Finite and affine Kac-Moody symmetric spaces and their buildings

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Lecture 1: Introduction to symmetric spaces
Lecture 2: Introduction to buildings
Lecture 3: Structure of symmetric spaces
Lecture 4: Affine Kac-Moody symmetric spaces
Lecture 5: Twin cities
Philosophy:

F. Klein: Erlanger Programm: Equivalence between

Algebra

(semi-)simple Lie groups

\[ \text{structural generalization} \]

affine KM Groups

Geometry

Symmetric spaces / buildings /

isoparam. submanifolds

\[ \text{geometric generalization} \]

affine KM sym. spaces / twin cities /

isoparam. submanifolds in H.S.

flats / apartment / section

admit a Weyl group action

- Symmetric spaces, buildings, isoparam submanifolds and polar action share their structure properties.
- (Simple) Lie groups are their symmetry groups.
- Simple Lie groups have a closely related structural generalization: affine KM groups.
- Similarly, symmetric spaces e.t.c. admit a closely related generalization: affine KM-sym spaces, twin cities e.t.c.
This is called \"Kac-Moody geometry\"
Lecture I: Symmetric spaces

Definition: (Riem. Manifold)

A Riem. manifold \((M, g)\) is a pair consisting of a smooth manifold \(M\) together with a 'metric' \(g\).

Hence, for \(p \in M\): \(g_p : T_p M \times T_p M \to \mathbb{R}\) is a positive-definite symmetric bilinear form, varying smoothly with \(p\).

Example:

- generic example:
  - difficult to analyse
  - not much meaning

- special examples with symmetry

\(S^n, \mathbb{H}^n, \mathbb{R}^n\)

- curvature \(\kappa = +1\)
- curvature \(\kappa = 0\)
- curvature \(\kappa = -1\)

⟹ Question: Find generalization of these manifolds!
Definition: (Symmetric space)

\((M, g)\) is a symmetric space if, for all \(p \in M\), \(\exists G_p : M \to M\) isometry

such that \((dG)_p|_{T_p M} = -Id_{T_p M}\)

\(v \in T_p M\)

Easy consequences:

Lemma:

A symmetric space is geodesically complete.

Proof:

Geodesics admit shifts:

\(\bar{\gamma}_q \cdot \bar{\gamma}_p\) defines a shift along \(\gamma(t)\) of distance \(2d(p, q)\)

Lemma:

A symmetric space is homogeneous, i.e., \(Isom(M)\) acts transitively.

Proof:

Hopf-Rinow \(\Rightarrow\) \(p, q \in M\) \(\exists\) geodesic \(s.t.\) \(p, q \in \gamma(t)\)

Then \(\bar{\gamma}_{\bar{m}}(p) = q\).
Proposition:
Let $M$ be a symmetric space. $G = \text{Isom}(M)$, the isometry group of $M$, is a lie group.

Consequence:

\[ M = \frac{G}{G_p} \quad \text{where} \quad G = \text{Isom}(M) \text{ and } G_p := \{ g \in G \mid g(p) = p \} \]

$G_p$ is called the 'isotropy group' of $M$.

Proposition:

\[ p, q \in M \Rightarrow G_p \cong G_q \]

⇒ The isomorphism class of the isotropy group of $M$ is independent of the base point chosen.

Definition:
The representation $G_p : T_p M \rightarrow T_p M$ is called the 'isotropy representation' of $M$.

Proposition:

$M$ Riemannian $\Rightarrow G_p$ is compact.

Proof:

$G_p$ acts by isometries; hence $G_p : T_p M \rightarrow T_p M$ preserves the metric $g_p$ at $p$; in consequence $G_p \subset \text{SO}(g_p)$.
Example:

Sphere: \( S^n = \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \} \equiv \frac{SO(n+1)}{SO(n)} \)

Define \( SO(n) \hookrightarrow SO(n+1) \)

\[
A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}
\]

\[\Rightarrow SO(n) = \operatorname{Isot}(e_{n+1}) = \operatorname{Isot}(-e_{n+1})\]

Isotropy representation:

\( SO(n): \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{acts by rotation} \quad (g, x) \mapsto g \cdot x \)

\( SO(n) \)-orbits
Example:

Hyperbolic space $H^2 = SL(2, \mathbb{R})/SO(2)$; $H^2 = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$

$SL(2, \mathbb{R}) : H^2 \rightarrow H^2$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ \hspace{1cm} $z \mapsto A \cdot z = \frac{az + b}{cz + d}$ 'Möbius transformations'

$i \in H$; Claim: $\text{Isotr}(i) = SO(2)$

Proof: Let $A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2)$

$A \cdot i = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot i = \frac{[\cos \theta i + \sin \theta (1)] [\cos \theta + i \sin \theta]}{[\cos \theta i + \sin \theta (i)] [\cos \theta + i \sin \theta]}

= \frac{\cos^2 \theta + \sin^2 \theta (1) + \cos \theta (1) i^2 + \cos \theta \sin \theta (i) (i)\sin \theta}{
\cos^2 \theta + \sin^2 \theta}

= i$

$\Rightarrow SO(2) \subset \text{Isotr}(i)$

$\bullet A i = i \Rightarrow \frac{ai + b}{ci + d} = i \Rightarrow ai + b = ci - d \Rightarrow a = d, b = -c \quad \Rightarrow A \in SO(2)$

using $a, b, c, d \in \mathbb{R}$ \hspace{1cm} $\det(A) = 1$

$\Rightarrow \text{Isotr}(i) \subset SO(2)$

$\Rightarrow SO(2) = \text{Isotr} \cdot i$

Isotropy representation:

$SO(2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Remark:

$H^2$ and $S^2$ have the same isotropy representation.
Let us describe isotropy representations more closely: We define:

**Definition:** (polar representation)

A representation $G: V \rightarrow V$ is called polar iff $\exists \Sigma \in V$, a 'section' such that: Each $G$-orbit $G \cdot x$ intersects $\Sigma$ orthogonally.

**Example**

1) $so(2): \mathbb{R}^2 \rightarrow \mathbb{R}^2$

2) $K$ a compact Lie group: $\forall: k \rightarrow k$ ; $\Sigma := h$, a Cartan subalg.

$(k, x) \rightarrow k \times k^{-1}$

**Lemma:**

The isotropy representation of a symmetric space is polar.

**Proof:** Easy, direct check!

**Theorem:** (Dadok)

Every polar representation is orbit-equivalent to one, which is the isotropy representation of a symmetric space.

**Proof:** difficult!
Classification of symmetric spaces

Idea: reduce the classification to the classification of lie algebras and lie groups!

Main tool: Orthogonal symmetric lie algebras (OSLA)

Definition:
An orthogonal symmetric lie algebra \((\mathfrak{o}_g, \mathfrak{k})\) is a pair, consisting of

- a real lie algebra \(\mathfrak{o}_g\)
- an involution \(\sigma: \mathfrak{o}_g \to \mathfrak{o}_g\) whose fixed point algebra is a lie algebra of the compact type.

Then:

1) Classify OSLA's

2) For each \((\mathfrak{o}_g, \mathfrak{k})\) find all \((G, K)\)

\[ \sigma = \text{Lie}(G), \quad \mathfrak{k} = \text{Lie}(K), \quad K \subset G \]

3) For all \((G, K)\) form the space \(M = G/K\)

4) Find isomorphisms / isometries.

\(\Rightarrow\) Classification of simple complex lie algebras

\(\mathfrak{o}_{n, n, 24}, \mathfrak{b}_{n, 22}, \mathfrak{c}_{n, n, 25}, \mathfrak{d}_{n, n, 24}, \mathfrak{e}_{6, 7, 8}, \mathfrak{f}_4, \mathfrak{o}_{7, 2}\)

\(\Rightarrow\) Classification of real forms \(\Rightarrow\) more complex, many cases!
Example: \( \text{Type } \alpha_1 \cong \text{SL}(2, \mathbb{C}) \)

Symmetric spaces of the compact type | Symmetric spaces of the non-compact type

Type II \[ \text{SU}(2) \cong S^3 \] \( \frac{\text{SL}(2, \mathbb{C})}{\text{SU}(2)} \cong H^3 \) Type IV

Type I \[ \text{SU}(2)/\text{SO}(2) \cong S^2 \] \[ \text{SL}(2, \mathbb{R})/\text{SO}(2) \cong H^2 \] Type III

\[ \text{sect}(X, Y) \geq 0 \]
precisely: \( \text{sect}(X, Y) = 1 \)

\[ \text{sect}(X, Y) \leq 0 \]
precisely \( \text{sect}(X, Y) = -1 \)

Duality! Same isotropy subgroups

- There are 'irreducible' symmetric spaces of the compact type and of the non-compact type.
- There is a duality relation between symmetric spaces of the compact type and of the non-compact type.

A compact symmetric space and a non-compact symmetric space have the same isotropy representation.
Classification of symmetric spaces

- each symmetric space is a quotient of a simply connected symmetric space
- each simply-connected symmetric space is a direct product of irreducible (simply-connected) symmetric spaces.

Classification of irreducible simply connected sym. spaces

\[ \begin{align*}
\text{Type II} & \quad K \quad \text{compact simple Lie group} \\
\mathbb{R} & \quad G/K \quad \text{complex simple Lie group} \\
& \quad \text{Type IV} \\
\text{Type I} & \quad K/_{K_R} = \text{Fix}(G) \\
& \quad \text{Type III} \\
& \quad \text{involution} \\
\text{sect}(X,Y) \geq 0 & \quad \text{duality} \\
\text{sect}(X,Y) \leq 0 & \\
\end{align*} \]

- Classification of Type II/IV: one for each simple, complex Lie algebra
- Classification of Type I/III: for each compact real form one for each involution
Metrics:

To make these Lie group quotients into symmetric spaces, we need invariant scalar products.

**Definition:**

Let $\mathfrak{g}$ be a Lie algebra. Define $B(X,Y) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ by

$$x, y \mapsto \text{tr}(\text{ad}(x) \circ \text{ad}(y))$$

**Lemma:**

- $B(X,Y)$ is ad-invariant
- $B(X,Y)$ is nondegenerate $\iff \mathfrak{g}$ is semisimple
- $B(X,Y)$ is negative-definite $\iff \mathfrak{g}$ is of compact type

**Definition:**

Define a metric on $G$ a simple Lie group by left translation of

$$\langle X, Y \rangle_{\text{Id}} = \varepsilon B(X,Y) \quad \text{where } \varepsilon \in \pm 1$$

- Choose $\varepsilon = -1$ if $G$ is compact
- $\varepsilon = 1$ if $G$ is noncompact.

Define on $M = \frac{G}{\mathcal{K}}$ a metric via the submersion

$$G \rightarrow M = \frac{G}{\mathcal{K}}$$
Example:
\[ \mathcal{P}_R (V^\prime) = \{ A \in \text{Mat}(n \times n, R) \mid A = A^t \text{ and } v^t A v > 0 \} \]
^ space of ellipsoids.

Hence we have the following geometric intuition

1. Each ellipsoid has principal axes, each element in \( \mathcal{P}_R (V^\prime) \)
   is diagonalizable

2. There is a global scaling factor: 
   
   \[ \mathcal{P}_R (V^\prime) = \mathcal{P}_{R,1} (V^\prime) \times R \]

   where \( \mathcal{P}_{R,1} (V^\prime) = \{ x \in \mathcal{P}_R (V^\prime) \mid \det (x) = 1 \} \)

3. Rescaling of principal axes, without rotation \( \cong \) n-dimensional flats

4. All n-principal axes are equivalent \( \cong \) Sym(n)-permutation group acting on flats

5. If two principal axes have the same length, then the choice of principal axes is not unique \( \cong \) these elements lie in more than one axis

6. At \( S^n \) there is asymmetry:
   \[ \mathcal{P}_R (V^\prime) \rightarrow \mathcal{P}_R (V^\prime), \]
   \[ (a_1, \ldots, a_n) \mapsto \left( \frac{1}{a_1}, \ldots, \frac{1}{a_n} \right) \]
Definition (Tangent space of $\mathbb{P}_R(V^n)$)

$T_{\text{Id}}(\mathbb{P}_R(V^n)) := \{ A \in \text{Mat}(nn, R) \mid A = A^+ \} = S^+_R(V^n)$

"symmetric matrices"

Definition:

The metric at $p \in \mathbb{P}_R(V^n)$ is given by

$$\langle X, Y \rangle = \text{trace}(p^{-1} X p^{-1} Y)$$

There is an action: $\varphi: \text{GL}(n, R): \mathbb{P}_R(V^n) \to \mathbb{P}_R(V^n)$ (*)

$$(g, p) \mapsto gp g^t.$$ 

Let us now describe this action more precisely:

Theorem:

The action (*) is

1) transitive

2) by isometries

3) $\text{Isotr(Id)} = \text{SO}(n)$

4) $\{ \pm \text{Id} \} \leq \text{GL}(n, R)$ acts trivially, $\text{GL}(n, R)/\{ \pm \text{Id} \}$ acts effectively

5) $\mathbb{P}_R(V^n)$ is a symmetric space. The symmetry $S_p$ is defined by: $S_p(q) = pq^{-1}p$ for $p, q \in \mathbb{P}_R(V^n)$
Proof:

1) Show $\forall p \in \mathbb{P}_R(V^n) \exists g s.t. p = gg^t$

Use that $p$ is diagonalizable, hence $p = ODO^t$, $O$ orthogonal, $D$ diagonal.

Put $g = O\sqrt{D} O^t$

$\Rightarrow gg^t = O\sqrt{D} O^t O\sqrt{D} O^t = ODO^t = p$

2) Let $X, Y \in T_pM \Rightarrow$

$\langle dg(X), dg(Y) \rangle_{qp} = \text{trace} \left( (g^t)^{-1} p^{-1} g X g^t (g^t)^{-1} p^{-1} g^t Y g^t \right) = \text{trace} \left( (g^t)^{-1} p^{-1} X p^{-1} Y g^t \right) = \text{trace} \left( p^{-1} X p^{-1} Y \right) = \langle X, Y \rangle_p$

3) Clear

4) Clear

5) Have to check:

i) $\delta_p(p) = p$

ii) $\delta_p$ is isometry

iii) $d\delta_p|_{T_pM} = -Id$