Buildings: surprising simple objects, versatile concept
many different definitions & ways to look at them

Definition I: (simplicial complex definition)
A building $\Sigma$ is a simplicial complex with a collection $A = \{A_i | i \in I\}$ of subcomplexes, called apartments, such that

1. each apartment is a Coxeter complex,
2. $x, y \in \Sigma \exists A_{xy} \subseteq \{x, y\}$
3. $A \cap A' \neq \emptyset \Rightarrow \exists \varphi: A \rightarrow A', \varphi|_{A \cap A'} = \text{Id}$

Variants are possible, i.e. 1 can be weakened.

For details see: Abramenko & Brown: Buildings
Ronan: Buildings
Explanation:

There are small pieces, buildings are constructed from: Apartments.

1. Describes the structure of these building blocks
2. Assures there are enough building blocks
   \[ \Rightarrow \text{as each pair of simplices is contained in a joined apartment, we can measure distances inside apartments} \]
3. Assures that apartments are pieced together in a systematic way; especially, distances are the same, independently in which apartment they are measured.

Towards a description of apartments:

Definition: (Coxeter group)

A Coxeter group \( W \) is a group admitting a presentation

\[ W = \{ s_\kappa \mid s_\kappa^2 = \text{Id} = (s_\kappa s_\lambda)^{m_{\kappa\lambda}} \}, \kappa \neq \lambda \]

\[ M := (M_{\kappa\lambda}) \text{ is called the Coxeter matrix.} \]

\[ m_{\kappa\kappa} = 1 \quad m_{\kappa\lambda} \in \{2,3,4,6,\infty\} \]
Examples:

\[ M_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow W_1 = \langle s \mid s^2 = 1 \rangle \]

\[ M_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow W_2 = \langle s, t \mid s^2 = t^2 = (st)^2 = 1 \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \]

\[ M_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \Rightarrow W_3 = \langle v, s, t \mid v^2 = s^2 = t^2 = (vs)^2 = (vt)^2 = Id \rangle = \text{PGL}(2, \mathbb{Z}) \]

Definition (Chamber complex):

An \( n \)-dim chamber complex \( \mathcal{C} \) is a simplicial complex, such that

* each simplex is contained in the boundary of an \( n \)-simplex (called a chamber).

\[ \text{No!} \quad \text{Yes!} \]

(hence, it is possible to talk about dimension!)

* for each pair of chambers \( x \) and \( y \), there is a sequence \( x_0, \ldots, x_n \)

s.t.

* \( x_i \) is a chamber

* \( x_0 = x \), \( x_n = y \)

* \( \mathcal{C} \langle x_i \rangle \cap \mathcal{C} \langle x_{i+1} \rangle \) contains an \( (n-1) \)-simplex ("wall")

\[ \text{No!} \quad \text{Yes!} \]

Such a sequence is called a 'gallery'.

Definition: (Coxeter complex)

A Coxeter complex is a chamber complex, equipped with a simply-transitive action of a Coxeter group on the chambers.
Example: \( W = \Sigma m(3) = \{ s, t | s^2 = t^2 = (st)^3 = Id \} \)

- 6 vertices
- Walk \( H_6 \)
- \( s(c) \)
- \( t(c) \)
- \( (sot)(c) \)
- \( (tos)(c) \)
- \( (tosot)(c) \)
- \( (tosots)(c) \)
- \( C = \text{fundamental chamber} \)
- \( 6 \text{ edges} \Rightarrow \text{chambers} \)
- 1 orbit of chambers
- 2 orbits of walls

Example: \( W = D_{oo} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle s, t | s^2 = t^2 = Id \rangle \)

- Action of generators gets more and more nonlocal!
Example of building: n-valent tree $T(n)$, $n \in \mathbb{N}$

There are different apartment systems possible.

Examples:

1. Minimal apartment system:
   - Choose 1 apartment for each chamber
   - Count apartments by counting chambers and hence pairs of chambers
   - Remove all apartments that are constructed from a pair of chambers in an earlier apartment.
   - $\Rightarrow$ countable apartment system!

2. Maximal apartment system
   - Take all 'ends' of $T(n)$ and choose one apartment for each pair of ends
   - As the pair of ends consists of a tree form a Cantor set and Cantor sets are uncountable, this apartmentsystem is uncountable
Theorem:
The two definitions for buildings coincide:

Some remarks to the proof.

• Simplicial def. $\Rightarrow$ W-metric def.

Construct a W-metric by taking for each pair of chambers $x_1, y \in C$ some apartment $A_{xy}$ and defining $d_w(x_1, y) = w \times y = w(x)$. This is independent of the apartment chosen by axiom $(3)$.

The check of the axioms for the W-metric definition is easy.

• Start with the W-metric definition.

Define an apartment to be a subset of $(C, d_w)$, isometric to $(W, d_w)$ where $d_w(w_1, w_2) = w_1^{-1}w_2$

Thus apartments can be viewed as isometries $\phi_A : (W, d_w) \to (C, d_w)$.

A proof that there are collections of these apartments, satisfying the simplicial complex definition is nontrivial.

Example:
Let $G$ be a simple Lie group. Put $C = G/B$ where $B$ is Borel and $d_w(fb, gb) = w \times Bf^{-1}gB = BwB$. 

(22)
Example: Type $A_{n-1}$: Flag complex description

The Lie group of type $A_{n-1} = \text{SL}(n, \mathbb{F})$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

Let $\text{SUB}(\mathbb{F}^n)$ denote the collection of subspaces of $\mathbb{F}^n$

Definition:
A flag is a sequence of subspaces $0 + V_1 \subset V_2 \subset \ldots \subset V_\ell \subset \mathbb{F}^n$
$\ell$ is called the length.

A flag is maximal if $\ell = n$; $\mathcal{FL}_n(\mathbb{F}^n)$ is the space of maximal flags in $\mathbb{F}^n$.

Example: (standard flag)

Let $e_1, \ldots, e_n$ be a basis of $\mathbb{F}^n$, then define

$V_i = \text{span}_\mathbb{F} \{e_1, \ldots, e_i\}$ and $V_1 \subset V_2 \subset \ldots \subset V_n$ defines a maximal flag.

Lemma:

- $\text{SL}(n, \mathbb{F})$ acts transitively on $\mathcal{FL}_n(\mathbb{F}^n)$
- The standard Borel subgroup stabilizes the standard flag.

Proof:

easy calculation!
Define a building as a simplicial complex, whose simplices correspond to flags and the boundary relation is given by inclusion.

Hence:

- $n$-all $\rightarrow (n+1)$-flag
- chamber $\rightarrow$ full flag
- vertex $\rightarrow$ subspace

Let us define apartments in this language.

A frame $F$ is a collection of 1-dim subspaces $\langle f_1 \rangle, \ldots, \langle f_n \rangle$ such that $V = \mathbb{F}^n = \mathbb{F} \langle f_1 \rangle \oplus \ldots \oplus \mathbb{F} \langle f_n \rangle$.

A flag is covered by a frame if each subspace is a union of subspaces $\langle f_{i_1} \rangle \ldots \langle f_{i_k} \rangle \in F$.

**Theorem:**
Frames cover Coxeter complexes for $\text{Sym}(n)$; they are in bijection to apartments.

**Proof:**
Calculation.

**Example:** $\text{SL}(3); \ W = \text{Sym}(3); \ F = \{\langle f_1 \rangle, \langle f_2 \rangle, \langle f_3 \rangle\}$