Integrable highest weight representations and Dominant K-theory

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Representation Theory

Assume that the GCM $A$ is symmetrizable.

Recall the "lattice" $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} a_i$
and define a partial order on $h^*_R$

$m \leq \lambda \iff m - \lambda \in Q^+_+$

Def: The dominant weights (in $h^*_R$) are:

$D := \{ \lambda \in h^*_R \mid \lambda(h_i) \geq 0 \text{ for } h_i \in \Delta \}$

$= h^*_R \cap C$, $C := \text{Weyl chamber}$
Def: A rep'n $L$ of the Kac-Moody Lie algebra $\mathcal{G}(A)$ is called a highest weight integrable rep'n with highest weight $\lambda$ if

(a) restricted to $\mathcal{J}$, its weights $\mu$ satisfy $\mu \leq \lambda$ and $\lambda$ has multiplicity one
(b) $L$ decomposes as a sum of finite dim'l rep's $\mathcal{G}_J(A)$ for $J \in \mathcal{J}(A)$ (It is sufficient to demand this for $J=\{i\}$.)

Thm 1 $L$ exists $\iff \lambda \in \mathcal{D}$

2 If $L$ exists, it is unique.

3 $L$ extends to a rep'n of $K(A)$

4 $L$ admits a $K(A)$-invariant Hermitian inner product

5 The weights of $L(\lambda)$ lie in the convex hull of $W \cdot \lambda$ (= Tits Cone)

6 $L$ is irreducible
Character Formula:

Given \( \lambda \in \mathcal{D} \), the character of \( L(\lambda) \) is:

\[
\text{ch} (L(\lambda)) = \sum_{w \in \mathbb{W}} (-1)^w e^{w(\rho + \lambda)} \prod_{\alpha \in \Delta^+} \left( 1 - e^{-\alpha} \right)
\]

for

\[
\rho = \sum_{i \in I} h_i^* \quad \text{i.e.} \quad \rho(h_i) = 1
\]

Here \( (-1)^w = (-1)^{\text{length of } w} \).
Two examples:

Ex 1: If \( \lambda = \rho \),

\[
\text{ch}(L(\rho)) = \sum_{w \in W} (-1)^w e^{w(2\rho)} \frac{e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}{e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}
\]

(\text{L}(0) \text{ implies the denominator identity:})

\[
e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^w e^{w\rho} \quad (\ast)
\]

Putting (\ast) into (\text{t}) gives

\[
\text{ch}(L(\rho)) = e^{2\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-2\alpha}) \frac{e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}{e^{\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}
\]

\[
= e^{\rho} \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha})
\]

Claim: \( L(\rho) \cong \Lambda^+(\mathfrak{h}_-^{\perp}) \) projectively
Ex2: \[ A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \] so that

\[ \mathfrak{g}(A) \cong 5 \mathbb{L}_2 \langle C[2], z^{-1} \rangle \oplus C[2] \oplus C[2] \]

\[ L = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \]

\[ L^* = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \]

\[ \rho = h^* + 2 \Lambda = (1, 2, 0) \]

\[ \Delta_+ = \{ n \mathbf{8} + \varepsilon \mathbf{2} h^* \mid n > 0, \varepsilon \in \{-1, 0, 1\} \} \]

\[ \text{or } n = 0, \varepsilon = 1 \]

Here the denominator identity (a) becomes:

\[ \sum_{m \in \mathbb{Z}} (-1)^m \frac{m(m-1)}{2} \frac{m(m+1)}{2} v^m \]

\[ = \prod_{n \in \mathbb{N}} \left( 1 - u^{n^2} v^{m+1} \right) \left( 1 - u^n v^{m+1} \right) \left( 1 - u^{n+1} v^m \right) \]
Dominant $K$-theory

**Question:** Is there a $K(A)$-equivarant $K$-theory of $K(A)$-spaces?

**Problems:** Can't be equivarant vector bundles on $X$ for some $K(A)$-space

($X$ may have a fixed pt)

**Solution:** Assume $X$ is a proper $K(A)$-complex (i.e., isotropy compact)

**Problem 2:** How does one stabilize rep's over vector bundles $\times X$?

**Solution:** Work inside an a priori stable space.

**Def.** Let $H$ be the Hilbert space given by completing sums of $L(\lambda)$, $\lambda \in D$

$$H = \bigoplus_{\lambda \in D} N L(\lambda)$$

(between a sum and product)
Def: $\mathcal{F}(H)$ := space of Fredholm operators on $H$
(with suitable norm topology)
This is a $\mathcal{K}(A)$-space.

Def: Given a proper, finite $\mathcal{K}(A)$-complex $X$, the dominant $K$-theory of $X$

\[ \mathcal{K}(X) = \sum \{ \sum_{n} \mathcal{F}(H) \} \]

Coefficients in dominant $K$-theory

Let $X(A)$ := the topological Tits building
(a classifying space of proper $\mathcal{K}(A)$-actions)

\[ = \underset{\text{hocolim}}{\text{colim}} \mathcal{K}(A) / \mathcal{K}_{\mathcal{S}(A)} \]

Since $X(A)$ is terminal in proper $\mathcal{K}(A)$-complexes

3 canonical $\mathcal{K}(X(A)) \to \mathcal{K}(X)$
Then assume that $A$ is of compact type i.e.

$$\Sigma(A) = \xi J | J \leq \mathbb{I}, \mathbb{I} = \mathbb{I}$$

then $\exists$ an isomorphism

$$K^g \circ \mathbb{K}^{\mathbb{I}}(X(A)) = \bigoplus_{\lambda \in \mathbb{D}} \mathbb{Z} \times S \mathbb{Z}_{\lambda}^*$$

If $\lambda \in \mathbb{D}$, $\exists a cubic Dirac operator:

$$D \subseteq L^p \otimes S((y^* \otimes L(\lambda))$$

$$= \wedge^*(\eta^*) \otimes S((y^* \otimes L(\lambda))$$

$$D^* := J + J^* + K$$

$$J \in \wedge^*(\eta^*) \otimes L(\lambda) \rightarrow \wedge^*(\eta^*) \otimes L(\lambda)$$

is the Koszul differential

$J^*$ is its adjoint

$K :=$ the Dirac operator for $S((y^*)$