Consider the antilinear involution of $\mathcal{Y}(A)$ defined by:

$$w|_{j^2} = -\text{Id}, \quad w|_{f_i} = -e_i, \quad w|_{e_i} = -f_i$$

Note: $w$ preserves $\mathcal{Y}(A_J)$ for $J \in I$.

Recall: $J(A) = \{ J \in I | \mathcal{Y}(A_J) \text{ is finite dim'd} \}$

For $J \in J(A)$, let $G(A_J) \leq \text{Aut}(\mathcal{Y}(A))$ be the corresponding cplx Lie gp, i.e.,

$$G(A_J) = \exp(\mathcal{Y}(A_J))$$

Notice $w$ acts on $G(A_J)$; let $K(A_J)$ be the group of fixed points.

$K(A_J)$ is the unitary form:

$K(A_J) \leq G(A_J)$ is a compact Lie gp.

**Def.** Let $\overline{K_J}(A) = \langle T, K(A_J) \rangle$

$$T = \langle \exp(2\pi i h) | h \in h_J \rangle$$

The gp we really want, $K_J(A)$, is a finite cover of $\overline{K_J}(A)$ by a fixed abelian gp depending on $A$.  

Def The unitary form of a
Kac–Moody gp is the colimit
in the category of topological gps:

$$\lim_{\mathcal{L}^A} K_j(A) = K(A).$$

The topology is the weakest st. all

$$K_{i_1}(A) \times \cdots \times K_{i_n}(A) \xrightarrow{\mu} K(A)$$

are continuous.

Bruhat decomposition:

Note that

$$K_i(A)/T \cong CP^1 \cup \cup$$

$$K_i(A)/T - [T] : \{ kT | k \in K \} \cong \mathbb{C}$$

Let $Z_i \subset K_i(A)$ be

a section of

$$K_i(A) \supset Z_i$$

restricted to $K_i(A)/T - [T] \cap \mathbb{C}$.

Thm 3 a CW decomposition of $K(A)/T$

$$K(A)/T = \biguplus_{w \in W(A)} \mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_k}$$

for $w = s_{i_1} \cdots s_{i_k}$ a reduced expression.
Facts:

1. $Z_w = Z_{i_1} \times \cdots \times Z_{i_k}$ is independent of the reduced expression.

2. $Z_w = \prod_{v \leq w} Z_v$ in the Bruhat order.

3. This is a CW-structure.

4. There is a generalization

$$\frac{K(A)}{KS(A)} = \prod_{w \in W^I} Z_w$$

for $W^I = \text{min coset reps of } W/W_I$.

5. $H_{2k}(K(A)/T; \mathbb{Z}) = \bigoplus_{l(w) = k} \mathbb{Z} \langle s_w \rangle$

$H^{2k}(K(A)/T; \mathbb{Z}) = \bigoplus_{l(w) = k} \mathbb{Z} \langle s_w \rangle$

$s_w$ is called a Schubert basis.
Topological Tits building

Def The Topological Tits Building $X$ is the $K(A)$-space

\[ X = \text{hocolim}_{J \in \mathcal{J}(A)} \frac{K(A)}{K_{J}(A)} \]

\[ \cong \frac{1}{|\mathcal{J}(A)|} \times \frac{K(A)}{\mathcal{T}} \]

$|\mathcal{J}(A)|$ is the subcomplex of the barycentric subdivision of the $(d-1)$ simplex whose simplices correspond to

\[ \phi \leq J_0 \leq \ldots \leq J_m \leq I \quad J_i \in \mathcal{J}(A) \]

e.g. $d = 2$

\[ \begin{array}{c}
\{1,3\} \\
\{2,3\} \\
\{1,2,3\}
\end{array} \]

\[ \begin{array}{c}
\text{cone} \\
\{1,2,3\} \\
\{1,3\} \cup \{2,3\} \cup \{1,2,3\}
\end{array} \]

if $K(A)$ is not Lie.

The relation $\sim$ is

\[ (x, kT) \sim (y, hT) \iff x = y \in \mathcal{J}(A) \text{ and } k K_{J_0(A)} = h K_{J_0(A)} \]

where $J_0$ is first in the chain $(*)$. 

Then \( X \) is the classifying space of proper \( K(A) \)-actions:

- if \( G \leq K(A) \) is compact \( \Rightarrow X^G \simeq \ast \).
- if \( \chi^H \neq \emptyset \Rightarrow H \leq K(A) \) is compact

Ex: Let \( T \leq K(A) \) be the maximal torus

\[
X^T = \text{hocolim}_{J \leq \bar{S}(A)} \frac{W(A)}{W_J(A)} = \text{hocolim} \left( \frac{K(A)}{k^{T}} \right)^{T}
\]

for \( d = 2 \)

\[
\frac{W(A)}{W_1} = \frac{(K/E_1)^{T}}{(E_1 ; W(A))} = \frac{(K(A)/\mathfrak{Z}_{2})^{T}}{W(A)/W_2}
\]

Fact: \( X \) imbeds in \( Y(A)^{\infty} \).
Consequences of the

\[ X = \operatorname{hocolim}_{J \in \mathcal{J}(A)} \frac{K(A)}{K(J(A))} \]

1. \[ B K(A) = E K(A) \times_{K(A)} X = \operatorname{hocolim}_{J \in \mathcal{J}(A)} E K(A) \times_{K(A)} K(J(A)) \]

\[ = \operatorname{hocolim}_{J \in \mathcal{J}(A)} B K_J(A) \]

2. \[ \operatorname{hocolim}_{J \in \mathcal{J}(A)} B W_J(A) = B W(A) \]

3. \[ \operatorname{hocolim}_{J \in \mathcal{J}(A)} B N_J(A) = B N(A) \]

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Fact: 3 spectral sequences

\[ \lim_{s \to t} H^i_j(F) \Rightarrow H^{i+j}(\operatorname{hocolim}(F)) \]

\[ \lim_{s \to t} \pi^i_j(F) \Rightarrow \pi^{i+j}(\operatorname{hocolim}(F)) \]
Thm: $BK(A)$ is a stable retract of $BN(A)$

Proof: "Lie retracts can be extended."

Thm: $\tilde{H}^*(BK(A); F_p)$ is noetherian for all primes $p$.

Def: A prime $p$ is good if there is no element of order $p$ in $W(A)$, (All primes $p > l+1 =$ size of $A + 1$ are good)

Thm: ① $\tilde{H}^*(BW(A); F_p) = 0$ if $p$ is good for $* > 0$.

② $\tilde{H}^*(BK(A); F_p) \leq H^*(BN(A); F_p)$ if $p$ is good.

Differentials in the Bousfield-Kan SS:

$$\lim_{i \to j} \tilde{H}^i(BK_j; F_p) \Rightarrow H^{i+j}(BK; F_p)$$

are trivial if $p > l+1$ and there are no additive extension problems.