1. Hochschild cochains

Notation 1.
- $k$-a commutative algebra with unit, $\mathbb{Q} \subset k$
- $A$-a flat $k$-algebra with unit
- $C^n(A,A) = \text{Hom}_k(A^\otimes n, A)$ - Hochschild cochains
- $\delta : C^n(A,A) \to C^{n+1}(A,A)$,
  $$(\delta D)(a_1, \ldots, a_{n+1}) = a_1 D(a_2, \ldots, a_{n+1}) +$$
  $$\sum_{i=1}^{n} (-1)^i D(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} D(a_1, \ldots, a_n) a_{n+1}$$

Lemma 2. $\delta^2 = 0$. The groups $H^n(A,A) = \frac{\ker(\delta : C^n \to C^{n+1})}{\text{im}(\delta : C^{n-1} \to C^n)}$ are called the Hochschild cohomology groups. In particular
- $H^0(A,A) = Z(A)$, the center of $A$
- $H^1(A,A) = \text{Der}(A)/\text{ad}(A)$
- $H^2(A,A) = \frac{\text{Infinitesimal deformations of } A}{\text{isomorphisms}}$

Here, an infinitesimal deformation of $A$ is an associative $k[\epsilon]$-linear product $\ast$ on $A[\epsilon]/\epsilon^2$ which coincides with the original product on $A$ modulo $\epsilon$ and an isomorphism is a $k[\epsilon]$-linear map of the form $T(a) = a + X(a)\epsilon$ which intertwines the $\ast$-products.
1.1. **Algebraic structures on Hochschild cochains I.** Given $D \in C^n$ and $E \in C^m$, the cup product $D \cup E \in C^{n+m}$ is given by

$$D \cup E(a_1 \ldots a_{n+m}) = D(a_1 \ldots a_n)E(a_{n+1} \ldots a_{n+m})$$

**Lemma 3.** $(C^*(A,A), \cup, \delta)$ is a differential graded algebra. The cup product is commutative up to homotopy, in fact the following holds

$$D \cup E - (-1)^{|D||E|} E \cup D = (-1)^{|D||E|} \delta(D\{E\}) - \delta(D)\{E\} - (-1)^{|D|-1}D\{\delta E\}.$$

Here the brace is defined by

$$D\{E\}(a_1, \ldots, a_{n+1}, \ldots, a_{n+m}, \ldots) = \sum_{i=0}^{n} (-1)^{(m-n)} D(a_1, \ldots, a_i, E(a_{i+1}, \ldots, a_{i+m}, \ldots), \ldots).$$

Hence $H^*(A,A)$ is a graded commutative algebra.

**Lemma 4.** The bracket $[D, E] = D\{E\} - (-1)^{|D|(|E| - 1)} E\{D\}$ defines on $C^{*+1}(A, A)$ a structure of DGLA (differential graded Lie algebra).

**Proof.** The brace $D\{E\}$ is not quite associative, but

$$(D\{E\})\{F\} - D\{E\{F\}\} = D\{F, E\} - (-1)^{|E|-1(|F|-1)} D\{E, F\},$$

where

$$D\{E, F\}(a_1, \ldots) = \sum_{i<j} (-1)^{|E|-1+i+|F|-1-j} D(a_1 \ldots E(a_i \ldots), \ldots, F(a_j \ldots), \ldots).$$

Hence

$$[D, [E, F]] + \bigcirc = (D\{E\})\{F\} - D\{E\{F\}\} - D\{F, E\} \pm D\{E, F\} + \bigcirc$$

cancel out.

Let $*$ be any product on $A$ and set $M(a, b) = a * b$. Then

$M$ is associative $\iff [M, M] = 0$.

In particular, if $m(a, b) = ab$, then $\delta D = (-1)^{|D|}[m, D]$ and $\delta^2 = 0$ follows from $[m, m] = 0$ and the Jacobi identity

$$\delta[D, E] = [\delta D, E] + (-1)^{|D|-1}[D, \delta E].$$

Hence

- $C^{*+1}(A, A)$ is a DGLA
- $H^{*+1}(A, A)$ is a Lie algebra
Moreover,

\[ [D, E \cup F] = [D, E] \cup F + (-1)^{(|D|-1)|E|} E \cup [D, F] + \delta D\{E, F\} \]

Recall

**Definition 5.** A Gerstenhaber algebra is a graded vector space \( A^* \) with binary operations \( m \) and \( [\cdot, \cdot] \), such that

- \((A^*, m)\) is a graded commutative algebra
- \((A^{*+1}, [\cdot, \cdot])\) is a Lie algebra
- 

\[ [a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c] \]

The above discussion says that \( H^*(A, A) \) has the structure of a Gerstenhaber algebra.

**Example 6.** Let \( A \) be a commutative algebra. Then \( \text{Der} A \) is an \( A \)-bimodule and

\[ \Lambda_A^* \text{Der}(A) \]

is a graded commutative algebra (with the wedge product). We set

- \([v, a] = v(a)\) for \( a \in \Lambda_A^0(A) = A \) and \( v \in \text{Der}(A) = \Lambda_A^1 \text{Der}(A) \)
- \([v, w] = vw - wv\) for \( v, w \in \text{Der}(A) \)

\([\cdot, \cdot]\) has a unique extension to \( \Lambda_A^* \text{Der}(A) \) such that \((\Lambda_A^* \text{Der}(A), \wedge, [\cdot, \cdot])\) is a Gerstenhaber algebra. The map

\[ \Lambda_A^n \text{Der}(A) \ni v_1 \ldots v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} v_{\sigma_1} \cup \ldots v_{\sigma_n} \in C^n(A, A) \]

is a morphism of complexes \( I_{\text{HKR}} : ((\Lambda_A^* \text{Der}(A), 0) \to (C^*(A, A), \delta). \)

**Example 7.** Similarly, for \( C^\infty(X) \) (\( X \) a smooth manifold) \( \Gamma(X, \Lambda^* T_X) \) is a Gerstenhaber algebra. The bracket here is called the Schouten-Nijenhuis bracket. One can obviously replace polyvector fields on a smooth manifold by the sheaf of holomorphic polyvector fields on an analytic manifold.

In this example, if set \( A = C^\infty(X) \) and from now on \( C^*(A, A) \) will be the cochains given by polydifferential operators, i.e. of the form

\[ D(f_1, \ldots, f_n) = \sum_{\text{finite}} D_1(f_1) \ldots D_n(f_n), \]

where \( D_i \) are differential operators. We again get a morphism of complexes

\[ I_{\text{HKR}} : ((\Lambda_A^* \text{Der}(A), 0) \to (C^*(A, A), \delta). \]

**Theorem 8** (Hochschild, Kostant, Rosenberg). For a regular commutative algebra, or \( A = C^\infty(X) \) (or sheaf of holomorphic functions...)

\( I_{\text{HKR}} \) is a quasiisomorphism.
Note that $I_{HKR}$ preserves neither the product, nor the bracket. But the induced maps 
\[(\Lambda^*_A \text{Der}(A), \wedge, [\cdot, \cdot]) \to (H^*(A, A), \cup, [\cdot, \cdot])\]
is an isomorphism of Gerstenhaber algebras.

1.2. Formality.

**Theorem 9** (Tamarkin). For an associative algebra $A$ there exists a DG Gerstenhaber algebra $\mathcal{C}^*(A)$, natural in $A$ and

1. a DGLA quasiisomorphism $\mathcal{C}^*(A) \to C^*(A, A)$
2. for $A$ regular commutative or $C^\infty(X)$ (or...) there exists a quasiisomorphism of DG Gerstenhaber algebras

\[C^*(A) \to \Lambda^*_A \text{Der}(A).\]

**Remark 10.** In other words, Hochschild cochains carry a structure of a DG Gerstenhaber algebra, if one chooses the model "correctly". This should be compared to the algebra of singular cochains on a topological space which carries a cup-product commutative up to cohomology. But if one chooses instead the model (over $\mathbb{Q}$) of Sullivan forms, it becomes graded commutative on the level of cochains.

A DGLA $\mathcal{A}^*$ is formal, if there exists a chain of DGLA quasiisom. of the form

\[
\begin{array}{ccc}
\mathcal{B}^* & \xrightarrow{\delta} & \mathcal{A}^*_i \\
\downarrow & & \downarrow \\
\mathcal{A}^*_i & & \mathcal{A}^*_{i+1}
\end{array}
\]

with $\mathcal{A}^*_i = \mathcal{A}^*$ and $\mathcal{A}^*_n = H^*(\mathcal{A}^*)$. As a "corollary"

**Theorem 11** (Kontsevich). $C^\infty(X)$ is formal.

1.3. Braces. For $D, E_1, \ldots, E_n \in C^*(A, A)$, we can form braces

\[
D\{E_1, \ldots, E_n\}(a_1, \ldots, a_N) = \sum \pm D(a_1, \ldots, E_1(a_{i_1+1}, \ldots, a_{i_1+n_1}), a, E_2(\ldots, \ldots, E_n(\ldots, a_N)
\]
The sum is over all possible insertions of $E_k$'s, where the order of $a$'s and $E$'s is preserved. The sign is dictated by the rule:

"transposition $a \leftrightarrow E$ contributes the sign $(-1)^{|a|-1(|E|-1)}$"
Lemma 12 (Brace relations).
\[ (D\{E_1, \ldots, E_m\})\{F_1, \ldots F_n\} = \]
\[ \sum \pm D\{F_1, \ldots, E_1\{E_{i_1}, \ldots E_{i_m}F_{i_{m+1}}, \ldots, E_2\{F_1, \ldots \}, \ldots, E_m\{F_{i_m}, \ldots \}, \ldots, F_n\} \]

Recall what we used already:

- \( \delta = [m, \cdot] \) respects \( \cup \) and \([\cdot, \cdot] \), where \( D \cup E = (-1)^{|D|m} D \cup E \)
- \([\cdot, \cdot] \) is a graded Lie bracket;
- \( [D, E \cup F] \sim [D, E] \cup F \pm E \cup [D, F] \) with the homotopy given by \( D\{E, F\} \).

The basic result says that \( C^* (A, A) \) can be given a structure of a DG Gerstenhaber algebra, up to a quasi of DGLA’s. The idea of the proof is as follows.

\[ \begin{array}{ccc}
C^* (Disc) & \xrightarrow{\text{Deligne conjecture}} & \text{Braces} \\
\downarrow & & \downarrow \\
H^* (Disc) & \sim & Gerst
\end{array} \]

\( G_\infty \) is a cofibrant resolution of the operad \( Gerst \), and the dotted map exists by general principles. As a corollary, \( C^* (A, A) \) is a \( G_\infty \)-algebra and DG Gerstenhaber algebra \( C^* (A) \) is its rectification.

From algebra to topology

Brace operations act on \( C^* (A, A) \)

\[ \begin{array}{ccc}
\downarrow & & \downarrow \\
C^* (Disc(n)) \otimes C^* (A, A)^\otimes n & \rightarrow & C^* (A, A)
\end{array} \]

Existence of the bottom action is the "Deligne conjecture".

Back from topology to algebra

- \( H^* (Disc(n)) \) is the natural operad \( Gerst(n) \) of \( n \)-ary operations on a Gerstenhaber algebra (Arnold, Cohen)
- \( C^* (Disc(n)) \rightarrow H^* (Disc(n)) = Gerst(n) \) is a quasiisomorphism of operads, i.e. the chain operad of the little disc operad is formal (Tamarkin) - this step involves an associator.

1.4. Applications to deformation theory. A deformation of an associative algebra \( A \) is an associative, \( k[[t]] \)-linear product \( * \) on \( A[[t]] \) of
the form
\[ a \ast b = ab + \sum_{i>0} t^i D_i(a,b), \quad D_i \in C^2(A,A). \]

An isomorphism of two deformations \((A[[t]], \ast)\) and \((A[[t]], \ast')\) is a \(k[[t]]\)-linear bijection \(T : A[[t]] \to A[[t]]\) of the form
\[ T(a) = a + \sum_{i>0} t^i T_i(a,b), \quad T_i \in C^1(A,A) \]
satisfying \(T(a \ast b) = T(a) \ast T(b)\). If we set \(a \ast b = ab + D(a,b)\), then \(D\) satisfies \([m + D, m + D] = 0\), or
\[ \delta D + \frac{1}{2}[D,D] = 0. \]
So

**Definition 13.** Let \((\mathfrak{g}^*, d)\) be a pronilpotent DGLA. A Maurer-Cartan element of \(\mathfrak{g}\) is an element \(\omega \in \mathfrak{g}^1\) satisfying the Maurer-Cartan equation
\[ d\omega + \frac{1}{2}[\omega, \omega] = 0. \]
two Maurer-Cartan elements \(\omega_1\) and \(\omega_2\) are gauge equivalent, if there exist an element \(X \in \mathfrak{g}^0\) satisfying
\[ d + \omega_2 = e^X(d + \omega_1)e^{-X}. \]
We set
\[ \text{Def}(\mathfrak{g}) = \frac{\text{Maurer Cartan elements}}{\text{gauge equivalence}} (= \pi_0(\mathfrak{g})) \]

**Theorem 14** (Goldmann-Nilsson, Yekuteli). A quasiisomorphism \(\phi : \mathfrak{g} \to \mathfrak{h}\) of pronilpotent DGLA’s induces a bijection \(\text{Def}(\mathfrak{g}) \to \text{Def}(\mathfrak{h})\).

**Corollary 15.** For a regular, commutative algebra \(A\) (or \(A = C^\infty(X)\) or...)
\[ \text{Def}(C^{*-1}(A,A)) \xrightarrow{\simeq} \text{Def}(C^*(A)) \]
\[ \xrightarrow{\simeq} \text{Def}(A^* \text{Der}(A))). \]
In particular, we get a bijection
\[ \text{deformations of } C^\infty(X) \xrightarrow{\text{isomorphisms}} \text{formal Poisson structures } t\pi_1 + t^2\pi_2 + \ldots \]
\[ \xrightarrow{\text{formal diffeomorphisms}} \]
We get more precise information. Given \( \pi = \sum_1^\infty t^n \pi_n \), \( [\pi, \pi]_{Sch} = 0 \), \( \pi \) be the corresponding algebra. Then we get
\[
C^{*+1}(A, A) \xleftarrow{\quad} C^{*+1}(A) \xrightarrow{\quad} \Lambda^{*+1}_A Der(A)
\]
where \( D \in tC^2(A, A) \) satisfies
\[
\delta D + \frac{1}{2} [D, D] = 0, \quad a \ast b = ab + D(a, b)
\]
In particular, we get the following

**Corollary 16.** There exists a chain of quasiisomorphisms of DGLA’s
\[
C^{*+1}(A_\pi, A_\pi) \xleftarrow{\quad} C^{*+1}(A_\pi)[[t]] \xrightarrow{\quad} \Lambda^{*+1}_A Der(A)[[t]]
\]
\[
[m_\pi , \cdot ] \xleftarrow{\quad} \delta + [\Pi, \cdot ] \xrightarrow{\quad} [\pi , \cdot ]
\]
In particular,
\[
Z(A_\pi) \simeq \{ a \in A[[t]] \mid \pi(da, \cdot) = 0 \}
\]

**Remark 17.** [Duflo isomorphism] The above isomorphism of the center of \( A_\pi \) with the Poisson center of \( \pi \) is as vector spaces.

### 2. Deformation 2-groupoids

Let \( (g, [\cdot, \cdot], d) \) be a nilpotent DGLA starting from dimension -1:
\[
g^{-1} \oplus g^0 \oplus g^1 \oplus g^2 \oplus \ldots
\]
As above, a Maurer-Cartan element \( \mu \in g^1 \) satisfies the equation
\[
d\mu + \frac{1}{2} [\mu, \mu] = 0.
\]
We will think of it as a flat connection
\[
\nabla_\mu = d + [\mu, \cdot]
\]
and will denote the set of Maurer-Cartan elements by \( MC(g) \). \( g^0 \) is a nilpotent Lie algebra and we will denote the corresponding Lie group by \( G^0 \) - think of it as the gauge group. \( G^0 \) acts on the space of flat connections by "gauge transformations":
\[
d + ad\mu \to Ad(e^X)(d + ad\mu).
\]
This descends to an action on Maurer Cartan elements, by
\[
Ad(e^X)(d + ad\mu) = d + ad(\int_0^1 e^{adX}(dX)dt + e^{adX}(\mu)).
\]
Given $\mu$, the bracket

$$[\theta, \phi]_\mu = [\nabla_\mu \theta, \tau]$$

defines a structure of Lie algebra on $g^{-1}$, and the corresponding group $G^{-1}_\mu$ acts on $G^0$ by multiplication by $e^{\nabla_\mu \theta}$.

All together, we get a Deligne two-groupoid $MC^2(g)$:

- **Objects** - Maurer Cartan elements $\mu$.
- **1-morphisms** $e^X$, $X \in g^0$, acting by $\mu \rightarrow \int_0^1 e^{adX}(dX)dt + e^{adX}(\mu)$.
- **2-morphisms** acting on $Hom(\mu_1, \mu_2)$ by multiplication by $e^{\nabla_{\mu_2} \theta}$ for $\theta \in G^{-1}_{\mu_2}$.

**Theorem 18.** A $L_\infty$ quasiisomorphism of two DGLA’s vanishing in degrees below -1 induces an equivalence of the associated Deligne two groupoids.

In particular, the formality of, say, $C^* (C^\infty (X), C^\infty (X))$ says that the Deligne two groupoid of deformations of $C^\infty (X)$ is equivalent to the one, where

- **objects** are formal Poisson structures $\pi \in t \Lambda^2 T_X$
- **1-morphisms** are the formal diffeomorphisms $exp(X)$, $X \in t \Lambda^1 T_X$
- **2-morphisms** are the formal diffeomorphisms $exp(X_\theta)$ associated to Hamiltonian vector fields $[\pi, \theta]$, $\theta \in C^\infty (X) [[t]]$

To be more precise, let us define equivalence of two-groupoids of the form $MC^2$. Given a DGLA $g$ as above,

$$\Sigma (g) = \{ n \rightarrow MC (g \otimes \Omega^* (\Delta_n)) \}$$

is a Kan simplicial set with homotopy vanishing in dimensions above 2, and two $MC^2$’s are equivalent if the corresponding $\Sigma$’s are homotopy equivalent.