Computations in Equivariant Stable Homotopy

One group at a time.

Basics today. Computations, mainly tomorrow.

Rmk: \( G \) finite. Extension to not Lie?

**Def:** If \( G \) set \( X \) is a set \( X \) w/ homomorphism \( G \to \text{Aut}(X) \)

\[
\text{Set}^G : \quad \text{obj} = G \text{-sets} \\
\text{mor} = G \text{-maps } (X \to Y \text{ s.t. } f(gx) = gf(x) \forall g \in G, x \in X)
\]

Given \( x \in X \), the orbit \( G \cdot x = \{ gx | g \in G \} \). 

Stabilizer \( \text{Stab}(x) = \{ h \in G | h \cdot x = x \} \)

**Prop:** \( \text{G/\text{Stab}(x)} \xrightarrow{\pi} G \cdot x \) is a bijection of \( G \)-sets

\[ [g] \mapsto gx \]

Index everything by \( G \)-sets where we'd index by \( \mathbb{N} \) non-equivariantly.

**Space = simplicial set here.**

**Def:** A **simplicial \( G \)-set** is a simplicial object in \( \text{Set}^G \) or \( \text{Ens}^G = \text{Man}^G \).

Or, \( G \)-object in \( \text{Set}^G \), and they're the same.

**Ex:** If \( X \) is a \( G \)-space, i.e. space with its action of \( G \), then \( \text{Sing}^G(X) = \text{Map}(\Delta^n, X) \) defines a simplicial \( G \)-set.

**Geometric realization** of a simplicial \( G \)-set is a \( G \)-space. This provides a functor

\[
\text{Man}^G \to \text{Top}^G
\]

If we replace (in \( \Delta^n \)) the non-degen \( n \)-simplex by an orbit, ...
Def. A $G$-CW complex is a space which can be built by iteratively attaching cells of the form $T \times D^n$, $T$ a $G$-set, along their boundary. (Aesthetically simpler than writing $G/H \times D^n$—functoriality in $G$ is more evident.)

(Note attaching maps are $G$-maps.)

Classically, $\Sigma$ and $\text{Geom}$ realizations are Quillen inverses. This holds here.

If $H \leq G$ the restriction functor $i^*_H : \text{Set}^G \rightarrow \text{Set}^H$

$$\text{Top}^G \rightarrow \text{Top}^H$$

e tc.

Left adjoint of $i^*_H$ is induction: $X \mapsto G \times X \equiv G \times X / (g,h) \sim (g,hx)$, $\forall g, h, x$

So

$$\text{Maps}_H (G \times X, Y) \cong \text{Maps}_G (X, i^*_H Y)$$

$$\left( G \times X \rightarrow Y \right) \mapsto \left( \begin{array}{c} G \times X \rightarrow G \times X \rightarrow Y \\ x \mapsto e \times x \end{array} \right)$$

Aside:

I also a right adjoint, coinduction $F_H (G, X)$.

As sets

Note $G \times X$ is a disjoint union $\bigsqcup_G G / \sim \times X$

$$F_H (G, X) \equiv \prod G \times X$$

Build a $G$-object from an $H$-object by taking some smash mon product.

(In spectra, can use smash product & get norms)

For a $G$-space $X$, $\pi^G_0 (X) = \pi_0 (X^G)$.

If $H \leq G$ then $X^G \rightarrow X^H$, inducing map on $\pi_0$. Better,

$$\pi^G_0 (X) \rightarrow \text{Set}^G \rightarrow \text{Set}$$

by $\pi_0 (X) (T) = \pi_0 (\text{Map}_G (T, X))$.
usually called a coefficient system.

$$\Pi_0(X)(G/H) = \Pi_0 Map_G(\ast \times X, X) = \Pi_0 Map_H(\ast, X) = \Pi_0(X^H)$$

Def: A G-map $X \xrightarrow{f} Y$ is a weak equivalence if $f^H$ is a w.equiv $\forall H \leq G$.

Def: $\Pi_n(X)(T) = \mathbb{F}[T \wedge S^n, X]^G_G$ pointed, i.e. $[T \wedge S^n, X]^G$ where basept $n \in X$ is $G$-fixed.

[Worry: what if $X$ has more $G$ components than $H$ components.]
[Some controversy erupted at this point.]
[Stabilization erases the controversy.]

--- BREAK ---

For stabilization, we need the Spanier-Whitehead Category
Spectra, of all sorts, are factorizations of this.

Obj: finite $G$-CW complexes
Mor: "stable homotopy classes of maps" has dualizability up to shift, so add negative spheres, & limits & colimits.

Let $U$ be a universe for $G$, then

1. $\{X,Y\}^G_U = \varinjlim_{V \subset U} [S^V \wedge X, S^V \wedge Y]^G_V, \ S^V = V \cup \{\ast\}, \ 1\ point\ compactification.$

cfinally, $V \subset U$ indep of how $V$ sits in $U$

Note: trivial 1-dim nps: $1 \subset V$, so $R^0 \subset V$, so we get the usual properties of stabilization: these are abgps
- composition is linear
- $\{X, Y \wedge Z\}^G_U \cong \{X, Y\}^G_U \times \{X, Z\}^G_U$
So \( Y \times \mathbb{Z} \) is both product and coproduct now.

Can also introduce integer shifts

\[
\{ X,Y \}_{\mathbb{Z},n}^G = \begin{cases} \{ s^nX,Y \}_{\mathbb{Z}}^G & n \geq 0 \\ \{ X,s^nY \}_{\mathbb{Z}}^G & n \leq 0 \end{cases}
\]

Get a lot of other things, depending on choice of \( C, \mathbb{Z} \).

(2) If \( G/\Delta \) embeds in \( \mathbb{Z} \), a stable map \( \{ S^0, G/\Delta \}^G \rightarrow \{ \mathbb{Z}, \mathbb{Z} \}^G \)

which can be used to show

\[
\{ X, G/\Delta, Y \}^G_{\mathbb{Z}} = \{ s^*X, Y \}^{S^0}_{\mathbb{Z}}
\]

To understand this map, think about algebra: for \( m \equiv N \) we can get a fixed point \( \Sigma_{gm} \).

Would like to send base up to sum of points in its orbit \& stabilization is needed so we can add.

Picture for \( H = \mathbb{Z}/3 \leq G = C_3 \). Say

\[
\xrightarrow{V} \quad \xrightarrow{1}
\]

Rep\( \uparrow R \): triv \& 2-dim are the irreps.

\[
\begin{array}{c|c}
\bullet & \bullet \\
\end{array}
\]

Choose equivariant nhood \& use Thom-Pontrjagin map

\[
S^V \rightarrow C^+_3 \wedge S^V
\]

i.e. a stable map

\[
S^0 \rightarrow C^+_3
\]

(Why \( S^V \), not \( S^3 \)?

Ans: Can always write target as

\( G^+_3 \wedge S^V \) \& nhood \& fp. is \( H \)-equivariantly homeo to \( \Sigma_{gm} \).

(See: freeing up by \( G^+_3 \), it really gives the same result.)

(Verbal description of the isomorphism, no time to write.)
Cor: (Wirthmüller Isomorphism) Induction & Coinduction are stably equivalent. (Just as coproduct & product.)

Stabilization: chose a universe. Contained trivial. Get Wirthmüller, etc. Forced equivalences. To choose universe, decide which isomorphisms one wants to enforce. Getting certain kinds of limits as limits.

**Ex:** $\{S^n, \mathbb{C}^*\}_{n \in \mathbb{N}}$ is now more than a point.

Ex: $[S^p, S^p]^{G_p}$, $p = \text{regular rep'n}$, i.e. $C \cong C$. (A $G_p$ gives group structure)

$$C_{2^+} \wedge S^p \to S^p \to S^p$$

is a $C_{2^+}$-equivariant cof. sequence

$$C_{2^+} \wedge S^p \to [S, S^p] \to \mathbb{Z}$$

continue $C_{2^+}$-equivariant cof. sequence

$$[S_{2^+}, S^p]^{G_p} \leftarrow [S^p, S^p]^{C_{2^+}} \leftarrow [S^p, S^p]^{C_{2^+}} \leftarrow [S_{2^+}, \mathbb{C}^*, S^p]^{C_{2^+}} \leftarrow [S^p, S^p]^{G_p}$$

Key points:
1. $S^p \to C_{2^+} \wedge S^2\ w\ $ is $S^p \to C_{2^+} \wedge S^p$

So to take underlying information & produce fixed point information.

2. The $S^p \to S^p$ is $(S^p)^{G_p} \to S^p$ so it induced restriction to fixed points.
Each finite $G$ set is recording an induced piece:

$$A(G) = \mathbb{Z} \left\{ [H/H] \mid H \leq G \text{ up to conj} \right\}$$

$$\uparrow$$

$$\text{Tr}_H^G (H/H)$$

$$\ni$$

$$G \times_\mathbb{Z} (H/H)$$

There is more structure: finite $G$-sets are self dual,

$$\left\{ G/H \times X, Y \right\}_U \cong \left\{ X, G/H \times Y \right\}_U$$

because both are just $\left\{ X, Y \right\}^H_U$ so long as $G/H$ embeds in $U$.

$$\Rightarrow \left\{ T \times X, Y \right\}_U \cong \left\{ X, T \times Y \right\}_U$$

So $\left\{ X, Y \right\} (T) := \left\{ T \times X, Y \right\}_U$ is a functor on finite $G$-sets in two different ways, one covariant, one contravariant.

1. E. A Mackey functor: two functors $M^*, M^\ast$ s.t.

1. $M^*(T) \cong M^\ast(T) = M(T)$

2. $M(T \times S) \cong M(T) \oplus M(S)$

3. If $\mathbf{s} \rightarrow \mathbf{t}$ is a pullback then $\mathbf{m}^*(\mathbf{s}) \rightarrow \mathbf{m}(\mathbf{t})$ $\mathbf{m}^\ast(\mathbf{s}) \rightarrow \mathbf{m}(\mathbf{t})$
expresses the two ways to decompose

\[(G \times H) \times (G/K)\]

N.B. Only needed the Wirthmüller isomorphism, that induction & coinduction have been made equal.