Lévy Processes in Finance

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Outline

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Price processes and trading strategies

- securities price process $S = (S_0(t), \dots, S_d(t))_{t\in[0,T]}$
  - $\mathbb{R}^{d+1}$-valued semimartingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$
  - $S_i(t)$ is price of security i at time t
  - Example:
    Bond: $S_0(t) = e^{rt}$
    Stock: $S_1(t) = S_1(0)e^{L(t)}$ with Lévy process L
- trading strategy or portfolio $\varphi = (\varphi_0(t), \dots, \varphi_d(t))_{t\in[0,T]}$
  - $\mathbb{R}^{d+1}$-valued predictable process
  - $\varphi_i(t)$ is number of shares of security i at time t
  - value process $V_\varphi(t) := \varphi(t)^\top S(t) = \sum_{i=0}^d \varphi_i(t)^\top S_i(t)$
  - φ is self financing if S-integrable and $V_\varphi = V_\varphi(0) + \varphi \bullet S$

Discounting

for simplified bookkeeping

- Idea: express prices in multiples of numeraire security S_0.
  - Assume $S_0, (S_0)_- > 0$.
- discounted price process $\hat{S} := \frac{1}{S_0} S = (1, \frac{S_1}{S_0}, \dots, \frac{S_d}{S_0})$
- discounted value process $\hat{V}_\varphi := \frac{V_\varphi}{S_0} = \varphi^\top \hat{S}$
- Example: for $S_0(t) = e^{rt}$ we have
  $\hat{S}(t) = e^{-rt} S(t)$ and $\hat{V}_\varphi(t) = e^{-rt} V_\varphi(t)$.
- Lemma:
  φ self financing $\Leftrightarrow$ φ $\hat{S}$-integrable and $\hat{V}_\varphi = \hat{V}_\varphi(0) + \varphi \bullet \hat{S}$
- Lemma: For $(\hat{S}_1, \dots, \hat{S}_d)$-integrable $(\varphi_1, \dots, \varphi_d)$ and $v \in \mathbb{R}$ ex. unique φ_0 such that $\varphi = (\varphi_0, \dots, \varphi_d)$ self financing with $V_\varphi(0) = v$.
- Consequence: do not worry about self financability and φ_0.

Arbitrage and consequences

An admissible self financing strategy φ is an arbitrage if $V_\varphi(0) = 0$, $V_\varphi(T) \geq 0$, $P(V_\varphi(T) > 0) > 0$.

- Without admissibility constraint Black-Scholes model is not arbitrage free.
- We take careless engineering point of view here. Precise statements are difficult.

Rule (law of one price)

Let φ, ψ be admissible strategies in an arbitrage-free market.
If $V_\varphi(T) = V_\psi(T)$, then $V_\varphi = V_\psi$.

Rule (fundamental theorem of asset pricing, FTAP)

The market is arbitrage free $\iff$
there exists some equivalent martingale measure (EMM) Q
(i.e. $Q \sim P$ is probability measure and $\hat{S}$ is Q-martingale).

Derivatives

- European option: $\mathscr{F}_T$-measurable random variable X
  - X stands for payoff at time T.
  - Examples:
    European call $X = (S_1(T) - K)^+$ or put $X = (K - S_1(T))^+$
    Lookback call $X = (\sup_{t \in [0,T]} S_1(t) - K)^+$
  - Discounted payoff is $\hat{X} := \frac{X}{S_0(T)}$.
- Futures and American options are more involved.
- What is a reasonable price for the option at time $t < T$?
- We distinguish liquidly traded and over-the-counter options.

Option pricing

for liquidly traded derivatives

- Consider underlyings S_0, S_1 (given processes) and derivative price process S_2 with $S_2(T) = X$ (yet to be determined).
- The FTAP implies:
  no arbitrage $\Longrightarrow$ $\hat{S}_2(t) = E_Q(\hat{X}|\mathscr{F}_t)$ for some EMM Q

Rule

S_2 does not depend on the chosen EMM $Q \iff$
X is attainable (i.e. $X = V_\varphi(T)$ for some admissible strategy φ).

Corollary (second fundamental theorem of asset pricing)

An arbitrage free market is complete (i.e. any option is attainable)
$\iff$ *there exists only one EMM.*

- Most models involving non-Gaussian Lévy processes are incomplete.

Problem of incompleteness

for liquidly traded options

- Problem: range of possible prices typically too large
  - e.g. trivial arbitrage bounds for European call in geometric Lévy model for standard Lévy processes (VG, NIG, . . .)
  - Away from Black-Scholes case arbitrage theory does not yield useful information.
- Approach 1:
  consider some derivatives as underlyings, e.g. vanilla options
  - get fewer EMM's, hence tighter price bounds for other derivatives
  - Problem: how to get reasonable and arbitrage free model for new underlyings?
  - Approach seems rather cumbersome.
- Approach 2 (martingale modelling):
  model prices directly under market's pricing measure
  - used in practice

Martingale modelling
for liquidly traded options

- Choose parametric class of models for underlyings S_0, S_1 under market's EMM Q.
  - e.g. $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with variance-gamma Lévy process L
- Consider martingale constraint, i.e. $\hat{S}^1$ Q-martingale.
  - here: restrict variance-gamma parameters such that $E(e^{L(1)-r}) = 1$
- Determine remaining parameters by calibration to observed prices of liquid options at $t = 0$.
  - choose parameters such that

    $$\text{discounted observed price} = E_Q(\text{discounted payoff})$$

    for all observed vanilla options
  - use e.g. least squares procedure if more options than parameters
- Use model e.g. for pricing of new non-liquidly traded derivatives.

Martingale modelling

for liquidly traded options (ct'd)

- Question: is the parametric class of models appropriate?
  - Statistical tests are useless because they concern real world probabilities P.
- Answer: inconsistencies may indicate inappropriate class or model.
  - e.g. no choice of parameters leads to reasonable fit of observed option prices
  - or recalibration leads to heavily time-varying models
- However: Entirely different classes may produce reasonable fit, but differ substantially for further e.g. path-dependent options (model risk).

Martingale modelling
for liquidly traded options (ct'd)

Two different situations:

- only interested in valuation of new options
  - modelling under pricing measure Q suffices, physical measure P not needed
- real-world probabilities needed as well (e.g. for value at risk assessments etc.)
  - model dynamics of S_0, S_1 under both P and Q
  - choose typically same parametric class under both P and Q
  - e.g. $S_1(T) = S_1(0)e^{L(t)}$, L VG Lévy process under P and Q
  - determine P-parameters by estimation using stock price data, Q-parameters by calibration using option prices
  - OBS! condition $Q \sim P$ leads to constraints on possible parameters

Individual pricing
of over-the-counter derivatives

- Situation:
  - underlyings S_0, S_1 liquidly traded, but not the option
  - client wants to buy option X from bank at time $t = 0$
  - reasonable prices π?
- $\pi \leq \pi_{\text{high}} := \inf\{\pi \in \mathbb{R} : \exists \text{ adm. } \varphi \text{ with } V_0(\varphi) = \pi \text{ and } V_\varphi(T) \geq X\}$
  - justification: otherwise client can do better by investing in such superhedge φ
- $\pi \leq \pi_{\text{low}} := \sup\{\pi \in \mathbb{R} : \exists \text{ adm. } \varphi \text{ with } V_0(\varphi) = \pi \text{ and } V_\varphi(T) \leq X\}$
  - justification: otherwise bank should sell such subhedge φ rather than option
- How to determine $\pi_{\text{high}}, \pi_{\text{low}}$?

Individual pricing
of over-the-counter derivatives (ct'd)

Rule (superhedging)

$\hat{\pi}_{\text{high}} := \frac{\pi_{\text{high}}}{S_0(0)} = \sup\{E_Q(\hat{X}) : Q \text{ EMM for } S_0, S_1\}$

$\hat{\pi}_{\text{low}} := \frac{\pi_{\text{high}}}{S_0(0)} = \inf\{E_Q(\hat{X}) : Q \text{ EMM for } S_0, S_1\}$

- Price bounds coincide with price interval for liquid options.
- hence same problem as before: range of prices too large
- alternatives:
  - pricing based on quadratic hedging
  - utility indifference pricing

Quadratic hedging
and pricing over-the-counter derivatives

- quadratic hedging of option X:
  - minimize expected squared hedging error

$$\varepsilon^2(\varphi) := E((\hat{V}_\varphi(T) - \hat{X})^2)$$

    over all admissible strategies φ
  - optimal strategy $\varphi^\star$, minimal hedging error $\varepsilon^2 := \varepsilon^2(\varphi^\star)$
  - $\hat{V}_{\varphi^\star}(T)$ is orthogonal projection in L^2 of $\hat{X}$ on $\{\hat{V}_\varphi(T) : \varphi \text{ admissible strategy}\}$.
- application to over-the-counter trade
  - charge e.g. $\pi = V_{\varphi^\star}(0) + \lambda\varepsilon^2$ as option's premium
  - use $V_{\varphi^\star}(0)$ to buy hedging strategy $\varphi^\star$.
  - take multiple of ε^2 as compensation for unhedgeable risk (parameter λ stands for risk aversion)
- How to determine $\varphi^\star$, $V_{\varphi^\star}(0)$, ε^2?

Quadratic hedging (ct'd)

Theorem (quadratic hedging)

Assumption: $\hat{S}$ *is martingale (general case is more involved).*
Then

$$V_{\varphi^\star}(0) = E(\hat{X})$$

$$\varphi^\star(t) = \frac{d\langle \hat{V}, \hat{S}_1\rangle(t)}{d\langle \hat{S}_1, \hat{S}_1\rangle(t)}$$

$$\varepsilon^2 = E\Big(\langle \hat{V}, \hat{V}\rangle(T) - (\varphi^\star)^2 \bullet \langle \hat{S}_1, \hat{S}_1\rangle(T)\Big)$$

with $\hat{V}(t) := E(\hat{X}|\mathscr{F}_t)$.

- How to determine angle brackets $\langle \hat{V}, \hat{S}_1\rangle$ etc.?

Utility indifference pricing
of over-the-counter derivatives

- Problem of approach via quadratic hedging:
  gains are penalized, hence economically questionable
- Alternative:
  increasing utility functions and indifference pricing
- Idea:
  sell option only if this is profitable for the bank (i.e. utility rises)

Expected utility maximization

- fix increasing, strictly concave utility function u.
- Examples:
  - $u(x) = \frac{x^{1-p}}{1-p}$ for $p \in (0, \infty) \setminus \{1\}$
  - $u(x) = \log(x)$ (behaves like $p = 1$ above)
  - $u(x) = 1 - e^{-px}$ for $p > 0$
- maximize expected utility $E(u(\hat{V}_\varphi(T)))$ over all admissible strategies φ with fixed initial value $V_\varphi(0) = v$
- solution approaches:
  - dynamic programming
  - martingale methods (here)

Rule (dual characterization)

φ admissible strategy such that $\frac{dQ}{dP} = \frac{u'(\hat{V}_\varphi(T))}{E(u'(\hat{V}_\varphi(T))}$ density of EMM
$\Longrightarrow \varphi$ is optimal.

Utility indifference pricing
of over-the-counter derivatives

- Question:
  - underlyings S_0, S_1 liquidly traded, but not the option
  - client wants to buy option X from bank at time $t = 0$
  - reasonable prices π for the bank?
- Answer:
  - utility of bank without option trade:
    $U_0 := \sup\{E(u(\hat{V}_\varphi(T))) : \varphi \text{ adm. with } V_0(\varphi) = v\}$
  - utility of bank with option trade:
    $U_X(\pi) := \sup\{E(u(\hat{V}_\varphi(T) + \hat{\pi} - \hat{X})) : \varphi \text{ adm. with } V_0(\varphi) = v\}$
  - require $U_X(\pi) > U_0$ (trade raises bank's utility)
  - utility indifference price: limiting price with $U_X(\pi) = U_0$
  - reasonable prices are those above indifference price
- Problem: utility indifference prices are hard to compute
- Alternative:
  compute approximate solution for small number of options

Approximate utility indifference pricing

of over-the-counter derivatives

- Notation:
  - $\pi(n)$: utility indifference price per unit of X for n options
  - $\varphi(n)$: optimal strategy for n sold options, i.e. for $U_{nX}(n\pi(n))$
- Goal: approximate $\pi(n)$, $\varphi(n)$ for small n
- Expansion:

$$\begin{aligned} \hat{\pi}(n) &= \hat{\pi}(0) + n\delta + o(n) \\ \varphi(n) &= \varphi^\star + n\eta + o(n) \end{aligned}$$

- Interpretation:
  - $\pi(0)$: limiting price for very small number of options
  - δ: risk premium per option that is to be sold
  - $\varphi^\star$: optimal strategy for pure investment problem without options
  - η: hedging strategy per option
- How to determine $\pi(0)$, δ, η?

Approximate utility indifference pricing
for exponential utility

Rule

Consider $u(x) = 1 - e^{-px}$.
Let $\varphi^\star$, Q be optimal pair from dual characterisation.
Define $\hat{V}(t) := E_Q(\hat{X}|\mathscr{F}_t)$. Then

$$\begin{aligned}
\hat{\pi}(0) &= \hat{V}(0) = E_Q(\hat{X}), \\
\delta &= \tfrac{p}{2} E_Q\Big(\langle \hat{V}, \hat{V}\rangle^Q(T) - \eta^2 \bullet \langle \hat{S}_1, \hat{S}_1\rangle^Q(T)\Big), \\
\eta(t) &= \tfrac{d\langle \hat{V}, \hat{S}_1\rangle^Q(t)}{d\langle \hat{S}_1, \hat{S}_1\rangle^Q(t)}.
\end{aligned}$$

- Q is the minimal entropy martingale measure.
- δ, η solve quadratic hedging problem for X under Q.
- Slightly more involved results hold for power/logarithmic utility with additional numeraire change.

Integral transform methods

for pricing and hedging in general

- Consider e.g. $X = (S_1(T) - K)^+$.
  We have encountered objects like
  - $\hat{V}(t) = E(\hat{X}|\mathscr{F}_t)$,
  - $\varphi(t) = \frac{d\langle \hat{V}, \hat{S}_1\rangle(t)}{d\langle \hat{S}_1, \hat{S}_1\rangle(t)}$,
  - $\varepsilon^2 = E\Big(\langle \hat{V}, \hat{V}\rangle(T) - \varphi^2 \bullet \langle \hat{S}_1, \hat{S}_1\rangle(T)\Big)$.
- How to compute them?
  - typically no closed-form solution available
  - often available: closed-form solutions for $\hat{X} = \hat{S}_1(T)^z$, $z \in \mathbb{C}$.
  - e.g. $E(\hat{X}) = E(\hat{S}_1(T)^z) = e^{-zrT}E(exp(zL(T)))$,
    i.e. extended characteristic function of $L(T)$ for $S_1 = S_1 e^L$

Integral transform methods
for pricing and hedging (ct'd)

- Idea: use integral representation

$$\hat{X} = \int_{R-i\infty}^{R+i\infty} \hat{S}_1(T)^z \ell(z) dz$$

  with explicitly given $\ell(z)$, R.
- linearity yields
  - $\hat{V}(t) = \int_{R-i\infty}^{R+i\infty} \hat{V}_z(t)\ell(z)dz$ with $\hat{V}_z(t) := E(\hat{S}_1(T)^z | \mathscr{F}_t)$,
  - $\varphi(t) = \int_{R-i\infty}^{R+i\infty} \varphi_z(t)\ell(z)dz$ with $\varphi_z(t) := \frac{d\langle \hat{V}_z, \hat{S}_1\rangle(t)}{d\langle \hat{S}_1, \hat{S}_1\rangle(t)}$,
  - $\varepsilon^2 = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \varepsilon^2_{yz}\ell(y)\ell(z)dydz$ with $\varepsilon^2_{yz} = E\Big(\langle \hat{V}_y, \hat{V}_z\rangle(T) - \varphi_y\varphi_z \bullet \langle \hat{S}_1, \hat{S}_1\rangle(T)\Big)$.
- determine $\ell(z)$ via inverse Laplace transform
  - for $\hat{X} = (\hat{S}_1(T) - K)^+$ we have $\ell(z) = \frac{K^{1-z}}{z(z-1)}$, $R > 1$
  - for $\hat{X} = (K - \hat{S}_1(T))^+$ we have $\ell(z) = \frac{K^{1-z}}{z(z-1)}$, $R < 0$

Outline

Basic concepts

of the general theory of stochastic processes

filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$
process $X = (X(t))_{t\in\mathbb{R}_+}$
càdlàg: right-continuous with left-hand limits $X_- = (X(t-))_{t\in\mathbb{R}_+}$
of finite variation: difference of two increasing processes
adaptedness: $X(t)$ $\mathscr{F}_t$-measurable for all t
predictability: slightly stronger than "$X(t)$ $\mathscr{F}_{t-}$-measurable for all t"
stopping time: random time τ with $\{\tau \leq t\} \in \mathscr{F}_t$
stopped process $X^\tau = (X(\tau \wedge t))_{t\in\mathbb{R}_+}$ with stopping time τ
martingale: $E(X(t)|\mathscr{F}_s) = X(s)$ for $s \leq t$
submartingale: $E(X(t)|\mathscr{F}_s) \geq X(s)$ for $s \leq t$
supermartingale: $E(X(t)|\mathscr{F}_s) \leq X(s)$ for $s \leq t$
density process: $Z(t) = E(\frac{dQ}{dP}|\mathscr{F}_t) = \frac{dQ}{dP}|_{\mathscr{F}_t}$
localized classes $\mathscr{C}_{\text{loc}}$ (local martingales etc.): $X^{\tau_n} \in \mathscr{C}$ for all n
special semimartingale: $X = X(0) + M^X + A^X$ with predictable A^X
compensator: finite variation process A^X in above decomposition
semimartingale: $X = X(0) + M + A$ with adapted A

Lévy processes
Definition

- processes of constant growth in stochastic sense
- random counterpart of linear functions
- continuous-time counterpart of random walks

Definition (Lévy process)

adapted càdlàg process X with $X(0) = 0$,
$X(t) - X(s)$ independent of $\mathscr{F}_s$ (independent increments),
$\mathscr{L}(X(t) - X(s))$ depends only on $t - s$ (stationary increments)

- Examples:
  - linear function $X(t) = bt$
  - Brownian motion with drift $X(t) = \mu t + \sigma W(t)$
  - Poisson process
  - compound Poisson process $X(t) = \sum_{i=1}^{N(t)} Y_i$

Lévy processes
Characterization

Theorem (Lévy-Khintchine formula)

Characteristic function is $\varphi_{X(t)}(u) = E(e^{iu^\top X(t)}) = \exp(t\psi(u))$ *with* *characteristic exponent*

$$\psi(u) = iu^\top b^h - \frac{1}{2}u^\top cu + \int (e^{iu^\top x} - 1 - iu^\top h(x))K(dx)$$

and *Lévy-Khintchine triplet* (b^h, c, K)*.*
h *denotes some* *truncation function* *as e.g.* $h(x) = x1\{|x| \leq 1\}$*.*

- Interpretation:
  - Drift coefficient b stands for linear function.
  - Diffusion coefficient c stands for Brownian motion.
  - Lévy measure K stands for jumps.
- $b^{\tilde{h}} = b^h + \int(\tilde{h}(x) - h(x))K(dx)$

- p-th moments exist if $K|_{\{|x|>1\}}$ has p-th moments.
- $E(X(t)) = b^h t = -i\psi'(0)t$ for $h(x) = x$ (i.e. $h = \mathrm{id}$)
- $\mathrm{Var}(X(t)) = (c + \int x^2 K(dx))t = -\psi''(0)t$ for dimension $d = 1$
- p-th exponential moments exist if $K|_{\{|x|>1\}}$ has p-th exponential moments.
- $E(e^{p^\top X(t)}) = \exp(t(p^\top b^h + \frac{1}{2}p^\top cp + \int(e^{p^\top x} - 1 - p^\top h(x))K(dx)))$, which equals $\exp(t\psi(-ip))$ if ψ has analytic extension.
- X is (local) martingale iff $E(X(1)) = 0$.
- e^X is (local) martingale iff $E(e^{X(1)}) = 1$.

Lévy processes
Path properties

- X has differentiable paths $\Leftrightarrow c = 0$, $K = 0$
  $\Leftrightarrow X$ is linear function.
- X has continuous paths $\Leftrightarrow K = 0$
  $\Leftrightarrow X$ is Brownian motion with drift.
- X is piecewise constant $\Leftrightarrow c = 0$, $K(\mathbb{R}) < \infty$, $b^0 = 0$
  $\Leftrightarrow X$ is compound Poisson process.
- X has finitely many jumps on $[0, t] \Leftrightarrow K(\mathbb{R}) < \infty$
- X is of finite variation $\Leftrightarrow c = 0$, $\int_{|x|\leq 1} |x| K(dx) < \infty$

Lévy processes
Pathwise representation

Theorem (Lévy-Itô decomposition)

Let X have finite expectation and triplet (b^{id}, c, K).

$$\begin{aligned} X(t) &= b^{id}t + \sqrt{c}W(t) \\ &\quad + \lim_{\varepsilon\to 0}\left(\sum_{s\le t}\Delta X(s)1_{\{|\Delta X(s)|>\varepsilon\}} - \int x1_{\{|\Delta X(s)|>\varepsilon\}}K(dx)s\right) \\ &= b^{id}t + \sqrt{c}W(t) + \int_{[0,t]\times\mathbb{R}} x(\mu^X - \nu^X)(d(s,x)) \end{aligned}$$

with random measure of jumps of X

$$\mu^X([0,t]\times B) := \#\{(s,x)\in[0,t]\times B : \Delta X(s) = x \neq 0\}$$

and compensated random measure of jumps of X

$$\nu^X([0,t]\times B) := K(B)t.$$

Lévy processes in finance
for modelling of stock prices etc.

- Brownian motions with drift
  - most important and the only continuous Lévy processes
  - density, characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 2 parameters
  - all exponential moments exist
- stable Lévy motions
  - the only self-similar Lévy processes
  - either Brownian motion or pure jump
  - characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 4 parameters
  - infinite variance (except for Brownian motion)

Lévy processes in finance
for modelling of stock prices etc. (ct'd)

- Merton model
  - Brownian motion + drift + Gaussian jumps
  - finitely many jumps on $[0, t]$
  - characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 5 parameters
  - all exponential moments exist
- Kou model
  - Brownian motion + drift + exponential jumps
  - finitely many jumps on $[0, t]$
  - characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 6 parameters
  - some exponential moments exist

Lévy processes in finance

for modelling of stock prices etc. (ct'd)

- variance gamma (VG) processes
  - pure jump Lévy process
  - infinitely many jumps on $[0, t]$ but of finite variation
  - density, characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 4 parameters
  - some exponential moments exist
- normal inverse Gaussian (NIG) processes
  - pure jump Lévy process
  - of infinite variation
  - density, characteristic exponent, Lévy-Khintchine triplet known in closed form
  - 4 parameters
  - some exponential moments exist
- etc.

Quadratic variation
and friends

- quadratic variation of semimartingale X:
$[X,X](t) = \lim_{\sup|t_i - t_{i-1}|\to 0} \sum_{i\geq 1} (X(t_i \wedge t) - X(t_{i-1} \wedge t))^2$
  - covariation of X and Y:
$[X,Y] = \frac{1}{4}([X+Y, X+Y] - [X-Y, X-Y])$
- Example: X Lévy process with triplet (b^h, c, K)
  - $[X,X]$ is Lévy process.
  - Triplet of $[X,X]$ is $(c, 0, \tilde{K})$ relative to $\tilde{h} = 0$,
where $\tilde{K}(B) = \int 1_B(x^2)K(dx)$.
- predictable quadratic variation of X:
$\langle X, X\rangle = A^{[X,X]}$ (compensator of $[X,X]$)
  - predictable covariation of X,Y: $\langle X, Y\rangle = A^{[X,Y]}$
- Example: X Lévy process with finite variance
  - $\langle X, X\rangle$ is linear function.
  - $\langle X, X\rangle(t) = \mathrm{Var}(X(t))$

Continuous martingale part
of a semimartingale

- Local martingales X, Y are orthogonal if XY local martingale.
- Local martingale X is purely discontinuous if orthogonal to all continuous local martingales.
- X local martingale $\Rightarrow$
  there is unique decomposition $X = X(0) + X^c + X^d$ with
  X^c continuous, X^d purely discontinuous, $X^c(0) = 0 = X^d(0)$.
- $X = X(0) + M + A$ semimartingale $\Rightarrow$
  continuous martingale part $X^c := M^c$ does not depend on decomposition of X
- $[X, Y](t) = \langle X^c, Y^c \rangle(t) + \sum_{s \leq t} \Delta X_s \Delta Y_s$
- X Lévy process with Lévy-Itô decomposition
  $X(t) = b^{\mathrm{id}} t + \sqrt{c} W(t) + \int_{[0,t] \times \mathbb{R}} x(\mu^X - \nu^X)(d(s, x))$
  $\Rightarrow X^c = \sqrt{c} W$

Stochastic integral
Heuristics

- Informal "definition":
  - φ predictable proces
  - X semimartingale
  - $\varphi \bullet X(t) :\approx \lim_{\sup|t_i - t_{i-1}| \to 0} \sum_{i \geq 1} \varphi(t_{i-1})(X(t_i \wedge t) - X(t_{i-1} \wedge t))$
  - alternative notation: $\int_0^t \varphi(s) dX(s) = \varphi \bullet X(t)$
  - $\varphi \bullet X(t) := \sum_{i=1}^d \varphi_i(t) \bullet X_i(t)$ for $\mathbb{R}^d$-valued φ, X
- Interpretation:
  - φ trading strategy
  - X stock price process
  - $\varphi \bullet X(t) = \int_0^t \varphi(s) dX(s)$ gains due to price changes

Stochastic integral
Properties

- $\varphi \bullet X$ is linear in φ and X
- $\varphi \bullet (\psi \bullet X) = (\varphi\psi) \bullet X$
- $[\varphi \bullet X, Y] = \varphi \bullet [X, Y]$
- $\langle \varphi \bullet X, Y \rangle = \varphi \bullet \langle X, Y \rangle$
- $\Delta(\varphi \bullet X) = \varphi \bullet \Delta X$
- $(\varphi \bullet X)^\tau = \varphi \bullet X^\tau$
- $(\varphi \bullet X)^c = \varphi \bullet X^c$
- X sigma-martingale $\Rightarrow \varphi \bullet X$ sigma-martingale
  (slightly more general than local martingale)
- integration by parts:
  $XY = X(0)Y(0) + X_- \bullet Y + Y_- \bullet X + [X, Y]$

Stochastic integral
Properties (ct'd)

Theorem (Itô's formula)

Let X semimartingale in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ of class C^2.

$$
\begin{aligned}
f(X(t)) &= f(X(0)) + Df(X_-) \bullet X(t) + \frac{1}{2}\sum_{i,j=1}^{d} D_{ij}f(X_-) \bullet \langle X_i^c, X_j^c\rangle(t) \\
&\quad + \sum_{s\le t}\Big(f(X(s)) - f(X(s-)) - Df(X(s-))^\top \Delta X(s)\Big)
\end{aligned}
$$

For $d = 1$:

$$
\begin{aligned}
f(X(t)) &= f(X(0)) + f'(X_-) \bullet X(t) + \frac{1}{2} f''(X_-) \bullet \langle X^c, X^c\rangle(t) \\
&\quad + \sum_{s\le t}\big(f(X(s)) - f(X(s-)) - f'(X(s-))\Delta X(s)\big).
\end{aligned}
$$

Random measures

and their integration

- recall random measure of jumps of semimartingale X:
  $\mu^X([0,t] \times B) := \#\{(s,x) \in [0,t] \times B : \Delta X(s) = x \neq 0\}$
- random measure here means: measure on $\mathbb{R}_+ \times \mathbb{R}^d$ for fixed ω
- integral process $\xi * \mu$
  - $\xi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ in some sense predictable function
  - μ random measure
  - $\xi * \mu(t) := \int_{[0,t] \times \mathbb{R}^d} \xi(s,x)\mu(d(s,x))$ pathwise integral
  - For $\mu = \mu^X$ we have $\xi * \mu^X(t) = \sum_{s \leq t} \xi(s, \Delta X(s)) 1_{\{\Delta X(s) \neq 0\}}$.
- compensator ν^X of μ^X:
  the in some sense predictable random measure such that
  $\xi * \mu - \xi * \nu$ local martingale for most predictable ξ
- stochastic integral $\xi * (\mu^X - \nu^X)$:
  extension of $\xi * \mu^X - \xi * \nu^X$ to some ξ where this is not defined

- X Lévy process with triplet $(b^h, c, K) \Rightarrow \nu^X(d(t,x)) = K(dx)dt$
- $\xi * \mu$ and $\xi * (\mu^X - \nu^X)$ are linear in ξ.
- $\varphi \bullet (\xi * \mu) = (\varphi\psi) * \mu$
- $\varphi \bullet (\xi * (\mu^X - \nu^X)) = (\varphi\psi) * (\mu^X - \nu^X)$
- $\xi * (\mu^X - \nu^X)$ is local martingale,
  typically (but not always) with jumps
  $\Delta(\xi * (\mu^X - \nu^X))(t) = \xi(t, \Delta X(t)) 1_{\{\Delta X(t) \neq 0\}}$
- $\langle \eta * (\mu^X - \nu^X), \xi * (\mu^X - \nu^X) \rangle = (\eta\xi) * \nu^X$
- Itô's formula:

$$\begin{aligned} f(X(t)) &= f(X(0)) + f'(X_-) \bullet X(t) + \tfrac{1}{2} f''(X_-) \bullet \langle X^c, X^c \rangle(t) \\ &\quad + \big(f(X_- + x) - f(X_-) - f'(X_-)x\big) * \mu^X(t) \end{aligned}$$

- X special semimartingale:

$$X = X(0) + A^X + X^c + x * (\mu^X - \nu^X)$$

  generalizes Lévy-Itô decomposition

Stochastic exponential

as solution to linear stochastic differential equation

- The solution $\mathscr{E}(X)$ to $Z = 1 + Z_- \bullet X$ is called Doléans exponential or stochastic exponential of X.
- Solution for $\Delta X > -1$:

$$\begin{aligned}\mathscr{E}(X) &= \exp\Big(X(t) - X(0) - \frac{1}{2}\langle X^c, X^c\rangle(t) \\ &\quad + \sum_{s\leq t}\big(\log(1 + \Delta X(s)) - \Delta X(s)\big)\Big)\end{aligned}$$

- Yor's formula:

$$\mathscr{E}(X)\mathscr{E}(Y) = \mathscr{E}(X + Y + [X, Y])$$

Geometric Lévy processes

modelling constant relative growth

- Idea: constant *relative* rather than absolute growth
  - more appropriate e.g. for stock prices
- geometric Lévy process:
  - $\mathbb{R} \setminus \{0\}$-valued semimartingale with $X(0) = 1$
  - $\frac{X(t)}{X(s)}$ independent of $\mathscr{F}_s$
  - $\mathscr{L}(\frac{X(t)}{X(s)})$ depends only on $t - s$
- equivalence for positive X:
  - X is a geometric Lévy process.
  - $X = e^L$ for some Lévy process L.
  - $X = \mathscr{E}(\tilde{L})$ for some Lévy process $\tilde{L}$.
- moreover:
  - can relate triplets of L and $\tilde{L}$
  - multivariate analogue exists

Semimartingale characteristics
as local Lévy-Khintchine triplets

- local dynamics of functions and processes

	deterministic functions	stochastic processes
constant growth	linear function $X(t) = bt$ slope b	Lévy process $X(t)$ triplet (b^h, c, K)
general case	function $X(t)$ local slope $b(t) = \frac{dX(t)}{dt}$	semimartingale $X(t)$ local triplet $(b^h, c, K)(\omega, t)$?

- Goal: definition of local triplet $(b^h, c, K)(\omega, t)$
- Interpretation:
  semimartingale X resembles locally around t
  a Lévy process with triplet $(b^h, c, K)(\omega, t)$.

Semimartingale characteristics
Definition

- Lemma: X Lévy process with triplet $(b^h, c, K) \Rightarrow$ $M(t) := e^{iu^\top X(t)} - \int_0^t e^{iu^\top X(s-)}\psi(u)ds$ is local martingale for any $u \in \mathbb{R}^d$, where $\psi(u)$ characteristic exponent of (b^h, c, K).

Definition (differential characteristics)

A predictable triplet $(b^h, c, K)(\omega, t)$ is called differential characteristics or local triplet of semimartingale X if $M(t) := e^{iu^\top X(t)} - \int_0^t e^{iu^\top X(s-)}\psi(s, u)ds$ is a local martingale for any u, where $\psi(t, u) := iu^\top b^h(t) - \frac{1}{2}u^\top c(t)u + \int(e^{iu^\top x} - 1 - iu^\top h(x))K(t, dx)$ denotes the characteristic exponent of $(b^h, c, K)(\omega, t)$.

- dependence on truncation function: $b^{\tilde{h}}(t) = b^h(t) + \int(\tilde{h}(x) - h(x))K(t, dx)$

Semimartingale characteristics

Alternative characterization

- X semimartingale with local triplet (b^h, c, K). Define
  - $X^h := X - (x - h(x)) * \mu^X$
  - A^{X^h} compensator of X^h
    (i.e. $X^h - A^{X^h}$ local martingale)
  - ν^X compensator of measure of jumps of X
- Then

$$
\begin{aligned}
A^{X^h}(t) &= \int_0^t b^h(s)ds, \\
\langle X_i^c, X_j^c \rangle &= \int_0^t c_{ij}(s)ds, \\
\nu^X([0,t] \times dx) &= \int_0^t K(s, dx)ds.
\end{aligned}
$$

Semimartingale characteristics
Rules

Lemma (Lévy processes)

X is a Lévy process $\Leftrightarrow$ local triplet of X does not depend on (ω, t).

Lemma (stochastic integrals)

Let (b^h, c, K) be the local triplet of X. Then $\varphi \bullet X$ has local triplet

$$\begin{aligned}
\widetilde{b}^{\tilde{h}}(t) &= \varphi(t)^\top b^h(t) + \int \left(\tilde{h}(\varphi(t)x) - \varphi(t)h(x)\right) K(t, dx), \\
\widetilde{c}(t) &= \varphi(t)c(t)\varphi(t)^\top, \\
\widetilde{K}(t, B) &= \int 1_B(\varphi(t)x)K(t, dx).
\end{aligned}$$

Semimartingale characteristics

Rules (ct'd)

Lemma (C^2-functions)

Let (b^h, c, K) *be the local triplet of* X *and* $f : \mathbb{R}^d \to \mathbb{R}^n$ *of class* C^2*. Then* $f(X)$ *has local triplet*

$$\widetilde{b}_i^{\tilde{h}}(t) = Df_i(X(t-))^\top b^h(t) + \frac{1}{2}\sum_{k,l=1}^{d} D_{kl} f_i(X(t-))c_{kl}(t)$$
$$+ \int \left(\tilde{h}_i\Big(f(X(t-)+x) - f(X(t-))\Big) - Df_i(X(t-))^\top h(x)\right) F(t,dx),$$

$$\widetilde{c}_{ij}(t) = \sum_{k,l=1}^{d} D_k f_i(X(t-))c_{kl}(t) D_l f_j(X(t-)),$$

$$\widetilde{K}_t(B) = \int 1_B\big(f(X(t-)+x) - f(X(t-))\big) K_t(dx).$$

Semimartingale characteristics

Rules (ct'd)

Lemma (change of measure)

Let (b^h, c, K) *be the local triplet of* X. *Suppose that* $Q \sim P$ *has density process* $\mathscr{E}(\varphi \bullet X^c + \psi * (\mu^X - \nu^X))$. *Then* X *has* Q*-local triplet*

$$\begin{aligned}
\widetilde{b}^h(t) &= b^h(t) + \varphi(t)^\top c(t) + \int h(x)\psi(t,x)K(t,dx), \\
\widetilde{c}(t) &= c(t), \\
\widetilde{K}(t)(B) &= \int 1_B(x)(1+\psi(t,x))K(t)(dx).
\end{aligned}$$

Lemma (predictable covariation)

Let (b^h, c, K) *be the local triplet of* X. *Then* $d\langle X_i, X_j\rangle(t) = \left(c_{ij}(t) + \int x_i x_j K(t,dx)\right) dt$.

Outline

1. Mathematical finance
2. Stochastic calculus
3. Applications to finance

Option pricing in geometric Lévy model
Integral transform method

- Model:
  - $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with Lévy process L
  - ψ characteristic exponent of L under pricing measure Q
  - $\hat{S}_1$ Q-martingale (i.e. $\psi(-i) = r$)
  - recall that we need $\hat{V}_z(t)$
- Solution:

$$\hat{V}_z(t) = \hat{S}_1(t)^z \exp((T-t)(\psi(-iz) - rz))$$

- Price of European call at t:

$$S_1(t) \int_{R-i\infty}^{R+i\infty} \left(\frac{K}{S_1(t)}\right)^{1-z} \frac{e^{(T-t)(\psi(-iz)-r)}}{2\pi iz(z-1)} dz$$

Utility maximization in geometric Lévy model

Power and logarithmic utility

- Model:
  - $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with Lévy process L
  - initial endowment v
  - utility function $u(x) = \frac{x^{1-p}}{1-p}$ or $u(x) = \log(x)$
  - maximize $E(u(V_\varphi(T)))$ over admissible φ
- Solution:
  - $\varphi^\star(t) = \gamma \frac{V_{\varphi^\star}(t-)}{S_1(t-)}$ with $\gamma \in \mathbb{R}$
  - $\hat{V}_{\varphi^\star}(t) = v\mathscr{E}(\gamma(\hat{S}_1)_-^{-1} \bullet \hat{S}_1)$
  - Density process of EMM Q from dual characterization is $Z(t) = e^{-\alpha(T-t)} \frac{u'(\hat{V}_{\varphi^\star}(t))}{E(u'(\hat{V}_{\varphi^\star}(t)))}$ with $\alpha \in \mathbb{R}$.
  - γ is determined as root from an equation.
  - α is known in terms of γ.
- Properties of the solution:
  - Fixed fraction of current wealth is invested in the stock.
  - Value of the optimal strategy is geometric Lévy process.
  - Density process of Q is geometric Lévy process.
  - S_1 is geometric Lévy process under Q as well.

Utility maximization in geometric Lévy model

Exponential utility

- Model:
  - $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with Lévy process L
  - initial endowment v
  - utility function $u(x) = 1 - e^{-px}$
  - maximize $E(u(\hat{V}_\varphi(T)))$ over admissible φ
- Solution:
  - $\varphi^\star(t) = \frac{\gamma}{\hat{S}_1(t-)}$ with $\gamma \in \mathbb{R}$
  - $\hat{V}_{\varphi^\star}(t) = v + \gamma(\hat{S}_1)_-^{-1} \bullet \hat{S}_1$
  - Density process of EMM Q from dual characterization is $Z(t) = e^{-\alpha(T-t)} \frac{u'(\hat{V}_{\varphi^\star}(t))}{E(u'(\hat{V}_{\varphi^\star}(t)))}$ with $\alpha \in \mathbb{R}$.
  - γ is determined as root from an equation.
  - α is known in terms of γ.
- Properties of the solution:
  - Fixed discounted amount of money is invested in the stock.
  - Value of the optimal strategy is Lévy process + v.
  - Density process of Q is geometric Lévy process.
  - S_1 is geometric Lévy process under Q as well.

Quadratic hedging in geometric Lévy model

Martingale case

- Model:
  - $S_0(t) = e^{rt}$, $S_1(t) = S_1(0)e^{L(t)}$ with Lévy process L
  - ψ characteristic exponent of L
  - $\hat{S}_1$ martingale (i.e. $\psi(-i) = r$)
  - recall that we need $\hat{V}_z(t)$, $\varphi_z(t)$, ε^2_{yz}
- Solution:
  - $\hat{V}_z(t) = \hat{S}_1(t)^z \exp((T-t)(\psi(-iz) - rz))$
  - $\varphi_z(t) = \frac{\hat{V}_z(t-)}{\hat{S}_1(t-)} \frac{\psi(-i(z+1)) - \psi(-iz) - r}{\psi(-2i) - 2r}$
  - $$\begin{aligned} \varepsilon^2_{yz} &= S_1(0)^{y+z}(e^{\kappa(y+z)T} - e^{(\kappa(y)+\kappa(z))T}) \\ &\quad \times \left(1 - \frac{(\kappa(y+1)-\kappa(y))(\kappa(z+1)-\kappa(z))}{\kappa(2)(\kappa(y+z)-\kappa(y)-\kappa(z))}\right) \end{aligned}$$

  with $\kappa(z) := \psi(-iz) - rz$
- can now apply results to approximate indifference pricing