Lecture 3: Soergel bimodules, Soergel's conjecture and other conjectures.

\((W, S)\) Coxeter system with length function \(l : W \to \mathbb{N}\)

Bruhat order \(\leq\) on \(W\): transitive closure of

\[ s_1 \ldots s_i \ldots s_k \leq s_1 \ldots s_k \]

where \(s_i \in S\) and \(l(s_i \ldots s_k) = k\).

\[ A = \mathbb{Z}[v, v^{-1}] \]

\[ \mathcal{H} = \text{Iwahori-Hecke algebra of } (W, S) \]

\[ = A\text{-algebra with basis } \{ H_w \mid w \in W \} \text{ such that} \]

\[ H_s H_w = \begin{cases} H_{sw} & \text{if } sw \geq w \\ H_{sw} + (v^{-1} - v) H_w & \text{if } sw < w \end{cases} \quad (s \in S, w \in W) \]

Bar operator on \(\mathcal{H}\): \(\mathbb{Z}\)-linear map \(h \mapsto \overline{h}\) with \(v^n H_x = v^{-n}(H_{x^{-1}})\)

Standard form on \(\mathcal{H}\): \(\mathbb{Z}[v, v^{-1}]\)-sesquilinear inner product with \((H_x, H_y) = \delta_{xy}\)

Thm [KL] \(\mathcal{H}\) has a unique basis \(\{ c_w \mid w \in W \}\) which is "Canonical" (see [WebI]) in sense that

1. \(\overline{c_w} = c_w\)
2. \(c_w \in H_w + \sum_{y \leq w} vZ(v) H_y\)
3. \((c_x, c_y) \in \delta_{xy} + vZ(v)\)

\(\{ c_w \} = KL\text{-basis}.\) Positivity conjectures: (a) \(c_w \in \mathbb{N}[v, v^{-1}]\text{-span } \{ H_y \}\).

(b) \(c_x c_y \in \mathbb{N}[v, v^{-1}]\text{-span } \{ c_z \}\).
Explaining existence and properties of a canonical basis is a generally difficult problem, but by design the most obvious approach is to look for category $\mathcal{C}$ with an isomorphism

$$ H \xrightarrow{\phi} [\mathcal{C}] = \text{split Grothendieck group} $$

which transfers

bar operator $\rightarrow$ duality functor

standard form $\rightarrow$ graded dimension of $\text{Hom}(\cdot, \cdot)$

{$C_w$} $\rightarrow$ representative set of indecomposable objects {$B_w$}

In case when $W$ is Weyl group of complex reductive group $G$ with Borel subgroup $B$, original proof of KL-conjectures described such an isomorphism with $\mathcal{C}$ given by additive and grading closure of semisimple complexes in

$$ D^b_{B \times B} (G, \mathcal{C}) = \text{equivariant derived category of } B\text{-bi-equivariant sheaves on } G. $$

Soergel [S92] found a more elementarily defined category which serves the same purpose and which makes sense for any Coxeter system. This will be the category of Soergel bimodules.
All categories of interest will be full subcategories of \( R\{-\text{Bim} = \{(\mathbb{Z}\text{-graded } R\text{-bimodules } M = \bigoplus_{i \in \mathbb{Z}} M^i \text{ w/ graded homomorphisms)}\}
\]

where \( R \) is some polynomial ring over integral domain \( K \) with characteristic zero.

Given \( \mathcal{E} \subset R\{-\text{Bim} \), define \([\mathcal{E}] = \) abelian group generated by symbols \([M]\) for \( M \in \mathcal{E} \) subject to relations \([M] = [A] + [B] \) if \( M \cong A \oplus B \)

Write \( \otimes \) for \( \otimes_R \). Then \([\mathcal{E}] \) is ring wrt. \([M][M'] = [M \otimes M'] \)

(up close under \( \otimes \))

In turn, \([\mathcal{E}] \) becomes \( A \)-algebra by setting \( v^d [M] = [M(d)] \)

where \( M(d) \) is grading shift such that \( M(d)^i = M^{i + d} \)

**Example** (Soergel bimodules for \( W = S_2 \))

Let \( W = S_2 = \{1, s\} \)

\( S = \{s = (1, 2)\} \)

\( R = IR[x] \)

\( W \) acts on \( R \) by \((s \cdot f)(x) = f(-x)\)

Let \( R^S = S\text{-invariants} = IR[x^2] \)

\( \text{Grade } R \) and \( R^S \) so that \( x^n \) has degree \( 2n \)
Define $R$-bimodules

\[ B_1 \overset{\text{def}}{=} R \]
\[ B_s \overset{\text{def}}{=} R \otimes_R R \overset{\text{def}}{=} R_s \]
\[ B_{[k]} \overset{\text{def}}{=} R \otimes R \otimes \cdots \otimes R \overset{\text{def}}{=} R_s \overset{k \text{ factors}}{\rightarrow} \] (gradings shift down)

Observe $B_1 = B_{[0]}$ and $B_s = B_{[1]}$ and $B_{[k]} = B_s \otimes \cdots \otimes B_s \overset{k \text{ factors}}{\rightarrow}$

\[ 1 \in B_1 \] has degree 0
\[ 1 \otimes 1 \in B_s \] has degree -1
\[ 1 \otimes \cdots \otimes 1 \in B_{[k]} \] has degree -k
\[ x \otimes x^2 = x^3 \otimes 1 \in B_s \] has degree 5

Claim (1) $B_1$ and $B_s$ are indecomposable

(2) $B_{[k]} \cong B_s(1) \oplus B_s(-1)$

Proof (1) $B_1$ and $B_s$ are generated as $R$-bimodules by single elements $1$ and $1 \otimes 1$ of lowest degree.

(2) $B_s$ is direct sum of sub-bimodules generated by

\[ \alpha = 1 \otimes 1 \otimes 1 \text{ in degree } -2 \]
\[ \beta = 1 \otimes x \otimes 1 \text{ in degree } 0 \]

Check $\langle \alpha \rangle \cong B_s(1)$ and $\langle \beta \rangle \cong B_s(-1)$.
Let $SBim C R-Bim$ be full subcategory formed by additive, grading closure of bimodules $B_{[k]}$.

**Facts**

(a) $SBim$ closed under $\otimes$ so $[SBim]$ is A-algebra.

(b) Indecomposable objects (up to grading shift and isomorphism) are $B_1$ and $B_5$.

(c) Map $\mathcal{Y}_r \rightarrow [SBim]$

$$C_w \mapsto [B_w]$$

is isomorphism.

**Proof**

(a) Note that $B_{[k]} \otimes B_{[e]} = B_{[k+e]}$

(b) Clear from claim.

(c) Recall $C_1 = 1$ and $C_5 = H_S + V$.

Clearly $C_1 C_w = C_w$ and $B_1 \otimes B_w = B_w$.

Check that $C_5 C_5 = (V + W') C_5$ and note that

$$[B_5][B_5] = [B_5 \otimes B_5] = [B_r(1) \otimes B_r(-1)] = (V + W') [B_5].$$

$SBim$ for general $(W, S)$: define analogous category of bimodules $B_{[k]}$, then pass to Karoubian envelope.
Setup in detail:

- Let $h$ be geometric repn of $W$ defined over commutative ring $k$, or more generally let $h$ be any "realization" of $W$ in sense of [Ew2, Def. 3.1].

- Let $R$ be graded ring of polynomial functions

$$R = \bigoplus_{n \geq 0} s^n(h^*)$$

(E.g., for $W = S_n$, think of $R$ as (quotient of) $R[x_1, \ldots, x_n]$)

Grade $R$ so that constant funs have degree 0
linear funs have degree 2
quadratic funs have degree 4 etc.

- Let $R^S$ = $s$-invariants = \{ $f \in R$ | $f(s \cdot x) = f(x)$ \} for $s \in S$.

Now, given any sequence $\alpha = (s_1, s_2, \ldots, s_k)$ with $s_i \in S$, define

$$B_{\alpha} \overset{\text{def}}{=} R \otimes R \otimes \cdots \otimes R (k) \in R-Bim$$

kill factors grading shift down

Note that $B_{\alpha} = B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_k}$ where $B_s = R \otimes R (1).$
View elements of $B_\alpha$ as sums of sequences

\[
\left( \begin{array}{c}
f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_k \\
\end{array} \right) \\
\begin{array}{ccc}
s_1 & s_2 & s_3 & \ldots & s_k
\end{array}
\]

where $f_i \in R$ and you can slide a scalar across porous wall $s_i$ if the scalar is $s_i$-invariant.

**Def:** Category $\text{SBim}$ of Soergel bimodules is full subcat of $R\text{-Bim}$ whose objects are direct sums & grading shifts of $M \in \ R\text{-Bim}$ such that $M \otimes N \cong B_\alpha$

for some $N \in R\text{-Bim}$, sequence $\alpha$.

$\rightarrow$ I.e., $\text{SBim}$ = Karoubian envelope of additive, grading closure of [bimodules $B_\alpha$]

$\rightarrow$ I.e., indecomposable objects of $\text{SBim}$ are direct summands of $B_\alpha$’s (up to grading shift)

If $M, N \in \text{SBim}$ then so are $M \otimes N$, $M[d]$, $M \otimes N$ so $[\text{SBim}]$ is $A$-algebra

**Soergel's Categorification Thm I.** There is a unique isomorphism

\[
\varphi : \mathcal{X} \cong [\text{SBim}] \quad (\text{of } A\text{-algebras})
\]

such that $\varphi([C_s]) = [B_s]$ for $s \in S$.

Moreover $(\varphi([M]), \varphi([N])) = \sum_{j \in \mathbb{Z}} \text{rank} \ \text{Hom}_R(M[j], N)$
Claim Sketch. Uniqueness is immediate, as $\mathcal{C}_a$ generate $\mathcal{A}$. Checking that $\mathcal{C}$ is homomorphism relatively not difficult; suffices to check certain isomorphisms in $\text{SBim}_{\mathcal{A}}$ corresponding to braid relations of $\mathcal{A}$. Showing that $\mathcal{C}$ is isomorphism follows from classification of indecomposable objects in $\text{SBim}_{\mathcal{A}}$, to be described next! $\square$

Remark. It is not at all obvious how to describe $\mathcal{C}(\mathcal{C}_w)$ for arbitrary $w \in \mathcal{W}$, or even if it is indecomposable, though we hope this is true!

Example. (Soergel bimodules for $W = S_3$)

Let $W = S_3 = \{1, s, t, st, ts, sts = st^3\}$

$S = \{s = (1,2), \ t = (2,3)\}$

$R = \mathbb{K}[x,y,z]$ graded so that $x^iy^jz^k$ has degree $2(i+j+k)$

$W$ acts on $R$ by $s \cdot f(x,y,z) = f(y,x,z)$

$t \cdot f(x,y,z) = f(x,z,y)$
Exercises

(a) The following are indecomposable Soergel bimodules:

\[ B_1 = R = \langle 1 \rangle \]
\[ B_s = \langle 1 \otimes 1 \rangle \]
\[ B_t = \langle 1 \otimes 1 \rangle \]
\[ B_{s+t} \overset{\text{def}}{=} B_s \otimes B_t = \langle 1 \otimes 1 \rangle \]
\[ B_{t+s} \overset{\text{def}}{=} B_t \otimes B_s = \langle 1 \otimes 1 \rangle \]

(b) If \( \delta = y - z \) and \( \Delta = \delta \otimes 1 + 1 \otimes \delta \) then

\[ B_s \otimes B_t \otimes B_s = \langle 1 \otimes 1 \rangle \otimes \langle 1 \otimes 1 \rangle \]

\[ \text{Call this } B_{s+t} \]
\[ \cong B_s \]

(c) If \( \delta' = x - y \) and \( \Delta' = \delta' \otimes 1 + 1 \otimes \delta' \) then

\[ B_t \otimes B_s \otimes B_t = \langle 1 \otimes 1 \rangle \otimes \langle 1 \otimes 1 \rangle \]

\[ \text{Call this } B_{t+s} \]
\[ \cong B_t \]

(d) \( B_{s+t} \cong B_{t+s} \) are indecomposable.
Facts (1) Indecomposable objects of $SBm$ are (up to isomorphism, grading shift)\[
\{ B_i, B_s, B_+, B_{st}, B_{ts}, B_{sts} \}\]

(2) Moreover, $\mathcal{C}(C_w) = B_w$.

Soergel [soe7] proves the following generalization of Fact (1):

**Soergel's categorification theorem II**: For each $w \in W$ there exists up to isomorphism a unique indecomposable bimodule $B_w$ which occurs as a direct summand of $B_\alpha$ for any sequence $\alpha = (s_i, \ldots, s_k)$ with $k = l(w)$ and $s_i \cdots s_k = w$,

and which appears in no $B_{\alpha'}$ with $\alpha'$ shorter sequence.

The set $\{ B_w \mid w \in W \}$ represents isomorphism classes of all indecomposable objects in $SBm$ (up to grading shift).

Soergel's conjecture is the generalization of Fact (2).
Soergel's conjecture. \[ E(C(w)) = \mathcal{B} w \forall w \in \mathcal{W} \text{ when } \text{char}(k) = 0 \]

Recall \( k = R_0 = \text{field of definition of } h \).

Elvis and Williamson prove this when \( k = \overline{\mathbb{F}} \) \cite{EW1}.

Immediately implies
\[
C_x \cap C_y \in \text{lin} [\{v,v^*\}] - \text{span} \{C_z\}
\]

Since by construction \( B_x \otimes B_y \approx \text{sum of grading shifts of } B_z \)'

Conjecture implies much more by means of explicit formula for inverse \( e^{-1} \).

(\text{Remark. Conjecture known to be false for some fields } \overline{\mathbb{F}} \text{ with positive characteristic. See } \text{[EW2, Remark 3.18].})

Soergel's characteristic map. \( ch: [\text{Sem}] \rightarrow \mathcal{X} \).

To define this we introduce \underline{standard bimodules}:

\text{Given } w \in \mathcal{W} \text{ define } \mathcal{R}_w \in \mathcal{R}\text{-Bim} \text{ such that }
- \text{As left } \mathcal{R}\text{-module } \mathcal{R}_w = \mathcal{R}
- \text{As right } \mathcal{R}\text{-module } b \ast r \overset{\text{def}}{=} w(r)b \text{ for } b \in \mathcal{R}_w, r \in \mathcal{R}.

Since \( \mathcal{W}\text{-action on } \mathcal{R} \) is repn, \( \mathcal{R}_w \circ \mathcal{R}_w \cong \mathcal{R}_{ww} \forall w, w' \in \mathcal{W} \)
\text{(Standard bimodules are not necessarily Soergel bimodules.)}
Given $f = \sum_{i \in \mathbb{Z}} q_i v_i \in N[v_i^{-}]$ define

$$(R_w)^f \overset{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} (R_w (-i))^{\oplus q_i} \uparrow$$

grading shift up

Prop. (Soergel)

(a) Every $B \in \mathbb{S}Bim$ has filtration

$$0 = B^0 \subset B^1 \subset \cdots \subset B^m = B$$

where $B^i \in \mathbb{S}Bim$ and $B^i / B^{i-1} = (R_{y_i})^{\oplus h_{y_i}}$ for some $y_i \in W$ and $h_{y_i} \in N[v_i^{-}]$.

(b) There is a unique such filtration for which $i < j$ implies $y_i < y_j$ in Bruhat order.

Filtration (b) $\overset{\text{def}}{=} \underline{\text{standard filtration of } B \in \mathbb{S}Bim}$

Characteristic map: $Ch : [\mathbb{S}Bim] \to \mathbb{K}$

$$[B] \mapsto \sum_{y \in W} a^{(i)}(y) h_y$$

coeffs in standard filtration of $B$

$\uparrow$
Example \( W = \mathbb{S}_2 = \{1, s\} \)

\( R = R[x] \) graded with \( x^n \) in degree \( 2n \).

\( R_s = R[x^2] \)

\[ B_s \overset{\text{def}}{=} R \otimes_{R_s} R(1) \in SBm \]

(Standard)

The Soergel bimodule \( B_s \) has filtration

\[ 0 = B_0 \subset B_1 = \langle x \otimes 1 + 1 \otimes x \rangle \subset B_2 = B_s \]

Check:

\( B_1/B_0 = B_1 \cong R_1(-1) = (R_1)^{\otimes v} \)

\( B_2/B_1 \cong R_s(1) = (R_s)^{\otimes v^{-1}} \) since \( B_1 \) is kernel of map

\[
\begin{cases}
B_s \rightarrow R(1) \\
_f \circ g \mapsto f(s \cdot g)
\end{cases}
\]

Thus \( h_1 = v \)

\( h_s = v^{-1} \)

\[ \Rightarrow c_h(B_s) = C_5 \Rightarrow + v = C_s \]
Soergel's Categorification Thm III.

Map \( \text{ch} : [SBim] \to \mathcal{X} \) is inverse to \( \mathcal{E} \).

Soergel's conjecture is thus equivalent to

\[
\text{Conjecture} \quad \text{ch}(Bw) = C_w \; \forall w \in \mathcal{W}.
\]

This implies \( C_w \in \mathbb{N}[v^*] - \text{span}\{H_i\} \) since by definition \( \text{ch}(Bw) \in \mathbb{N}[v^*] - \text{span}\{H_i\} \) and we know \( C_w \in \mathbb{Z}[v^*] - \text{span}\{H_i\} \).

Elias and Williamson [EW1, EW2] have proved Soergel's conjecture when \( K = \mathbb{R} \).

In few words: they show that \( Bx \) "looks like Cohomology of smooth projective variety" and apply Hodge theoretic ideas of de Cataldo and Migliorini.

More generally, Elias and Williamson have developed effective methods of doing computations in SBim, by giving this monoidal category a presentation as a 2-category, and developing a diagrammatic language to describe its morphisms.
What next? Some related open problems (as evidence that there are many others)

Positivity conjectures.

Let $P_{y,w} \in \mathbb{Z}[v^\pm]$ such that $C_w = \sum_{y \in W} v^{l(w) - l(y)} P_{y,w} \cdot H_y$

$h_{x,y} \in \mathbb{Z}[v^\pm]$ such that $C_x C_y = \sum_{z \in W} h_{x,y}^z C_z$

Conjecture

1. $P_{y,w} \in \mathbb{N}[v^\pm]$.

2. $P_{y,w} - P_{z,w} \in \mathbb{N}[v^\pm]$ if $y \leq z$.

3. $h_{x,y}^z \in \mathbb{N}[v^\pm]$.

4. $h_{x,y}^z$ symmetric unimodal, i.e. of the form

$$a v^3 + b v^{d+2} + c v^{d+4} + \ldots + c v^{d+4} + b v^{d+2} + a v^d$$

where $a \leq b \leq c \leq \ldots$

5. Combinatorial invariance (see [Inc]):

$$P_{y,w} = P_{y',w'}$$

whenever intervals $[y,w]$, $[y',w']$ in Bruhat order are isomorphic posets.

Only (1) & (3) are known for all Coxeter systems.
"Twisted" KL-theory

Let \(* \in \text{Aut}(W)\) such that \(s^* \in S \forall s \in S\) and \(*^* = 1\).

Set \(I_* = \{ w \in W \mid w^* = w^* \}

\[ X_* = A \cdot \text{span} \{ a_{w} \mid w \in I_* \} \text{ free } A\text{-module} \]

Thm (Lusztig and Vogan) [Lu]

(a) \(X_*\) has \(X\)-module structure (analogous to \(X\)-algebra structure)

(b) There is unique \(\overline{\text{bar operator}}\) \(a_1 \mapsto \overline{a_1}\) on \(X_*\) such that \(\overline{a_1} = \overline{a_1}\) and \(\overline{a_1 a} = \overline{a_1} \cdot \overline{a} \forall \overline{a} \in X_*\)

(c) There is unique basis \(\{ A_w \mid w \in I_* \}\) of \(X_*\) such that

\[ \overline{A_w} = A_w \in a_w + \sum_{y < w} \overline{A_y} a_y \]

Can view \(\{ c_w \}\) as special case of \(\{ A_w \}\)
when \(W = W' \times W'\) and \(*\) interchanges factors.

While \(A_w \& \text{IN}(v)\)-span \(\{ a_w \}\) there are analogous positivity conjectures and many mysterious questions attached to this construction. Elias and Williamson have announced work generalizing their methods to this twisted context.
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