Groups of local characteristic p

U. Meierfrankenfeld

Masterclass on Classification Problems in Groups and Fusion Systems June 10-14, 2013
Definition

Let $G$ be a group and $p$ a prime.

- A $p$-local subgroup of $G$ is the normalizer of a non-trivial $p$-subgroup of $G$.
- $G$ has characteristic $p$ if $C_G(O_p(G)) \leq O_p(G)$.
- $G$ has local characteristic $p$ if $p$ divides $|G|$ and all $p$-local subgroups of $G$ have local characteristic $p$. 
Notation

From now on $p$ is prime, $G$ is a finite $\mathcal{K}_p$-group of local characteristic $p$ with $O_p(G) = 1$ and $S$ is a Sylow $p$-subgroup of $G$.

Goal

Understand and classify the finite groups of local characteristic $p$ with $O_p(G) = 1$.

Disclaimer

For $p$ odd we do not expect to be able to achieve a complete classification. Some groups with a relatively small $p$-local structure will remain unclassified.
Definition

Let $L$ be a finite group. A $p$-reduced normal subgroup of $L$ is an elementary abelian normal $p$-subgroup $Y$ of $L$ with $O_p(L/C_L(Y)) = 1$.

$Y_L$ is the largest $p$-reduced normal subgroup of $L$.

Notation

$tC$ is a maximal $p$-local subgroup of $G$ with $N_G(\Omega_1 Z(S)) \leq tC$ and $E = O^p\left(F^*_p(C_t(Y_t))\right)$

Definition

A finite group $L$ is $p$-minimal if a Sylow $p$-subgroup of $L$ is contained in a unique maximal subgroup of $L$ but is not normal in $L$. 
Theorem (The Small World Theorem)

One of the following holds:

1. $S$ is contained in a unique maximal $p$-local subgroup of $G$.
2. $E$ is contained in at least two maximal $p$-local subgroups of $G$.
3. There exists a $p$-local subgroup $M$ of $G$ with $S \leq M$ and $Y_M \nsubseteq Q$.
4. There exists a $p$-minimal subgroup $P$ of $G$ with $S \leq P$ such that $Y_P \leq Q$ and $\langle Y^E_P \rangle$ is not abelian.
5. There exist $p$-minimal subgroups $P_1$ and $P_2$ of $G$ with $S \leq P_1 \cap P_2$, $O_p(P_i) \neq 1$, $P_1 \leq ES$ and $O_p(\langle P_1, P_2 \rangle) = 1$.
6. If $P$ is a $p$-minimal subgroup of $G$ with $S \leq P$ and $P \not\leq \tilde{C}$, then $O^p(P) \sim q^2SL_2(q)'$, where $q$ is a power of $p$.
7. $p = 3$ or $5$ and there exist $p$-local subgroups $M_1$ and $M_2$ of $G$ such that $S \leq M_1 \cap M_2$, $O_p(\langle M_1, M_2 \rangle) = 1$ and, for $i = 1, 2$, $M_i \sim p^{3+3}SL_3(p)$. 
If $\tilde{C}$ is the unique maximal $p$-local subgroup of $G$ containing $S$, then either $\tilde{C}$ is a strongly $p$-embedded subgroup of $G$ or one can apply the local $CGT$-theorem to obtain a $p$-local subgroup of a very restricted structure. But we currently do not know whether this information will be enough to identify $G$. To avoid this problem we will assume from now on that $S$ is contained in at least two maximal $p$-local subgroups of $G$. 
We now distinguish two cases:

\(\neg E!\) There exist two distinct maximal \(p\)-local subgroups \(M_1\) and \(M_2\) with \(E \leq M_1 \cap M_2\).

\(E!\) \(\tilde{C}\) is the unique maximal subgroup \(p\)-local subgroup of \(G\) containing \(E\).

In the \(\neg E!\) we choose suitable subgroups \(L_1\) and \(L_2\) with

\[ E \leq L_1 \cap L_2 \quad \text{and} \quad O_p(\langle L_1, L_2 \rangle) = 1. \]

We then use the amalgam method to determine the structure of \(L_1\) and \(L_2\). Given \(L_1\) and \(L_2\) one should be able to identify \(G\) up to isomorphism.
The $E!$-case

**Definition**

A $p$-subgroup $Q$ of $G$ is called large, if $C_G(Q) \leq Q$, and

$$N_G(A) \leq N_G(Q) \text{ for all } 1 \neq A \leq C_G(Q)$$

**Lemma**

*Suppose $E$ lies in a unique maximal $p$-local subgroup of $G$, then $O_p(\tilde{C})$ is a large $p$-subgroup of $G$,***
Theorem (Structure Theorem)

Let $Q$ be a large $p$-subgroup of $G$ and $M$ be a $p$-local subgroup of $G$ with $Q \leq S \leq G$ and $Q \nmid M$. Put $M^\circ = \langle Q^M \rangle$, $\overline{M} = M/C_M(Y_M)$ and $I = [Y_M, M^\circ]$.

Suppose that $Y_M \leq Q$. Then one of the following holds.

1. $\overline{M}^\circ \cong SL_n(q), Sp_{2n}(q)$ or $Sp_4(2)'$ and $I$ is the corresponding natural module.

2. There exists a normal subgroup $K$ of $\overline{M}$ such that
   - $K = K_1 \times \cdots \times K_r$, $K_i \cong Sl_2(q)$ and
   - $Y_M = V_1 \times \cdots \times V_r$

   where $V_i := [Y_M, K_i]$ is a natural $K_i$-module.

3. $Q$ permutes the $K_i$’s transitively.

4. There exists a $p$-local subgroup $M^*$ of $G$ with $M \leq M^*$ and $M^*$ fulfills the previous case.
Suppose that $Y_M \not\subseteq Q$. Then one of the following holds:

- There exists a normal subgroup $K$ of $\overline{M}$ such that $K = K_1 \circ K_2$ with $K_i \cong SL_{m_i}(q)$, $Y_M \cong V_1 \otimes V_2$ where $V_i$ is a natural module for $K_i$ and $\overline{M}^\circ$ is one of $K_1$, $K_2$ or $K_1 \circ K_2$.

- $(\overline{M}^\circ, p, I)$ is as given in the following table:

<table>
<thead>
<tr>
<th>$\overline{M}^\circ$</th>
<th>$p$</th>
<th>$I$</th>
<th>$\overline{M}^\circ$</th>
<th>$p$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL_n(q)$</td>
<td>$p$</td>
<td>nat</td>
<td>$O_4^+(2)$</td>
<td>2</td>
<td>nat</td>
</tr>
<tr>
<td>$SL_n(q)$</td>
<td>$p$</td>
<td>$\wedge^2(nat)$</td>
<td>$\Omega_{10}^\pm(q)$</td>
<td>2</td>
<td>spin</td>
</tr>
<tr>
<td>$SL_n(q)$</td>
<td>$p$</td>
<td>$S^2(nat)$</td>
<td>$E_6(q)$</td>
<td>$p$</td>
<td>$q^{27}$</td>
</tr>
<tr>
<td>$SL_n(q^2)$</td>
<td>$p$</td>
<td>nat $\otimes$ nat$^q$</td>
<td>$M_{11}$</td>
<td>3</td>
<td>$3^5$</td>
</tr>
<tr>
<td>$3 \text{ Alt}(6), 3 \text{ Sym}(6)$</td>
<td>2</td>
<td>$2^6$</td>
<td>$2M_{12}$</td>
<td>3</td>
<td>$3^6$</td>
</tr>
<tr>
<td>$\Gamma SL_2(4), \Gamma GL_2(4)$</td>
<td>2</td>
<td>nat</td>
<td>$M_{22}$</td>
<td>2</td>
<td>$2^{10}$</td>
</tr>
<tr>
<td>$Sp_{2n}(q)$</td>
<td>2</td>
<td>nat</td>
<td>$M_{24}$</td>
<td>2</td>
<td>$2^{11}$</td>
</tr>
<tr>
<td>$\Omega_n^\pm(q)$</td>
<td>$p$</td>
<td>nat</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Theorem (The $H$-Structure Theorem)

Suppose that $Q$ is a large $p$-subgroup of $G$ and let $M$ be a $p$-local subgroup of $G$ with $Q \leq S \leq G$ and $Y_M \not\leq Q$. Then there exists $H \leq G$ such that $M^\circ S \leq H$, $O_p(H) = 1$ and $H$ has the same residual type as one of the following groups:

- A group of Lie-type in characteristic $p$.
- For $p = 2$: $M_{24}, He, Co_2, Fi_{22}, Co_1, J_4, Fi_{24}, Suz, B, M, U_4(3)$ or $G_2(3)$.
- For $p = 3$: $Fi_{24}, Co_3, Co_1$ or $M$. 
Let $Q = O_p(\tilde{C})$. For $L \leq G$ put $L^\circ = \langle Q^g \mid g \in G, Q^g \leq L \rangle$. In view of the $H$-structure theorem we assume from now on that $Y_M \leq Q$ for all $p$-local subgroups $M$ of $G$ with $S \leq M$.

**Theorem (The P!-Theorem)**

Let $P \leq G$ such that

(*) $S \leq P \leq G$, $P$ is $p$-minimal, $O_p(P) \neq 1$ and $Q \ntriangleleft P$.

Put $P^* := P^\circ O_p(P)$ and $Z_0 := \Omega_1 Z(S \cap P^*)$. Then

- $Y_P$ is a natural $SL_2(p^m)$-module for $P^*$.
- $Z_0$ is normal in $\tilde{C}$.
- Either $P$ is unique with respect to (*) or $P^* \sim q^2 SL_2(q)$. 

Suppose that there exists more than one subgroup $\tilde{P}$ of $G$ such that $S \leq \tilde{P}$, $\tilde{P}$ is $p$-minimal, $\tilde{P} \not\subseteq N_G(P^\circ)$ and $O_p(M) \neq 1$, where $M = \langle P, \tilde{P} \rangle$. Then $p = 3$ or $5$ and $M^\circ \sim p^{3+3^*} SL_3(p)$ for any such $\tilde{P}$.
Theorem (The Rank 2 Theorem)

Suppose there exists $p$-minimal subgroups $P_1$ and $P_2$ of $G$ with $S \leq P_1 \cap P_2$, $P_1 \leq ES$, $O_p(P_i) \neq 1$ and $O_p(\langle P_1, P_2 \rangle) = 1$. Then one of the following holds:

- $(P_1, P_2)$ is a weak BN-pair.
- The structure of $P_1$ and $P_2$ is as in one of the following groups.
  - For $p = 2$: $U_4(3).2^e$, $G_2(3).2^e$, $D_4(3).2^e$, $HS.2^e$, $F_3$, $F_5.2^e$ or Ru.
  - For $p = 3$: $D_4(3^n).3^e$, $Fi_{23}$, $F_2$.
  - For $p = 5$: $F_2$.
  - For $p = 7$: $F_1$. 

Copenhagen, June 10th, 2013
Theorem (The Isolated Subgroup Theorem)

Let $H$ be a finite group, $T \in \text{Syl}_p(L)$ and $P^*$ be $p$-minimal subgroup of $H$ with $T \leq P^*$. Put $K = \langle O^p(P^*)^H \rangle$ and

$$L = \langle R \mid T \leq R \leq H, \text{R is p-minimal, R} \neq P^* \rangle.$$ 

Suppose that $O_p(L) \nsubseteq O_p(P^*)$ and $P^*$ is narrow. Then $K/O_p(K)$ is quasisimple.

Corollary

Put $K = \langle O^p(\tilde{P})\tilde{C} \rangle$. Then $K/O_p(K)$ is quasisimple.