Examples today.

- Orthogonal spectrum, $G$ opt Lie, $\pi^G_0(x) = \text{colim} \left[ \Sigma^y X'h v \right]_v \text{ for } y \in U_0$ up to preferred iso
- $X \rightarrow Y$ is a global equivalence $\iff \pi^G_k(p) \text{ iso } V_k \in U \forall G$ (in family)
- $\mathcal{D} = \Delta \left[ \text{gl equiv}^{-1} \right]$

Burnside category $A$: obj = cpt Lie gpd
- mor $A(G,K) = \text{Nat}(\pi^G_0, \pi^K_0)$

**Global functor:** $F : A \rightarrow \text{Ab}$ additive

\[ \Delta \pi^G_0 \rightarrow [\pi^G_0, k] \text{ empty, } \]

For calculations, need a basis:

\[ A(G, K) = \mathbb{Z} \langle \pi_0^G \alpha^k \mid [L, S], K \geq l \rightarrow G, \quad \|W_k(L)\| < \infty \rangle \]

Ex. $\mathcal{D} \rightarrow \text{Functor (Rep, Set)}$

- forget $\sim, \pi$ - $\Delta$ left adjoint

$\Sigma^+ \text{ freely builds in the Rep structure}$

For $Y \in \text{OrthoSpace}$, unit $Y \rightarrow \mathcal{L}(\Sigma^+ \rightarrow Y)$ induces a morph

- $\mathcal{D} \text{ Rep }$ functors

\[ \pi^+ Y \rightarrow \pi^0 (\mathcal{L} \Sigma^+ Y) = \pi^0 (\Sigma^+ Y) \]

adjoint to a morphism $\mathcal{D}$ global functors

**Thm:** $\Lambda(\pi^0 Y) \rightarrow \pi^0 \Sigma^+ Y$

**Pf:** tomDreck splitting

(Classically $\pi^0 \Sigma^+ Y$ is $\mathbb{Z}[\text{set } \pi^0 Y]$.)

More concrete, special case...
Ex: \( B_{\mathfrak{g}} G = L(V, -) / G \) is the global classifying space

\[ \pi_0^k(B_{\mathfrak{g}} G) = \text{Rep}(K, G) \quad \text{induced by}\quad \text{find char} \in \pi_0^G(B_{\mathfrak{g}} G) \]

i.e.

\[ \pi_0 B_{\mathfrak{g}} G \cong \text{Rep}(-, G) \]

\[ \Rightarrow \quad \pi_0(\Sigma^\infty B_{\mathfrak{g}} G) = A(G, -) \]

In other words, proj gens are the \( \Sigma^\infty B_{\mathfrak{g}} G \).

Fact: Choose a set of \( \text{an} \) ol. of pt lie \( \mathfrak{g} \), e.g. subgps of \( O(n) \)'s.

\( \{ \Sigma^\infty B_{\mathfrak{g}} G \} \) is a set of compact generators for \( \mathcal{Y} H \)

\[ \mathcal{Y} H(\Sigma^\infty B_{\mathfrak{g}} G, X) \cong \pi_0^G(X) \quad \text{has} \quad \mathfrak{g} \quad \text{as a corollary.} \]

More special case: \( G = e \), \( O = V \) is our faithful rep'n. Then \( B_{\mathfrak{g}} e = \star \)

So \( \Sigma^\infty B_{\mathfrak{g}} e = S \)

\[ V \rightarrow L(o, V)/e + \wedge S^V = S^V \]

So \( \pi_0 S = A(e, -) \) is the classical Burnside ring.

Next class of examples, \textcolor{red}{\underline{global Borel theories}}

Let \( E \) be a non-equivariant scheme theory. Then \( G \rightarrow E^0 B_6 \) is a global functor. (Feshbach for \( tr \)). This is realized by the right adjoint

\[ \mathcal{Y} H \xrightarrow{\text{Forget}} \mathcal{S} H \]

No closed formula for \( \pi_0 L_X \), but Thm: \( \pi_0^G RE \cong E^0 B_6 \) natural in \( G \), iso \( \pi_0^G RE \cong E^0 B_6 \) of global functors.

(requires a specific point set level construction of \( R \). No \( \rightarrow \))
\[ \mathcal{Y} \mathcal{H}(\Sigma_{+} B G, \text{RE}) = \mathcal{H}(U(\Sigma_{+} B G), E) \]

\[ \prod_{0}^{\text{RE}} E^{0} B G \quad \text{formally.} \]

N.B. L is strong symm monoidal
R is lax (at point set level)

(usual \& for orthog spectra is derivable w.r.t. global equivalences)

Point set level model for R

E an orthog. spectrum. Let bE be an orthog. spectrum by
\[
(bE)(V) = \text{map}(L(V, R^\infty), E(V)) \quad \text{w/ diag } O(V) \text{-action}
\]

Then
\[
(bE)(V) \wedge (bF)(W) \longrightarrow b(E \wedge F)(V \oplus W)
\]

is
\[
\text{map}(L(V, R^\infty), E(V)) \wedge \text{map}(L(W, R^\infty), E(W)) \quad \longrightarrow \quad \text{map}(L(V \oplus W, E(V)) \wedge F(W))
\]

\[
\text{map}(L(V, R^\infty) \times L(W, R^\infty), E(V) \wedge F(W))
\]

giving
\[
bE \wedge bF \longrightarrow b(E \wedge F).
\]

Thus: If E is a non-equiv. \& spectrum then bE is a global L-spectrum and b realizes R.

Adjunction unit: \[
E(V) \longrightarrow \text{map}(L(V, R^\infty), E(V))
\]

Note if V is faithful for G then \[L(V, R^\infty)\] is an EG.
If \( R \) is a comm orthog ring spectrum then
\[ \pi_0 R \] also has products, power operations, & norm maps.
(since \( \pi^i R \) is a c.o.m.s.)
\[
\text{map} \left( S^X, R(N) \right) \times \text{map} \left( S^X, R(M) \right) \to \text{map} \left( S^X S^Y, R(N) \times R(M) \right)
\]
\[ \Rightarrow \]
\[ \text{map} \left( S^X S^Y, R(N \times M) \right) \]

Norms & power operations come from comm orth monoid space structure on \( \pi^i R \).
\[ P^m \left( S^v \overset{f}{\to} R(v) \right) = \left[ \begin{array}{c}
\begin{array}{c} P^m \left( S^v \overset{f}{\to} R(v) \right) \\
\text{m}
\end{array}
\end{array} \right] 
\]

\( R \) comm orth ring spectrum \( \Rightarrow \) the power operations in \( \pi_0 \text{br} R \) are the power operations in \( G \overset{\text{m}}{\to} R^{G G} \).

Global power functor more convenient than Tambara functors because the power operations have better explicit formulas. [They are equivalent categories.]

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Example: Constant global factors. \( M \) an abel gp.

\( \) (like constant Mackey functors: exact in reach direction forces to direction)
\[ M \left( G \right) = M, \quad x^* = \text{Id}, \quad \pi^G_M = \text{mult. by} \ X(M/4) \]

Only check that isn’t obvious is double coset. It is true.

\[ \mathbb{E} \text{Eilenberg-MacLane spectrum } H M \in \mathbb{H} \quad \pi_k H M = \begin{cases} M & k = 0 \\ 0 & k \neq 0 \end{cases} \]

Most obvious point-set model is not the constant \( H M \) ! Is this the right thing? We define
\[ H_{\mathbb{Z}}(\nu) = \mathbb{Z}[s^v] = \text{reduced function on } s^v \]

\[ \uparrow \quad (M\text{-space easiest}) \quad (\text{basept = o as reduced}) \]

\[ \text{non-eg E.M. space} \quad \text{& type } (\mathbb{Z}, \dim(M)) \]

\[ \mathbb{Z}[s^v] \wedge \mathbb{Z}[s^w] \longrightarrow \mathbb{Z}[s^{v \wedge w}] \]

\[ \Sigma_{g;i} \wedge \Sigma_{b;w} \longrightarrow \Sigma_{g;b; (v; w)} \]

Segal, Shimakawa: \( G [\text{finite}] \Rightarrow \mathbb{Z}[s^v] \) is an equiv. E.M. space for the constant Mackey functor \( \mathbb{Z} \):

\[ \mathbb{Z}[0] \]

map \( G \times (s^v, \mathbb{Z}[s^v]) \rightarrow \mathbb{Z} \)

Proof fails for non-discrete \( G \), so

\( H_{\mathbb{Z}} \) is a \( \mathbb{Z} \)-global \( \mathbb{Z} \)-spectrum

\[ \pi_0^G(\mathbb{Z}) = \mathbb{Z} \quad \text{for } G \text{ finite but } \pi_0^G \mathbb{Z} \neq \mathbb{Z} \]

\[ \text{If: Consider } G = SU(\mathbb{C}) \cong T = \{e^{i\theta} x^i \} \text{ max torus} \]

\[ \Theta_{\mathbb{Z}}(T) \wedge 2 \in \pi_0^{SU(\mathbb{C})}(H_{\mathbb{Z}}) \]

\[ SU(\mathbb{C})/T \cong \mathbb{C}^1 \text{ w/ } \chi(SU(\mathbb{C})/T) = 2, \text{ so in } \mathbb{Z} \quad \Theta(1) = 2 \]

But here this fails: pass to geom f.p.

\[ \Phi: \pi_0^{SU(\mathbb{C})}(H_{\mathbb{Z}}) \longrightarrow \pi_0^{SU(\mathbb{C})}(H_{\mathbb{Z}}) \]

\[ = \text{cotim} \left[ s^{SU(\mathbb{C})}, H_{\mathbb{Z}}(s^\nu) \right] \]

\( \Phi \) kills all transfers from proper subgroups, so \( \Phi(\Theta(1)) = 0 \).
HZ(\mathbb{S}^y) points are finite sums of points, so to be \text{SU}(2) invariant, we must have (since \text{SU}(2) conn.)

\[ HZ(\mathbb{S}^y)_{\text{SU}(2)} = HZ(\mathbb{S}^{y_{\text{SU}(2)}}) \]

So

\[
\text{cdim} \left[ \mathbb{S}^{y_{\text{SU}(2)}}, HZ(\mathbb{S}^{y_{\text{SU}(2)}}) \right] \ni \mathbb{Z}
\]

and so \( \pi_2(2) = 2 \neq 0 \).

What does this mean? Comm product on \( HZ \) less explicit than on \( HZ \). Which is the good \( HZ \)?

In fact, here \( \text{tr} \frac{SU(2)}{\mathbb{F}_2} \) and 1 are lin indep. Result is rank 2.

\[ \text{general result} = \text{Burnside ring} \quad \text{finite index torsor} = \text{their index} \]

Closed form versions (e.g. \( HZ \)) should not be ignored.

So perhaps \( HZ \) is "wrong". Similar problems

\( HZ \) is the global delooping of \((\mathbb{Z},+)\) so have built in infinite index norms freely, ignoring the existing Euler characteristic \( \chi \) norms.

Note \( \mathbb{F}_2 \)

\[ HZ \rightarrow HZ \]

\underline{Global forms of top \text{K}-theory}

7 different global homotopy types

\[ \begin{array}{c}
L(\mathbb{K}u) \quad \text{homotopy ring spectrum but not pointed level ring spectrum} \\
L(\mathbb{K}u) \quad \text{not incomm ring spectrum}
\end{array} \]

\[ \begin{array}{c}
R(\mathbb{K}u) \quad \text{Grothendieck global level} \\
R(\mathbb{K}u) \quad \text{Borel K-theory}
\end{array} \]

\[ \text{ultra}- \text{comm ring spectrum} \]

[Diagram with arrows and labels]
Let $U = \text{ ex hermitian i.p. space if possibly infinite dim } (\equiv C^n, n \to \infty)$

For based space $X$ let

$C(X, U) =$ space of finite configurations on $X$ labelled by

pairwise orthogonal f.d. subspaces of $U$
\[ \prod_{n \geq 0} X^n \times \operatorname{sub}^n(U)/U \]

n-tuples of pairwise orthogonal subspaces

\[ \kappa u(v) = G(S^v, \operatorname{Sym}(V^o)) \text{ w/ diagonal } O(v) \text{-action} \]

new ingredient, \( v \) in both \( j \) One go at a time used fixed \( v \) in \( \operatorname{Sym}(V^o) \) which was a \( G \) -universe. If we try to do that globally we get \( C^0 \times C^0 \neq C^0 \). This is an easier fix than using subrings.

\[ \mu_{v,w} : \kappa u(v) \times \kappa u(w) \longrightarrow \kappa u(v \oplus w) \text{ is} \]

\[ [v_0, \ldots, v_n, E_1, \ldots, E_n] \times [w_0, \ldots, w_n, F_1, \ldots, F_m] \longrightarrow [v_0w_0, \ldots, E_1 \otimes F_j] \]

using canonical natural \( \operatorname{Sym}(V^o) \otimes \operatorname{Sym}(W^o) = \operatorname{Sym}((V \oplus W)^o) \)

\[ \dim : \kappa u \longrightarrow HZ \]

\[ [v_0, \ldots, v_n, E_1, \ldots, E_n] \longrightarrow \sum \dim(E_i) v_i \]

\( \text{morphism of ultra-comm ring spectrum.} \)

\[ \kappa u \longrightarrow \text{borel theory } \bigotimes \text{ HZ, not to HZ.} \]

**Theorem:** For finite \( G \), \( \kappa u(G) \) is \( G \)-equivariant connective \( K \)-theory.

**Proof:** Shima kaun's 0 loop space machine. \( \square \)

**Tomorrow:** \[ \operatorname{Rep}(G) \longrightarrow \mathcal{T}_0^G(\kappa u(G)) \]
Question session

What does this have to do with $K$-theory?

$k_0(R) = C(\text{ult}, C^\infty) \cong \mathbb{U}$

\[ [\lambda_1, \ldots, \lambda_n, E_1, \ldots, E_n] \rightarrow \phi \text{ with eigenvalue, eigenspace pairs } \lambda_i, E_i \]

(and I outside)

similarly $\leftarrow \phi$

(from a paper by Suslin, short paper, on $G$-equivariant alg. $K$-theory. probably known to Segal, etc.)

$\mathcal{F}$ c* alg version

Segal uses a constant universe: okay non-equivariantly & if you don't care about a product. These two issues seem to arise together.

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tom Dieck splitting: $X$ a based $G$-CW-spectrum

$$\pi_*^G(\Sigma^\infty X) \cong \bigoplus T_{\text{tomDieck}}^H(EW^+_H \wedge X^H)$$

$WH = W_G H$, Weyl group

115 (Adams 50)

$\bigoplus \pi_*^H(EW^+_H \wedge \Sigma^H \wedge X^H) / WH$

Diagonal $WH$ action

$X = S^0$ $\Rightarrow$ Equivalent $\pi_0$ is Burnside ring

$\pi_0^G(S) = \bigoplus \pi_0^H B(\text{WH})_+$

$\cong \bigoplus \mathbb{Z}$

The map uses Wirthmüller, transfer, ... Not elementary.

(Segal ICM talk) (may have simple description)

(Recommend tom Dieck's original paper for excellent exposition.)
$H = \mathcal{G}$ summand $= \text{Geom f.p.}$

others are transfers