Lecture 3
(2x45 min, 8 pages)

Start again w/ the stable situation.

Previous approaches:

Lewis-May § I section encoding compatibility; change [group ⨿ universe] functors
Greenlees & May: Completions ⨿ MU modules set up formalism to study norm maps,
indexing on (O, ν)
Böhmann: made more explicit, an equivalent category

Def: (formerly ⨿ ⨿ prefuntors)

An orthogonal spectrum consists of
- based spaces \( X_\nu \), \( \nu \geq 0 \)
- \( O(\nu) \) actions on each \( X_\nu \),
- structure maps \( \sigma_\nu : X_\nu \wedge S^1 \to X_{\nu+1} \)

such that \( \nu, m \geq 0 \):

\[
X_\nu \wedge S^m \to X_{\nu+m} \wedge S^{m-1} \to X_{\nu+2} \wedge S^{m-2} \to \cdots \to X_{\nu+m}
\]

is \( O(\nu) \times O(m) \)-equivariant.

Coordinate-free version requires Thom complexes over complements.

\( V \) an inner product space, \( \dim V = n \), \( X \) orthogonal spectrum,

Def: \( X(V) := L (\mathbb{R}^n, V)^+ \wedge_{O(n)} X_\nu \) as an \( E(V) \) space.

Of course \( X_\nu \) in disguise but now coordinate-free.

Generalized structure maps

\[
X(V) \wedge S^W \to X(V \oplus W) , \quad O(V) \times O(W) \text{ equivariant.}
\]

No ⨿ mentioned yet, but implicit via rep's:

If \( G \) is a Lie group and \( V \) is a \( G \)-rep, then \( X(V) \) has \( G \) action via \( G \to O(V) \).

Note \( X(\mathbb{R}^n) = X_n \) is fair statement since isomorphism is so canonical.
we get equivariant stable homotopy groups.

**Def:** For \( k \in \mathbb{Z} \)

\[
\pi_k^G(X) = \text{colim}_{V \in \mathcal{U}_G} \left[ s^{V \oplus k} X(v) \right] \quad (k \geq 0; \ [s^V X(v)] \text{ if } k<0)
\]

\( \pi_k^G \) is a complete project of \( s^G \).

Formalism allowing pairs \((G, \text{universe})\) possible, but applications unclear.

These are morphisms in approp category: \([s^k X]\).

**Def:** \( X \rightarrow Y \) morphism of orthog. spectra is a global equivalence if

\[
\pi_k^G(f) : \pi_k^G X \rightarrow \pi_k^G Y
\]

is an isomorphism \( \forall k \in \mathbb{Z}, \forall G \).

Can restrict \( G \) to various subsets (e.g., finite) to get other notions of equivalence.

**Def:** The global stable homotopy category is \( \mathcal{S}^H = \mathcal{S}^e [\text{gl.equiv.}] \).

**Thm:** The global equivalences are part of a proper, cofibrantly generated, topological stable monoidal model category. The fibrant objects are the global omega spectra \((G\text{-spectra})\), i.e. the \( \mathcal{X} \) s.t. \( \forall G \) and \( \forall V, W G\text{-rep's}, \forall \text{faithful}, \) the map

\[
\tilde{\alpha}_{V,W} : X(v) \rightarrow \text{map} (s^W, X(v \oplus W))
\]

is a \( G \) equiv equivalence. \((*)\)

\[(*)\] Don't require a good homotopy type on non-faithful rep's. Meaningful values are only attained on faithful rep's. Requiring \( \tilde{\alpha}_{V,W} \) is equiv ...
Can characterize $\Sigma$-obj's via latching; Martin Stolz's $\Sigma$-obj is flat model structure.

**Note:** For $X$ a global $\Sigma$-spectrum, then $V G$, $H$-faithful $G$-rep's $V$, the space map $(\Sigma^V, X(V))$ is independent of $V$ up to $G$-we equiv and a $G$-infinite loop space.

**Note:** Global $\Sigma$-spectrum $\Rightarrow$ non-$\Sigma$ $\Sigma$-spectrum.

**Note:** $H \& G$, $V$ $G$-faithful $\Rightarrow$ $H$-faithful so downward compatibility easy. Upward depends on compatibility.

**Note** Global $\Sigma$-spectra are intrinsically interesting w/o model structure.

Every global equivalence is a stable equivalence (non-$\Sigma$). So this is a finer notion. So $\mathcal{F}$ forgetful functor, part $\mathcal{G}$ a recollement $\mathcal{G}H @ \mathcal{G}H \overset{L}{\underset{R}{\leftrightarrow}} \mathcal{L}H = \mathcal{L}p\left[\text{st.equiv}^{-1}\right]$.

$\exists X : \pi_x X = 0$ (non-$\Sigma$).

$L, R$ are fully faithful (need model cat structures, flat & proper).

$\mathcal{L}H$ is a localization of $\mathcal{G}H$; thinking of it as a subcategory is possible via $L$ and $R$; good for mapping out of or into, resp.

For a fixed $G$, we have $\mathcal{L}p \rightarrow \mathcal{G}\mathcal{L}p$ (orthog to $p^*$ w/ external $G$-action $X \overset{\mathcal{L}p}{\rightarrow} X \times G$ triv. $G$-action). It no longer an equivalence on pointed set level (as above), instead fully faithful, but that vanishes upon localizing.
\[ X \xrightarrow{f} Y \text{ g.l. eq. } \Rightarrow f \langle \sigma \rangle : X \langle \sigma \rangle \rightarrow Y \langle \sigma \rangle \text{ is a } G\text{-stable eq.viv.} \]

giving
\[ \mathcal{U} : YH \leftrightarrow Ho (G - Sp) \]

not a recollement. Has both adjoints but they're not fully faithful.

Not every \( G \) homotopy type is part of a global family. The global spectra are always split, for example.

For \( G \), take all subgps & quotients & work relative to that family. Still not a part of a recollement, L&R not fully faithful.

Proof in 2 steps: first a level model structure. Then localize \( G \) it is no longer just about \( O(n) \) but about \( G \) sitting in, in many ways.

Have to produce generators: take all closed subgps \& \( O(n) \), and that means all opt Lie \( G \). So just accepting coordinate-free approach, use all \( G \), works best.

Global family: closed under iso, subgp (closed), \& quotients.

All Opt Lie
\[
\begin{array}{c}
\text{All Finite}.
\end{array}
\]

\[ \left\{ \begin{array}{c}
\text{All Abelian Lie 6ps}
\text{All trivial Lie 6ps}
\end{array} \right\} \text{ most used examples.} \]

Odd note: Can mix families to get perverse examples.
Examples to come, but we need language for their properties.

One now: usual adjoint functor pair

\[ \begin{array}{ccc}
\text{Orthog. Spaces} & \xleftarrow{\Sigma^*} & \mathcal{L}p \\
\Lambda & \xrightarrow{\Sigma^*} & \Gamma
\end{array} \]

\[(\Sigma^* Y)(V) := Y(V), \wedge S^V
\]

\[\text{w/ diagonal } O(V)-\text{action}
\]

\[\text{B structure map}
\]

\[\Phi : Y(V) \to Y(V \otimes W)
\]

\[S^V S^W \xrightarrow{\otimes} S^{V \otimes W}
\]

\[(\mathcal{L}^* X)(V) = \text{map} (S^V X(V))
\]

\[\text{structure maps induced by conj}
\]

\[V \xrightarrow{\alpha} W \text{ in } L \text{ gives}
\]

\[\alpha_X : \text{map} (S^V X(V)) \to \text{map} (S^W X(W))
\]

\[V \xrightarrow{\otimes} W \xrightarrow{\otimes} X(V \otimes W)
\]

\[S^V \otimes S^W \xrightarrow{\otimes} S^{V \otimes W}
\]

\[\alpha_{X(W)} : X(V \otimes (W \otimes W)) \to X(V) \times S^{W \otimes W}
\]

\[\Rightarrow (\otimes \alpha_{X(W)\otimes W})(\alpha_X \otimes W)
\]

\[X(V \otimes (W \otimes W)) \Rightarrow \]

\[X(V) \times S^{W \otimes W}
\]

\[\Rightarrow \]

\[\Rightarrow X(W)
\]

For first orthog spaces

\[\text{unst. gl. eq} \mathcal{H} \to \text{st. gl. eq},
\]

\[\text{Compute: } \Pi_0^G (\mathcal{L}^* X)
\]

\[\text{roughly } % G \text{ structure exists here}
\]

\[\text{& other } % G \text{ inherently stable}
\]

\[= \text{colim} \Pi_0 (\mathcal{L}^* (X)(V))^G
\]

\[V \in U
\]

\[= \text{colim} \Pi_0 (\text{map}^G (S^V X(V))) = \text{colim} [S^V X(V)]^G = \Pi_0^G (X)
\]

This is an equality, not just an iso!

\[\alpha : K \to G \text{ cont hom induces (an additive) } \Pi_0^G (X) \to \Pi_0^K (X) \text{ of orthog sp. } X.
\]

Restriction along all cont homs is something that is new here.

Next, transfer maps, as in Mike Hill's talks.

\[\text{for } H \leq G \text{ get } \tau_H^G : \Pi_0^H (X) \to \Pi_0^G (X) \text{ as usual}
\]

\[\text{(N.B. no } [B: H] \text{ finite restriction)} \text{ (Only the degree 0 transfer.)}
\]
The Burnside category has objects G-sp\textsuperscript{G} and morphisms G-Mackey categories, with the global functor \( A(\mathcal{G}, K) = \text{Nat}(\pi_0^G, \pi_0^K) \).

No other choice: use all the operations we have. Obviously (pre-) additive.

**Def.** A global functor is an additive functor \( f : A \to \text{Ab} \).

**Ex:** Let \( X \in \mathcal{L} \) give \( \pi_0 X(\mathcal{G}) = \pi_0^G(x) \).

**Fact:** Every global functor \( F \) has an Eilenberg-MacLane spectrum \( HF \) such that:

\[
\pi_k(F) = \begin{cases} 
F & k = 0 \\
0 & \text{other}
\end{cases}
\]

Use t-structure. Global functors are the heart:

\[
\mathcal{G}H_{\leq 0} = \{ X \mid \pi_k^G X = 0 \text{ for } k < 0 \}
\]

\[
\mathcal{G}H_{> 0} = \{ X \mid \pi_k^G X = 0 \text{ for } k > 0 \}
\]

heart \( \mathcal{G}F \).

General hearts may not have enough proj, inj but \( \mathcal{G}F \) does (representables).

Fortunately, we can describe \( A(\mathcal{G}, K) \) w/ a presentation & see Mackey functors.

**Thm:** The group \( A(\mathcal{G}, K) \) is free abelian w/ basis the transformations \( \text{tr}_L^K \circ \alpha^* \)

where \( (L, \alpha) \) range through all \( K \times G \) conjugacy classes of

- subgroup \( L \leq K \) with finite \( W_K(L) \)
- cont. hom \( \alpha : L \to G \)

(conj by \( K \times G \) acts same op)

\[
\begin{array}{ccc}
\pi_0^G(X) & \xrightarrow{\alpha^*} & \pi_0^L(X) \\
\downarrow & & \downarrow \\
\pi_0^K(X) & \xrightarrow{\text{tr}_L^K} & 
\end{array}
\]
$\pi_0^G : \mathcal{Y} H \to \text{Ab}$ is represented by $\Sigma_+^\infty B G$

Then $\pi_0^K(\Sigma_+^\infty B G) G$ is then calculated by the van-Dieck splitting.

The double coset formula is the expression for composition in this basis.

If $\alpha$ is surjective, $\alpha : K \to G$, it is simple:

\[ \alpha^* \circ \text{tr}_H^G = \text{tr}_L^K \circ (\alpha|_L)^* \]

\[ \text{Upshot: To specify a global functor } M \text{ you must give} \]
- the abelian group $M(G)$ for each $G$
- restrictions $\alpha^* : M(G) \to M(K)$
- transfers $\text{tr}_H^G : M(H) \to M(G)$

satisfying
- restrictions are contravariantly functorial
- transfers are transitive
- (*) above
- double coset formula for $\text{res}_K^G \circ \text{tr}_H^G$
- $\text{tr}_H^G = 0$ if $|W_0 H| = \infty$

Consider $C_2$-Mackey functors

\[ \begin{array}{c}
\text{res} \\
\text{M}(e)
\end{array} \]

\[ \text{M}(e) \]

\[ \text{res} \circ \text{tr} = 1 + \gamma \]

\[ \text{Global functor at } C_2 = e \]

\[ \begin{array}{c}
\text{M}(e) \\
\text{res} \left( \begin{array}{c}
\text{M}(e) \\
\text{M}(e)
\end{array} \right) \text{tr} \\
\text{res} \left( \begin{array}{c}
\text{M}(e) \\
\text{M}(e)
\end{array} \right) \\
\text{M}(e)
\end{array} \]

\[ \text{p : } C_2 \to e \]

s.t.

\[ \text{res} \circ p^* = 1 \text{ (split epi: obstruct for a Mackey-functor to be part of a global-functor)} \]

\[ \text{algebraic reflection of split homotopy type.} \]
Question session: global factor at $G_3$,

$$\tau: G_3 \to G_3 \text{ by } \tau(x) = x^{-1}$$

Peter Webb calls these inflation functors. (The global factors?)

D. Symonds defines "regular global Mackey functors" via generators & relations.

$$r = 0 \text{ for } \infty \text{ weyl } g_1$$

For finite groups $A_{fin} \cong A^{comb}$ defined as

$$A^{comb}(G, K) = G \text{ gp } G \text{-free finite } K \text{-G-biset } K \infty$$

Then additive functors $A^{comb} \to A_{Ab}$ are called inflation functors (Webb)

Algebraists have studied this a lot.

Groups also considered these.

Note $A(e, K) = A(K)$ Burnside ring

Choice of variance here fit well with $\pi_0^G \to \pi_0^K$. Algebraists often want to study the effect on contravariant functors, hence choose the other variance.

End. See tomorrow