt}, \ S_1(t) = S_1(0)e^{L(t)}$ with Lévy process $L$
  - $\psi$ characteristic exponent of $L$ under pricing measure $Q$
  - $\hat{S}_1$ $Q$-martingale (i.e. $\psi(-i) = r$)
  - recall that we need $\hat{V}_z(t)$

- **Solution:**
  $$\hat{V}_z(t) = \hat{S}_1(t)^{z} \exp((T-t)(\psi(-iz) - rz))$$

- **Price of European call at $t$:**
  $$S_1(t) \int_{R-i\infty}^{R+i\infty} \left( \frac{K}{S_1(t)} \right)^{1-z} \frac{e^{(T-t)(\psi(-iz) - r)}}{2\pi iz(z-1)} \, dz$$
Utility maximization in geometric Lévy model
Power and logarithmic utility

Model:
- $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with Lévy process $L$
- initial endowment $\nu$
- utility function $u(x) = \frac{x^{1-p}}{1-p}$ or $u(x) = \log(x)$
- maximize $E(u(V_\varphi(T)))$ over admissible $\varphi$

Solution:
- $\varphi^*(t) = \gamma \frac{V_{\varphi^*}(t-)}{S_1(t-)}$ with $\gamma \in \mathbb{R}$
- $\hat{V}_{\varphi^*}(t) = \nu S_1(\gamma(\hat{S}_1)^{-1} \cdot \hat{S}_1)$
- Density process of EMM $Q$ from dual characterization is $Z(t) = e^{-\alpha(T-t)} \frac{u'(\hat{V}_{\varphi^*}(t))}{E(u'(\hat{V}_{\varphi^*}(t)))}$ with $\alpha \in \mathbb{R}$.
- $\gamma$ is determined as root from an equation.
- $\alpha$ is known in terms of $\gamma$.

Properties of the solution:
- Fixed fraction of current wealth is invested in the stock.
- Value of the optimal strategy is geometric Lévy process.
- Density process of $Q$ is geometric Lévy process.
- $S_1$ is geometric Lévy process under $Q$ as well.
Utility maximization in geometric Lévy model

Exponential utility

- **Model:**
  - \( S_0(t) = e^{rt}, \ S_1(t) = S_1(0)e^{L(t)} \) with Lévy process \( L \)
  - initial endowment \( \nu \)
  - utility function \( u(x) = 1 - e^{-px} \)
  - maximize \( E(u(\hat{V}_\varphi(T))) \) over admissible \( \varphi \)

- **Solution:**
  - \( \varphi^*(t) = \frac{\gamma}{\hat{S}_1(t-)} \) with \( \gamma \in \mathbb{R} \)
  - \( \hat{V}_{\varphi^*}(t) = \nu + \gamma(\hat{S}_1)^{-1} \cdot \hat{S}_1 \)
  - Density process of EMM \( Q \) from dual characterization is
    \[ Z(t) = e^{-\alpha(T-t)} \frac{u'(\hat{V}_{\varphi^*}(t))}{E(u'(\hat{V}_{\varphi^*}(t)))} \] with \( \alpha \in \mathbb{R} \).
  - \( \gamma \) is determined as root from an equation.
  - \( \alpha \) is known in terms of \( \gamma \).

- **Properties of the solution:**
  - Fixed discounted amount of money is invested in the stock.
  - Value of the optimal strategy is Lévy process + \( \nu \).
  - Density process of \( Q \) is geometric Lévy process.
  - \( S_1 \) is geometric Lévy process under \( Q \) as well.
Quadratic hedging in geometric Lévy model
Martingale case

Model:
- \( S_0(t) = e^{rt} \), \( S_1(t) = S_1(0)e^{L(t)} \) with Lévy process \( L \)
- \( \psi \) characteristic exponent of \( L \)
- \( \hat{S}_1 \) martingale (i.e. \( \psi(-i) = r \))
- recall that we need \( \hat{V}_z(t) \), \( \varphi_z(t) \), \( \varepsilon_{yz}^2 \)

Solution:
- \( \hat{V}_z(t) = \hat{S}_1(t)^z \exp((T-t)(\psi(-iz) - rz)) \)
- \( \varphi_z(t) = \frac{\hat{V}_z(t-)}{\hat{S}_1(t-)} \frac{\psi(-i(z+1)) - \psi(-iz) - r}{\psi(-2i) - 2r} \)
- \[
\varepsilon_{yz}^2 = S_1(0)^y+z(e^{\kappa(y+z)T} - e^{(\kappa(y) + \kappa(z))T})
\times \left(1 - \frac{\kappa(y+1) - \kappa(y))(\kappa(z+1) - \kappa(z))}{\kappa(2)(\kappa(y+z) - \kappa(y) - \kappa(z))}\right)
\]
- with \( \kappa(z) := \psi(-iz) - rz \)
- can now apply results to approximate indifference pricing