Lecture 5  
(2x45 min, 8 pages)

Global forms of K-theory, cont.

\[ L(ku) \rightarrow ku \rightarrow ku^c \rightarrow b(ku) \]

(b is better than R because defined on point set level)

\[ L(KU) \rightarrow KU \rightarrow b(KU) \]

\[ ku(n) = \mathcal{C}(S^0, Sym(V_n)) \]

gives \( ku(R) \cong U \) in particular, via eigenvalue/eigenspace decomposition.

\( \langle > : R(G) \rightarrow \pi_0^G(ku) \) next:

Let \( W \) be a complex \( G \)-rep in (unitary). Choose a real rep in \( V \) and a complex \( G \)-embedding \( W \rightarrow V_\delta \). Let

\[ \langle w \rangle \in \pi_0^G(ku) \]

be

\[ S^W \rightarrow ku(v) \]

\[ \sigma \rightarrow [\nu, W] \]

\[ W \circ V_\delta \in Sym(V_\delta) \]

**Facts:**

\[ \langle w \rangle \text{ only depends on the iso class of } W \]

\[ \langle w \rangle + \langle w' \rangle = \langle w \oplus w' \rangle \]

\[ \alpha^* \langle w \rangle = \langle \alpha^* w \rangle \]

If \( H \leq G \) and \([G:H] < \infty\) then

\[ \operatorname{tr}_H^G \langle w \rangle = \langle \operatorname{tr}_H W \rangle = \langle CG \circ W \rangle \]

\[ \langle w \rangle \cdot \langle w' \rangle = \langle w \otimes w' \rangle \]

(Not equal on representatives since \( W \otimes W \) lands in the linear part, but tensoring representatives lands in quadratic part, so a homotopy is needed.)

\[ P^m \langle w \rangle = \langle W^m \rangle \text{ in } \pi_0^{2m}(ku) \]

\[ N^G_H \langle w \rangle = \langle W^G H \rangle \text{ (tensor induction)} \]

Well defined additive

\[ R^G(G) \rightarrow \pi_0^G(ku) \]

\[ R(G) \]

induced.
Prop: $R(G) \rightarrow \Pi_0(\text{ku})$ is split mono & is iso for $G$ finite.

For $dim G > 0$ the discrepancy has to do with infinite index transfers, as with $H\mathbb{Z}$ and $H\mathbb{Z}$.

Segal: Let $H \hookrightarrow G$ be mono hom of prof. gps. There is a smooth induction

\[ i^*: R(H) \rightarrow R(G) \]

First Hour, that's $\infty$ dim, then work on it to get a virtual rep'.

A character formula!

Warning: $\langle \rangle: R(\cdot) \rightarrow \Pi_0(\text{ku})$ does not take smooth induction to the homotopy theoretic transfer. So not a hom of global functors. (Bob Oliver has a homotopy theoretic smooth induction. May be useful.)

Ex: $G = SU(2) \hookrightarrow T$. Then $i^*_1(1) = 2$ but in $\Pi_0^{SU(2)}(\text{ku})$.

$\text{tr}_{SU(2)}^T(1) \neq 2$; difference detected by $\text{dim: } \Pi_0(\text{ku}) = \Pi_0(\text{HZ})$.

N.B.

\[ H\mathbb{Z} \rightarrow b(H\mathbb{Z}) \]
\[ \text{ku} \rightarrow \text{ku}^c \]
\[ \text{KU} \]

Problem goes away here: really get $R(G)$ in degree 0.
Periodic global $K$-theory

Homotopy types go back to Segal (and Atiyah?) but the precise version is (almost) that of Michael Joachim. $C^*$-alg make the construction more natural.

**Ingredients:** $\mathcal{C}(V) = \text{Clifford algebra of } V$ (complexified)

$$\dim \mathcal{C}(V) = 2^\dim_{\mathbb{R}}(V)$$

$\mathcal{C}$ is a $2^{1/2}$-graded $C^*$-algebra

$$\mathcal{C}(V) \otimes \mathcal{C}(W) \cong \mathcal{C}(V \otimes W) \quad (\text{e.g. } \mathcal{C}(V^\otimes V) = \mathcal{C}(V))$$

Next need compact operators $\mathcal{C}$ the Hilbert space should vary nicely with $\mathcal{C}$.

$H_V = \text{Hilbert space completion of } \text{Sym}(V_0)$

$$= L^2(V) \quad (\text{Joachim's version})$$

Sym $(V_0) \otimes \text{Sym} (W_0) \cong \text{Sym} (V_0 \otimes W_0)$ induces $H_V \otimes H_W \cong H_{V \otimes W}$

**Def:** $s = C_0(\mathbb{R}) = \text{cont. complex-valued functions on } \mathbb{R}$ vanishing at $\infty$

(again, $2^{1/2}$ graded by even/odd)

**Def:** $KU(V) = \text{map}_{2^{1/2}, C^*} (s, H_V \otimes \mathcal{C}(V))$

(Eigenvalues can cluster at 0, otherwise discrete)

 TODO config space model (which looks unnatural?)

**NB.** $s$ is not commutative as a $2^{1/2}$-graded object.
N.B. Graded here means individual homogeneous parts, NOT their direct sum.

What Stefan would like to do is to write down the Bott class and its inverse, & multiply them to get 1. Even non-equivariantly.

Next, product of $\text{KU}$ and $\text{ku}^\mathbb{C} \to \text{KU}$.

**Multiplication:** $s \xrightarrow{\Delta} \text{seos}$ exists; meaning slightly obscure.

$\Delta$: morphism of $\mathbb{Z}/2$ graded $C^*$-algebras.

For $f \in C_0(\mathbb{R})_{\text{even}}$

$$
\Delta(f)(x,y) = \begin{cases} 
  f(x^2+y^2) & \text{f even} \\
  \frac{x+y}{\sqrt{x^2+y^2}} f(x^2+y^2) & \text{f odd}
\end{cases}
$$

**Coconum & Coassoc**!

(In the literature it is usually presented only on generating functions.)

$\mu_{\text{sw}} : \text{KU}(V) \times \text{KU}(W) \to \text{KU}(V \oplus W)$ is

$$
C^*(s, H_V \otimes \text{Cl}(V)) \otimes C^*(s, H_W \otimes \text{Cl}(W)) \xrightarrow{\otimes} C^*(\text{seos}, H_V \otimes \text{Cl}(V) \otimes H_W \otimes \text{Cl}(W)) \xrightarrow{\Delta^*}
$$

$$
C^*(s, H_{V+W} \otimes \text{Cl}(V+W))
$$

This gives an ultra-commutative ring spectrum $\text{KU}$. 

$\exists$
Thm (Joachim) Let $G$ be a compact Lie group, $V$ a faithful $G$-representation s.t. $\text{Sym}(V_G)$ is a complete universe $\,^{(\ast)}$. Then

$$\text{map} \,(S^V, KU(V)) \cong \mathcal{B}_{\mathcal{U}} \,(U_G)$$

(paraphrase of Joachim's statement)

Cor. $\pi^G_0(\mathbb{K}U) \cong [S^V, KU(V)]^G$ for $V$ large

$$\cong \pi^G_0 \mathcal{B}_{\mathcal{U}} \cong \mathcal{R}(G)$$

Then, the map $\mathcal{R}(G) \xrightarrow{\sim} \pi^G_0(\mathbb{K}U) \xrightarrow{\pi^G_0(\mathcal{U})} \pi^G_0(\mathbb{K}U)$ must be an iso.

Def. $\mathcal{U}$ the morphism of ultra-commutative ring spectra $\mathbb{K}U \to KU$.

($\text{à la Peter Teichner}$)

$$\mathbb{K}U(V) = C(S^V, \text{Sym}(V_G)) \quad \xrightarrow{\text{KU}(V)} \quad C^\ast(S, \mathbb{K} \otimes \mathcal{C}(V))$$

This requires functional calculus:

you have a function on the spectrum of an operator $\mathcal{U}$ and you produce the operator which has the effect on spectrum.

$$f_c : S^V \xrightarrow{\text{fc}} C^\ast(S, \mathcal{C}(V))$$

$$f_c(V)(g) = \begin{cases} g(m) \cdot 1 & \text{g even} \\ g(\nu) \cdot \frac{\nu}{|\nu|} & \text{g odd} \end{cases}$$

($\ast$) Is this automatic from the fact that $V$ is faithful?

Such are cofinal in any case.
Define \( j(V) : ku(V) \to KU(V) \) by
\[
[v, \ldots, v_n, E_1, \ldots, E_n] \mapsto \left\{ \begin{array}{c}
g \mapsto \prod_{i=1}^n \left( \text{Proj}_{E_i} \otimes \text{fo}(v_i)(g) \right) \\
\in C_k(V)
\end{array} \right. \\
\text{function of calculus applied to mult by } * \to \nu
\]

\( O(V) \)-equivariance not too hard.
Well defined used \( \text{Proj}_{E_i} + \text{Proj}_{E_i} = \text{Proj}_{E_i \oplus E_i} \) since they're orthogonal

Similarity of \( A(f) \) and \( f_c \) makes the map multiplicative.

Unit
\[
S^V \to ku(V)
\]
\[
v \mapsto [v, C] \mapsto \left\{ g \mapsto \text{Proj}_{E_i} \otimes \text{fo}(v_i)(g) \right\}
\]

Note: we're landing in finite rank, not just compact, operators.

Intuitive picture: over \( S^* \) we have Clifford submodules supported at a finite subset \( w \); symmetry: at \( t \) & \(-t\) you get opposite subspaces whose sum is a graded submodule. At \( 0 \) you have an actual graded Clifford module. At \( \infty \), new reps appear.

1. Real and unitary version ought to be possible
2. Completion map is naturally, but no help w/ proof

Dress-Atiyah complex? \( \text{Real} \)

Forget & right-adjoint
Snith splitting \( \Sigma^\infty_+ P \rightarrow KU \)

\( P^e(v) = P(\text{Sym} V_0) \) (avoids \( \mathcal{C}^\infty_+ + \mathcal{C}^\infty_+ = \mathcal{C}^\infty_+ \) problems, recall)

\( P = P^e = B_{ge} \mathcal{U}(i) \)

In literature false equivariant claims of Snith splitting.

\( P(\text{Sym}(V_0))_+ \wedge L^v \rightarrow \text{ku}(v) \)

\( L \wedge v \rightarrow [v, L] \)

Bott class

\( \beta \in \pi_2^e(\Sigma^\infty_+ P) = \pi_2(\Sigma^\infty_+ \mathcal{C}^\infty_+) \) classifies \( \gamma_e - 1 \)

\( \Sigma^\infty_+ P \) \([\beta^2] \cong KU \)

\( \times \text{dim} \) \( \left( \begin{array}{c} \Sigma^\infty_+ P \\ \Sigma^2 \Sigma^\infty_+ P \\ \Sigma^3 \Sigma^\infty_+ P \\ \vdots \end{array} \right) \)

A universal property: appropriate spectrum invariant this elt. 

Colim may not create this in global setting.

If we have norms & powers, we have to invert more than just one class, all its norms & powers as well.

Ostvaer & Spitzweck

\( G = A \) abelian opt Lie \( \Sigma^\infty_+ P(U_A)[\beta^2] \cong KU_A \)

Their proof would work if you inverted Bott classes for all irreducibles.

What should work globally?
Conj: \[ (\sum_{+} P) \left[ \frac{1}{\beta_{u,v,0}} \right] \rightarrow KU \] is an Ab-global equivalence

\[ \beta_{u,v,0} = \text{Bott class of tautological } U(1) \text{ bundle} \]

inverting in this category inverts all norms & powers, etc.

Expect on &p: it inverts all norms of all restrictions.

Another property: KU is right induced! (labeled May 8...?)

(*) \[ KU^*_G(X) \cong K^*_G(X \times E(G, cyclic)) \] (characters determine rings)

\[ L \quad \Rightarrow \quad R \]

\[ Y \quad \Rightarrow \quad YH_{cyc} \]

says it is in image of R.

Ab is closure of cyc under products

Conj would then give \[ KU \cong R_{H^e} \left( \sum_{+} P \left[ \frac{1}{\beta_{u,v,0}} \right] \right) \] as Real ultra comm ring spectra.

Extremely highly structured version of Snaith’s theorem.

(*) Does HKR theory say higher chromatic theories are right induced?

--- The End ---