Exercises

Exercise 1. Prove Lemma 1.3: Let $t$ be an involution in a finite group $G$ and $L$ a component of $C_G(t)$. Suppose $L(G) = E(G)$. Then there exists a component $D$ of $G$ such that one of the following holds:

1. $L = D$.
2. $L < D = [D, t]$.
3. $D \neq D^t$ and $L = E(C_D(t)) = \{ dd^d : d \in D \}$ is the image of $D$ under the map $d \mapsto dd^d$.

Exercise 2. Prove Lemma 5.4: Let $F, F'$ be fusion systems on $p$-groups $S, S'$ and $\alpha : S \to S'$ a homomorphism of groups.

1. Assume $\ker(\alpha)$ is strongly closed in $S$ with respect to $F$. Then for each $P, Q \leq S$ and $\phi \in \hom_F(P, Q)$, the map $\phi^\alpha : P\alpha \to Q\alpha$ defined by $\phi^\alpha : x\alpha \mapsto x\phi\alpha$ is an injective group homomorphism independent of the choice of representative in $\ker(\alpha|_P)$.

2. If $\alpha : F \to F'$ is a morphism of fusion systems then $\ker(\alpha)$ is strongly closed in $S$ with respect to $F$.

In addition prove Lemma 7.1: Assume $S_0$ is a subgroup of $S$ normal in $F$ and $\theta : S \to S^+ = S/S_0$ is the natural map $\theta : x \mapsto x^+ = xS_0$. For $P, Q \leq S$ set $\hom_+(P, Q) = \{ \phi^+ : \phi \in \hom_+(PS_0, QS_0) \}$, where $\phi^+ = \phi^\theta$. Define $F^+$ to be the category whose objects are the subgroups of $S^+$ and with morphism sets $\hom_+(P^+, Q^+)$. Prove $F^+$ is a fusion system on $S^+$ and $\theta : F \to F^+$ is a surjective morphism of fusion systems with kernel $S_0$.

Exercise 3. Prove the assertions in Example 5.6: let $S$ be a quaternion group of order $m \geq 16$. Then $S$ has two conjugacy classes $Q_i^2$, $i = 1, 2$, of quaternion subgroups of order 8. Let $U$ be the universal fusion system on $S$. We define four subsystems of $U$ on $S$. Write $S$ for the system $F_S(S)$. For $i = 1, 2$, set $F_i = (\Aut(Q_i))$. Finally set $F_{1,2} = (\Aut(Q_1), \Aut(Q_2))$. Prove $S$, $F_1$, $F_2$, and $F_{1,2}$ are the four saturated fusion systems on $S$. Also prove there is an automorphism $\alpha$ of $S$ with $Q_1\alpha = Q_2$, and $\alpha : F_1 \to F_2$ is an isomorphism, so up to isomorphism there are exactly three saturated fusion systems on $S$.

Let $q$ be an odd prime power with $(q^2 - 1)_2 = m$, where $n_2$ is the 2-share of an integer $n$. Then $S$ is Sylow in the group $G = SL_2(q)$ and we will write $SL_2(m)$ for the fusion system $F_S(G)$. Prove $F_{1,2} = SL_2(m)$. Similarly if $q = q_0^2$ is a square then $G$ has a subgroup $H$ with $O^2(H) \cong SL_2(q_0)$ and $S$ Sylow in $H$, and if we choose notation so that $Q_1 \leq O^2(H)$, prove $F_S(H) = F_1$.

Exercise 4. Prove part (2) of Lemma 6.3: Assume $F$ is a saturated fusion system on $S$ and $Q \leq S$ is strongly closed in $S$ with respect to $F$. Assume there is $Q_1 \leq Q$ such that $Q_1 \leq F$, $Q/Q_1$ is abelian, and $\Aut_F(Q_1)$ is a $p$-group. Prove $Q \leq F$. 1
Exercise 5. Prove Lemma 11.1: Let $G$ be a finite group and $K$ a subgroup of $G$ of even order. Then the following are equivalent:

1. $K$ is tightly embedded in $G$.
2. For each nontrivial 2-subgroup $X$ of $K$, $X^G \cap K = X^{N_G(K)}$ and $N_G(X) \leq N_G(K)$.
3. For each involution $x$ in $K$, $x^G \cap K = x^{N_G(K)}$ and $C_G(x) \leq N_G(K)$.

Exercise 6. Prove Lemma 12.9: Assume $F$ is a saturated fusion system on a finite 2-group $S$ and $C$ is a quasisimple subsystem of $F$ on $T$. Adopt Notation 12.1; in particular $Q_0 = C_S(T)$. Assume $T \in \mathcal{F}^f$ and set $\Sigma = N_{\text{Aut}_F(Q_0T)}(T)$. Then

1. $\text{Aut}_F(T) = \text{Aut}_{\Sigma}(T)$.
2. $\Sigma$ acts on $Q_0$.
3. If $\sigma \in \Sigma$ and $X \in \mathcal{X}$ with $X\sigma \in \mathcal{X}$, then $\sigma|_T \in \text{Aut}(C)$.
4. If some characteristic subgroup of $Q_0$ is in $\mathcal{X}$ then $\text{Aut}_F(T) \leq \text{Aut}(C)$.
5. Suppose $Q \leq Q_0$ with $Q^\# \subseteq \mathcal{X}$ and $|Q| > |Q_0 : Q|$. Then $\text{Aut}_F(T) \leq \text{Aut}(C)$.

Exercise 7. Prove Lemma 13.2: Let $\mathcal{F}_0$ be a saturated fusion system on a 2-group $S_0$ and define a split extension of $\mathcal{F}_0$ as in Definition 13.1. Assume $(\mathcal{F}, Q)$ is a split extension of $\mathcal{F}_0$ and $Q$ is weakly closed in $S$ with respect to $\mathcal{F}$. Prove $Q \trianglelefteq \mathcal{F}$, so $\mathcal{F} = Q \times \mathcal{F}_0$. 