Lie Groupoids
- group actions, foliations, differentiable stacks
  equivalence classes of Lie groupoids
- model of singular spaces
- models for NCC (convolution algebras).

**Definition:**
A groupoid is a small category in which every arrow is invertible.

- $M$, set of objects
- $G$, set of morphisms

$g \in G$, $y \xleftarrow{g} x$, $x = s(g)$, $y = t(g)$

Source target

$\forall x \in M$, $\exists 1_x \in G$

composition: $y \xleftarrow{g_1} x \xleftarrow{g_2} z$ if $s(g_1) = t(g_2)$

$\forall g \in G$, $\exists g^{-1}$ s.t. $gg^{-1} = 1$.

Let $G_2 = \{ (g_1, g_2) | s(g_1) = t(g_2) \}$ comparable arrows

Then $G_2 \xrightarrow{m} G$

multiplication
These maps should satisfy:

(i) associativity of \( m \) (whenever possible)
(ii) identity axioms for units.

**Definition:**

A groupoid \( \mathcal{G} \Rightarrow M \) is a Lie groupoid if \( M, \mathcal{G} \) smooth manifolds, \( s : \mathcal{G} \rightarrow M \) a smooth submersion and all other maps are smooth maps.

**Remark:**

(i) \( s \) smooth submersion \( \Rightarrow \mathcal{G}_2 \) smooth manifold

(ii) \( t = s \circ I \) \( \Rightarrow t \) is also a smooth submersion.

**Examples**

(i) \( M \) smooth manifold is a Lie groupoid up only unit arrows.

(ii) Pair groupoid.

\[ M \times M \xrightarrow{s,t} M, \quad s, t \text{ corresponding projections} \]

product \( (x, y)(y, z) \rightarrow (x, z) \)

(iii) \( f : Z \rightarrow M \) surjective submersion

\[ Z \times M \xrightarrow{s} Z \quad \text{subgroupoid of the pair groupoid of } Z \]

(iv) \( U = \bigcup U_i; \bigcup_{i \in I} \) open cover of \( M \), \( \bigcup_{i \in I} U_i \rightarrow M \).

Applying (i) we get \( \bigcup_{i,j} U_i \cap U_j \rightarrow \bigcup_{i} U_i \).
5. \( K \) Lie group. Then \( K \rightrightarrows \ast \) is a Lie groupoid.

6. \( K \times M, M \) manifold. Then\[
\begin{align*}
K \times M & \rightrightarrows M \\
\sigma(k, x) &= x \\
t(k, x) &= kx
\end{align*}
\]
\[ (k_1, m_1)(k_2, m_2) = (k_1 k_2, x) \]

7. Foliation groupoids.
\( F \subseteq TM \) integrable subbundle.
Frobenius Thm \( \implies M \) partitioned into leaves \( L \).

\underline{Monodromy groupoid} \( \text{Mon}(F) \rightrightarrows M \)

\( \forall x, y \in M \), an arrow between \( x, y \) is given by homotopy classes of paths in the leaves \( L \) of the foliation. This means that there are no arrows between \( x, y \) if they belong to different leaves.

\underline{Holonomy groupoid} \( \text{Hol}(F) \rightrightarrows M \)

\( \forall x, y \in M \), an arrow between \( x, y \) is given by holonomy classes of paths in the leaves.

\underline{Holonomy}:

1. A leaf of a foliation \( \gamma : [0,1] \rightarrow L \) a path, \( \gamma(0) = x, \gamma(1) = y \). Choose transversals \( T_x, T_y \) to the foliation near \( x, y \). The holonomy of \( \gamma \) is a germ of a diffeomorphism
\[
(T_x, x) \xrightarrow{\text{hol}(\gamma)} (T_y, y)
\]

Another way to describe the foliation is by giving a foliation atlas of \( M \). Let \( M = \bigsqcup U_i \), charts \( \phi : U_i \rightarrow \mathbb{R}^{n-i} \times \mathbb{R}^i \)

\( n = \dim M \), \( q = \text{codim} F \).
Transition functions $g_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y))$. These transition functions preserve the leaves $\mathfrak{g}_c^{-1}(\mathbb{R}^n \times \mathfrak{g}_S)$.

Remarks: homotopic paths have the same holonomy. This gives us a morphism between the monodromy and the holonomy groupoids:

$$\text{Mon}(F) \rightarrow \text{Hol}(F)$$

Let $G \rightarrow M$ be a Lie groupoid. This defines an equivalence relation on $M$ where $x \sim y \iff \exists g \in G$ s.t. $y = g \cdot x$.

$M/\sim$ "quotient space" or "coarse moduli space".

Actions of Lie groupoids:

An action of a Lie groupoid $G \rightarrow M$ on a manifold $Z$ is given by a map $\mu: Z \rightarrow M$ together with another map

$$Z \overset{\mu}{\underset{x}{\times}} M \rightarrow Z$$

$$(z, g) \mapsto zg$$

satisfying $(z, g_1)g_2 = z(g_1g_2)$ (whenever defined).

Example of an action

The groupoid acting on itself.

Take $Z = G$, $\mu = s: G \rightarrow M$. Then $G \overset{\mu}{\times} M \rightarrow G$.

We can take the action to be the multiplication map $G \overset{m}{\times} G \rightarrow G$.
Another example:

Consider $T^*G := \ker ds \subseteq TG$. Let $A = T^*G |_M$, a vector bundle over $M$.

Lemma: $G$ acts on $T^*G$ and $T^*G / G \cong A$.

Proof: Let $y \leftarrow^g x$ and consider $R_g : s^{-1}(y) \rightarrow s^{-1}(x)$, $h \mapsto hg$

Note that $R_g$ is a diffeomorphism.

By the submersion theorem, $(\ker ds)_h \cong T_h(s^{-1}(sh))$. This isomorphism gives us

$$dR_g : T^*G \rightarrow T^*G$$

This defines the action on $T^*G$. This action lifts the right action of $G$ on itself.

We have $G / G \cong M$ via the maps $G \xrightarrow{\gamma} M$.

By the lifted action, we get $T^*G / G \cong A$. \qed

$\mathfrak{k}_m^s(G) = \mathfrak{g}$-invariant vector fields on $G$ tangent to the fibers of $s : G \rightarrow M$.

Lemma. (1) $\mathfrak{k}_m^s(G) \subseteq \mathfrak{g}(G)$ as lie algebras

(2) all vector fields in $\mathfrak{k}_m^s(G)$ are projectable along $t : G \rightarrow M$ to $M$.

lie algebroids

$M$ manifold. A lie algebroid over $M$ is a vector bundle $A \rightarrow M$ together with a lie structure on $\Gamma(A)$ and a vector bundle map $\rho : A \rightarrow TM$ s.t. $\rho [X,Y] = [\rho(X), \rho(Y)]$, $\forall X, Y \in \Gamma(A)$, $[X, fY] = f[X,Y] + (\rho(X)f) Y$, $f \in C^\infty(M)$.  

Remark:
Associated to a Lie groupoid is a Lie algebroid:
\[ G \xrightarrow{\pi} M \xrightarrow{\phi} A \cong (\ker d\pi)|_M, \quad \rho = dt|_M. \]

Examples:
1. \( M \) manifold w/ only unit arrows. This has Lie algebroid \( \mathfrak{g}_0 \).
2. \( M \times M \xrightarrow{\pi} M \) pair groupoid:
   \[(x,y) \cdot (y,z) = (x,z)\]
   \[\Delta: M \xrightarrow{\pi} M \times M \]
   \[x \xrightarrow{} (x,x) \quad \text{(unit map = diagonal)}\]
   has Lie algebroid \( TM \xrightarrow{} M \).
3. \( K \) Lie group has Lie algebroid \( A = \text{Lie}(K) \).
4. Action groupoid \( K \times M \xrightarrow{\pi} M \).
   This has Lie algebroid the trivial vector bundle
   \[ \text{Lie}(K) \times M \xrightarrow{\pi} M \] w/ anchor \( \rho: \text{Lie}(K) \times M \xrightarrow{} TM \)
   \[ \text{the infinitesimal action} \]
5. \( F \subseteq TM \) foliation then \( \text{Mon}(F) \) and \( \text{Hol}(F) \) have the
   same Lie algebroid \( \mathfrak{f} \), \( \rho \) is the inclusion to \( TM \).
6. Let \( (P, \pi) \) be a Poincaré manifold \( \pi \in \Gamma(L^\infty P) \), \( [[\pi, \pi]] = 0 \)
   Schouten bracket
   Lie algebroid \( A = T^* P \). The anchor map is
   \[ \rho = i_{\pi}: T^* P \xrightarrow{} TP \]
   \[ \text{Lie derivative} \]
   \[ \alpha, \beta \in \Omega^1(P), \quad [\alpha, \beta] = \mathcal{L}_\alpha \beta - \mathcal{L}_\beta \alpha \cdot \alpha \cdot \beta = d(\pi(\alpha, \beta)) \]
   On exact forms, \( [df, dg] = d\{f, g\} \).
Example:
\[ g = \text{Lie algebra} \]
\[ g^* \text{ is a Poisson manifold of the Lie algebroid} \]
\[ T^*G \rightarrow g^* \]
\[ G \text{ is a Lie group integrating } g, \text{ s.t. are the left and right trivializations of } T^*G. \]

\[ \text{Symplectic groupoid.} \]
Fur maps:

\[ \partial_i : G_k \longrightarrow G_{k-1} \]

\[ \partial_i (g_1, \ldots, g_k) = \begin{cases} 
(g_2, \ldots, g_k) & i = 0 \\
(g_1, \ldots, g_i, g_{i+1}, \ldots, g_k) & 1 \leq i \leq k-1 \\
(g_1, \ldots, g_{k-1}) & i = k
\end{cases} \]

Remark:

\[ \partial_1 : G_2 \longrightarrow G_1 \quad \partial_0 = s \quad \partial_1 = t \]

**Groupoid cohomology**

cochains: \[ C_{\text{diff}}^{k}(G) = C^{\infty}(G_k) \]

differential: \[ \delta = \sum_{i=0}^{k} (-1)^i \partial_i \quad \implies \delta^2 = 0 \]

the cohomology of \( \left( \bigoplus_{k=0}^{\infty} C_{\text{diff}}^{k}(G), \delta \right) \) is called the groupoid cohomology, denoted by \( H_{\text{diff}}^{\cdot}(G) \).

**Lie algebroid cohomology**

1. vector bundle \( A \longrightarrow M \)
2. Lie bracket on \( \Gamma(A) \)
3. anchor map \( \rho : \Gamma(A) \longrightarrow TM \)

\[ \rho [X, Y] = [\rho(X), \rho(Y)] \quad [X, fY] = f [X, Y] + (\rho(X)f)Y \]

"de Rham forms" \( \Omega_A^{k} = \Gamma(A^k A^*) \) w/ differential \( d : \Omega_A^{k} \longrightarrow \Omega_A^{k+1} \)

\[ (d\omega)(X_1, \ldots, X_{k+1}) = \sum_{\{i\}} (-1)^{i+j-i} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) + \sum_{i} (-1)^{i} \rho(X_i) \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \]
Note that $d^2 = 0$ and hence $\bigoplus_{k \geq 0} \Omega^k_{A, d}$ is a complex $\mathbb{C}$-cohomology denoted by $H^*_\text{nc}(A)$.

The Weinstein-Xu map:

\[ \Phi : H^*_\text{nc}(G) \xrightarrow{\cong} H^*_\text{nc}(A), \quad A = \mathbb{C}[z] \text{ algebraoid }(G) . \]

$c \in C^k_{\text{diff}}(G), \, X_i \in A$

\[ \Phi(c)(X_1, \ldots, X_k) = \sum_{(\sigma) \in S_k} (-1)^{\sigma} R_{X_{\sigma(1)}} \cdots R_{X_{\sigma(k)}}(c) \]

\[ (R_x c)(g_2, \ldots, g_k) = L_x c(-, g_2, \ldots, g_k)(t(g_2)) \]

\[ \Rightarrow R_x c \in C^k_{\text{diff}}(G) \]

This defines a map \[ \Phi : C^k_{\text{diff}}(G) \rightarrow \Omega^k A \text{ if commutes w/ } A_S. \]

Examples: \[ M \times M \rightarrow M. \]

Groupoid cohomology:

\[ G_k = M^{k(k+1)} \rightarrow C^k_{\text{diff}}(G) = C^\infty(M^{k(k+1)}) \]

\[ (\delta f)(X_0, \ldots, X_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(X_0, \ldots, \hat{X}_i, \ldots, X_{k+1}) \], \[ f \in C^\infty(M^{k(k+1)}). \]

Claim: \[ H^*_\text{diff}(M \times M) = \begin{cases} R & = 0 \\ 0 & + 0 \end{cases} \]

Exercise: prove using the $\delta f$ contraction:

\[ h : C^k_{\text{adj}} \rightarrow C^{k-1}_{\text{adj}} \]

\[ (hf)(X_0, \ldots, X_{k+1}) = f(X_0, X_1, X_1, \ldots, X_{k+1}) \]

\[ \Rightarrow \delta h + h \delta = 1. \]
The Lie algebroid of $\mathcal{M} \rightarrow M$ is $TM$.

$\Rightarrow (\Omega^*_A, d)$ is the usual de Rham complex.

$\Rightarrow H^*_\text{deR}(TM) \cong H^*_\text{dR}(M)$

**Remark:**
The Wuanxin - Xu map factors as follows:

$$C^*_\text{diff}(\mathcal{M} \times \mathcal{M}) \xrightarrow{\partial} C^*_A(M) \rightarrow \Omega^*_\text{M}$$

- Alexander - Spanier- Eilenberg cochains complex,
- germs $\Delta^{k+1}(C^\infty(M^{(k+1)}))$, i.e.
- germs of smooth functions around $\Delta^{k+1}(M^{(k+1)})$
- diagonal

The differential of $C^*_A(M)$ is the "restriction" of $\partial$ to the diagonal, denoted by $\partial$.

**Proposition:** $H^*(C^*_A(M), \partial) \cong H^*_\text{dR}(M)$

**Theorem (Crawford):**
For a proper groupoid $G \rightarrow \mathcal{M}$,

$$H^k_{\text{diff}}(G) = \begin{cases} 
\{ C^\infty(M)^{k=0} \} & k=0 \\
0 & k>0 
\end{cases}$$
Ingredients of the proof:

1. Any Lie groupoid has a system, i.e., a family of measures \( \lambda = \{ \lambda_x | x \in M \} \) on the fibers of the target such that:
   
   \( \forall \phi \in C_c^\infty(\mathcal{O}), \quad \int_{t^{-1}(x)} \phi(g) \, d\lambda^x(g) \in C^\infty(M) \)

2. \( \lambda \) is left-invariant

   \( \int_{t^{-1}(x)} \phi(gh) \, d\lambda^x(h) = \int_{t^{-1}(y)} \phi(h) \, d\lambda^y(h) \)

3. On any proper Lie groupoid, there always exists a "cut-off" function \( c \in C^\infty(M) \) such that:
   
   \( \text{supp}(c \cdot s) \rightarrow M \text{ proper} \)

   \( \int_{t^{-1}(x)} c(s(g)) \, d\lambda^x(g) = 1 \quad \forall x \in M \)

With (1) and (2), a contraction of the complex is given by

\[ b(q, g_1, \ldots, g_{k-1}) = \int_{t^{-1}(t(g))} \theta' \cdot \Phi((q')^{-1}, g_1, \ldots, g_{k-1}) c(s(q')) \, d\lambda^{q'}(g_1) \]

Consider a Lie groupoid action \( G \rightarrow M \succ (\mu, \zeta) \) i.e.

\( \exists \mu : Z \rightarrow M \) with \( Z^{\mu^{-1}} G \rightarrow Z \). Assume \( Z^{\mu} \rightarrow M \) is "injective, submersion". We say that the action is proper if the map

\[ Z^{\mu^{-1}} G \rightarrow Z \times Z \]

\( (Z, g) \rightarrow (Z, zg) \) is proper.
Given \( G \rightarrow M \sim (\mu, Z) \), there is an associated groupoid \( Z^\mu \rtimes G \rightarrow Z \), the action groupoid.

**Theorem:** Associated to a proper action as above, there is a map \( \Phi_\mu : H^*_{df}(G) \rightarrow H^*_{fr, ino}(Z) \).

\( \mu \) surjective \( \smallfrown \) submersion \( \smallfrown Z \) is foliated along \( \mu \).

Call this foliation \( F = \text{killed}_\mu \). Consider \( \Omega^*_F := \Gamma(N^* F) \). There is a differential \( d_F \) (foliated cohomology).

\( G \rightarrow M \sim (\mu, Z) \rightarrow G \rightarrow M \sim (H_f, F) \).

Let \( \Omega_{F, ino} = \) invariant foliated forms. This gives \( H^*_{F, ino}(Z) \).

**Back to van Est**

\( H \) Lie group, \( K \) maximal compact subgroup.

\( K \backslash H \smallfrown H \) properly

\( \text{w/} \ Z = K \backslash H \) then \( F = T_Z \).

\( \Omega^*_\text{fr, ino}(Z) \cong \Lambda^*(H/K) \).

Consider the group \( G \rtimes Z \). Consider the \( \# \) double complex

\[
\begin{array}{ccccccccc}
C^\infty(Z^2) & \xrightarrow{d_1} & C^\infty(Z^2) & \xrightarrow{d_2} & C^\infty(Z^2 \times G) & \xrightarrow{\partial_1} & C^\infty(Z^2 \times G^2) & \rightarrow & \\
\downarrow{\delta_2} & & \downarrow{\delta_1} & & \downarrow{\partial_2} & & \downarrow{\partial_1} & & \downarrow{p} \\
C^\infty(Z) & \xrightarrow{d_1} & C^\infty(Z \times G) & \xrightarrow{\partial_1} & C^\infty(Z \times G^2) & \rightarrow & \\
\downarrow{\delta_2} & & \downarrow{\delta_1} & & \downarrow{\partial_2} & & \downarrow{p} & & \\
C^m(Z) & \xrightarrow{d_1} & C^m(G) & \xrightarrow{\partial_1} & C^m(G^2) & \rightarrow & \\
\end{array}
\]
$\beta_1$ is the groupoid cohomology differential of the action $\mathbb{G} \rtimes \mathbb{Z}^k$.

$\beta_2$ is the group cohomology differential.

Lemma: The inclusion $R \rightarrow C^\infty(\mathbb{Z})$ induces an isomorphism on cohomology.

Proof: Look at filtration by p-degree since cohomology of columns vanishes, except in deg 1. This proves the lemma.

Lemma: The inclusion $C^\infty_m(\mathbb{Z}) \rightarrow C^\infty(\mathbb{Z})$ induces an isomorphism on cohomology.

Proof: Look at filtration the rows. This is the groupoid complex for a proper action. This vanishes in cohomology by Champs's result.

These lemmata \( \Rightarrow H^*(\mathbb{G}) \cong H^*(C^\infty_m(\mathbb{Z}), d_0) \).

Last step: Localize the complex $C^\infty_m(\mathbb{Z})$ in the diagonal.

$$
\Phi_2 : H^*(\mathbb{G}) \cong H^*(C^\infty_m(\mathbb{Z}), d_0) \rightarrow H^*(C^\infty_m(\mathbb{Z}), d_0) \cong H^*_m(\mathbb{Z})
$$
Let $\mathcal{A} \rightarrow \mathcal{M}$ be a Lie algebroid with Lie bracket $[,]$ and anchor map $\rho: \mathcal{A} \rightarrow \mathcal{T}\mathcal{M}$. The vector space $\mathcal{C}^\infty(\mathcal{M}) \oplus \Gamma(A)$ has a Lie algebra structure with

$$ [(f, X), (g, Y)] = (\rho(X)g - \rho(Y)f, [X, Y]) $$

Consider the universal enveloping algebra, denoted by $U$. The universal enveloping algebra of $\mathcal{A}$ is defined as $U/I = U(\mathcal{A})$ with $I = \langle i(f)i(g) - i(fg), i(f)i(X) - i(fX), f, g \in \mathcal{C}^\infty(M), X \in \Gamma(\mathcal{A}) \rangle$.

**Definition**

(i) In $U(\mathcal{A})$, we have $i(f)i(X) - i(fX)$, $[i(X), i(Y)] = i[X, Y]$ and $[i(X), i(f)] = i(\rho(X)f)$.

(ii) $U(\mathcal{A})$ is universal in these relations.

(iii) The algebraic version of a Lie algebroid, as a Lie-Rinehart algebra. The universal enveloping algebra exists for such objects.
Examples
(1) $\mathfrak{g}$ as a trivial Lie algebroid, we get the usual enveloping algebra.
(2) $TM \rightarrow M$, $\mathcal{U}(TM) = \mathcal{D}(N)$ the algebra of differential operators on $M$.
(3) $\mathcal{F} \subseteq TM$ foliation, $\mathcal{U}(\mathcal{F})$ is the algebra of differential operators acting on the leaves.

**Theorem (Poincaré–Birkhoff–Witt)**
The symmetrization map
$$X_1 \otimes \cdots \otimes X_n \mapsto \sum_{\sigma \in S_n} (-1)^{\sigma} X_{\sigma(1)} \cdots X_{\sigma(n)}$$
defines an isomorphism of algebras
$$\Gamma'(\text{Sym } A) \rightarrow \text{Gr}(\mathcal{U}(A))$$
In particular, $\text{Gr}(\mathcal{U}(A))$ is commutative.

**Goal:** Find a lift $\mathcal{U}(A) \rightarrow \Gamma'(\text{Sym } A)$.

Another description of $\mathcal{U}(A)$:
$G \ni \mathbf{g} \rightarrow \mathbf{g}' \mapsto \mathbf{g}' \in M$ and this action lifts to $T_s G \rightarrow \ker ds \in T\mathcal{G}$.

$G/\mathcal{G} \cong M$, $T_sG/\mathcal{G} \cong A$
$$\rightarrow C^\infty(M) \rightarrow C^\infty_m(G), \Gamma'(A) \otimes X^s_m(G) \text{ invariant vector fields on the } s\text{-fiber}$$

**Definition:** $\mathcal{D}(G)$ algebra of invariant differential operators along $s$-fibers.

So $\mathcal{D} = \mathcal{D}(G)$ is a actually a smooth family of differential operators $\mathcal{D} = \bigotimes_{x \in M} D_x$, on $S^1(x)$. Turvianess means: $D_x = R_y \circ D_y \circ R_y^{-1}$ for all $x \rightarrow y$. 

By the isomorphisms (\#) we get maps \( \phi_i : C^\infty(M) \rightarrow \mathcal{D}(E) \) and 
\( \phi_i : \Gamma(A) \rightarrow \mathcal{D}(E) \) satisfying the relations \( \mu(A) \). 
Hence, we get an algebra morphism 
\( \mu(A) \xrightarrow{\cong} \mathcal{D}(E) \).

The exponential map on a Lie groupoid.

Choose a connection \( \nabla \) on \( A \). Since \( \xi^*A = T_s G \), we get a 
connection on \( T_s G \). This defines the exponential map

\[
\exp_{\nabla, x} : A_x \xrightarrow{\cong} (T_s G)_x \xrightarrow{\exp_T} s^{-1}(x) \subseteq G, \quad x \in M.
\]

Varying the base point, we get \( \exp_{\nabla} : A \rightarrow G \). It is a local

diffeomorphism from a neighborhood of the zero section \( m \) \( A \) to a 
neighborhood \( V \) of \( M \subseteq G \).

Let \( \xi \in C^\infty(G) \) s.t.

(a) \( \text{support} (\xi) \subseteq V \)
(b) \( \xi \equiv 1 \) in a neighborhood of \( M \).

Definition: \( \xi \in A_x^* \), 
\( \ell_\xi(g) = \langle x(g), \xi \rangle \), \( \langle \cdot, \cdot \rangle : A \times A^* \rightarrow \mathbb{R} \),

\[
\Rightarrow \ell_\xi \in C^\infty(G).
\]

Definition: The symbol \( \sigma_V(D) \in C^\infty(A^*) \) of \( D \in \mathcal{U}(A) \) is defined by

\[
\sigma_V(D)(\xi_x) = (D_x \ell_\xi)(x).
\]

Remark:

In fact \( \sigma_V(D) \) depends polynomially on \( \xi \), for \( D \in \mathcal{U}(A) \)
then \( \sigma_V(D) \in \bigoplus_{k=0}^{\infty} \Gamma^k(\text{Sym}^k A) \).

Theorem:
The map \( D \mapsto \sigma_V(D) \) defines a linear isomorphism \( \mathcal{U}(A) \xrightarrow{\cong} \Gamma(\text{Sym} A) \).

Proof: \( \sigma_V \) respects the filtration, to check that its graded quotient

\[
\sigma_k : \frac{\text{Fil}_k \mathcal{U}(A)}{\text{Fil}_{k-1} \mathcal{U}(A)} \rightarrow \Gamma^k(\text{Sym}^k A)
\]
We have the Hopf property:
\[ \sigma_k(D) \sigma_l(E) = \sigma_{k+l}(DE), \; D \in \text{Fun}_k(U(A)), \; E \in \text{Fun}_l(U(A)). \]

Consider \([D,E] = 0 \mod \text{Fun}_{k+l+1}(U(A))\). Define
\[ \{\sigma_k(D), \sigma_l(E)\} := \sigma_{k+l}(D,E) \]

Lemma: \(\{,\} \) defines a Poisson bracket on \(T^*(\text{Sym}A)\)
(Think of \(T^*(\text{Sym}A)\) as polynomials on \(A^*\)).

Proof:
\[ \text{Jacobi:} \; \{\{\sigma_k(D), \sigma_l(E)\}, \sigma_m(F)\} = \sigma_{k+l+m-2}[[D,E], F] \]
\[ \Rightarrow \text{Jacobi on } [,]. \]
\[ \text{Leibnitz:} \; \{\sigma_k(D) \sigma_l(E), \sigma_m(F)\} = \sigma_{k+l+m-1}[[DE,F]]. \]

Explicitly,
\[ \forall f, g \in C^\infty(M), \; X, Y \in T(A) \]
\[ \cdot \; \{f, g\} = 0 \]
\[ \cdot \; \{X, f\} = \rho(X)f \]
\[ \cdot \; \{X, Y\} = [X, Y] \]

Examples:
1. \(\mathfrak{g}\) Lie algebra, \(\mathfrak{g}^*\) Lie–Poisson structure.
2. \(TM\). In \(T^*M\), we have \(\{,\}\) as the canonical symplectic structure.

Remark: \(G\) acts on \(T_s^*G\) w/ quotient \(T_s^*G / G \cong A^*\).
\[ T_s^*(G) = \bigcup_{x \in M} T^*(s^1(x)) \text{ is a regular Poisson manifold w/ symplectic}
\]
\[ \text{leave } T^*(s^1(x)), \; x \in M. \]

The quotient map \(T_s^*(G) \rightarrow A^*\) is a Poisson map.
Deformation Quantization

Form the adiabatic lie algebra $\mathfrak{a}_h^\mathbb{R} \to M \times [0, \infty)$. As a vector bundle, $A_h = A$.

$\Gamma \mathfrak{a}_h^\mathbb{R} = \mathfrak{a}_h^\mathbb{R} \times [0, \infty) \rightarrow \Gamma(A)$

$[X, Y]_h = \hbar [X_h, Y_h]$

$p(X)_h = \hbar p(X_h)$

This defines a new algebra. Consider $\mathfrak{u}\mathfrak{f}_h^\mathbb{R}$. This algebra contains $C^\infty([0, \infty))$ in its center. Define

$$\lim_{\hbar \to 0} \left( \mathfrak{u}\mathfrak{f}_h^\mathbb{R} \right) \frac{\text{PBW}}{\hbar} \rightarrow \Gamma(\text{Sym} A) \llbracket \hbar \rrbracket$$

In this way, the associated $\ast$-product on $\Gamma(\text{Sym} A) \llbracket \hbar \rrbracket$ in $C^\infty(\mathbb{A}^\ast) \llbracket \hbar \rrbracket$. This defines a $\ast$-product at first order term the poisson bracket $\{\cdot, \cdot\}$ just defined.

Theorem

There is the algebra $\mathfrak{u}(g)$ of $\psi\text{DO}$'s on $G$. An element $P \in \mathfrak{u}(g)$ is a family of operators $P = \xi P_x$ $x \in M$ of $\psi\text{DO}$'s on $s^{-1}(x)$ in:

(a) $x \mapsto P_x$ is smooth

(b) $P$ is invariant i.e. $P_x = R_y \circ P_y \circ R_y^{-1}$, $y \leftarrow x$.

(c) Support condition: $\mu(\text{Supp}(P)) \subseteq G$ compact.

$$\text{Supp}(P) \subseteq \bigcup_{x \in M} (s^{-1}(x) \times s^{-1}(x)) \subseteq G^s x_m^s G$$

$\mu : G^s x_m^s G \to G$

$(q_1, q_2) \mapsto q_1 q_2^{-1}$
Remark:

\[ \psi^\infty(G) = \bigcup_m \psi^m(G) \]

\[ \psi^{-\infty}(G) = \bigcap_m \psi^m(G) \text{ is an ideal of } \psi^\infty(G). \]

\( \rho \in \psi^{-\infty}(G) \) has a smooth family of smooth kernels

\[ K_{\rho} \in C^\infty(s^{-1}(x) \times s^{-1}(x)) \]

which is invariant under the action

\[ K_{\rho}(z, g, z') = K_{\rho}(z', z). \]

Since \( G \overset{\sim}{\times} M \hookrightarrow \overset{\sim}{C} \) we can consider the reduced kernel

\[ \overset{\sim}{K}_{\rho} \in C^\infty_c(G). \]  So \( \psi^{-\infty}(G) \cong C^\infty_c(G) \) as algebras. The multiplication in \( C^\infty_c(G) \) is convolution

\[ (f_1 f_2)(g) = \int f_1(gh^{-1}) f_2(h) \, d\lambda(h) \]

\( A : \) convolution algebra.

Picture:

\( A \subset \psi^\infty(G) \)

\( UA \subset \psi^{-\infty}(G) \) as differential operators

Definition:

\( D \in UA \) is elliptic if it is a family of elliptic differential operators on \( s \)-fibers.

Ellipticity \( \Rightarrow \exists Q \in \psi^\infty(G) \) s.t. \( 1 - Q D, 1 - DQ \in A \).

Remark:

This is enough to define \( \text{ind}(D) \in K_0(A) \). We can pass this to \( H^*_{\text{diff}}(G) \). Goal: compute this pairing.
Index Theorems

the case of a proper action of a Lie group $G \to \mathbb{Z}$ properly s.t. $\mathbb{Z}/G$ is compact.

D elliptic $G$-invariant differential operator

Lemma: For a proper, cocompact action $\mathbb{Z} \times G \to \mathbb{Z}$, 
exists "cut off" function i.e a $c \in C_c(\mathbb{Z})$ s.t.

$$\int_{\mathbb{Z}} c(xq) dq = 1, \forall x \in \mathbb{Z}$$

If this function, we can average differential geometric objects to get invariants ones. For example, let $p$ be a Riemannian metric. Then

$$p_{\text{ave}}(x, y) = \int_{\mathbb{Z}} c(xq) p(xq, yq) dq$$

There exists an algebra of $G$-invariant PDO's on $\mathbb{Z}$,

$$\psi^\infty_{\text{inv}}(\mathbb{Z}) = \cup \psi^m_{\text{inv}}(\mathbb{Z}) \in P$$

(i) $P = R_g \circ P \circ R_{g^{-1}}$

(ii) $\text{supp}(P) \subset \mathbb{Z} \times \mathbb{Z}$ is $G$-compact

* Look at the paper of Connes & Moscovici about $K^2$-index

Choose a $G$-invariant Riemannian metric, so we get

$$\exp : T\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$

define $\phi_\xi(x, y) = \chi(x, y) e^{i \xi \cdot \exp_x(y)}$

$\Rightarrow$ a nice neighborhood of $\Delta$

$\emptyset$ outside another one.
Let $\alpha \in C^\infty(T^*Z)$,
\[
[Op(\alpha)(\varphi)](x) = \int_{T_x^*Z} \int_Z e_\varphi(x,y) \alpha(x,\xi) f(y) \, dy \, d\xi
\]

For a suitably chosen $\alpha \in C^\infty(T^*Z)$, this defines a $G$-invariant $\Psi DO$. The kernel of this operator is
\[
K_{\alpha}(x,y) = \int_{T^*_y Z} e_\varphi(x,y) \alpha(x,\xi) \, d\xi
\]

$Op(\alpha) \in \Psi^{\infty}_{inv}(Z) \implies K_{\alpha}(x,y) = K_{\alpha}(x,y)$.

Smoothing operators: $\Psi^{\infty}_{inv}(Z) \subset C^\infty_{G-\text{comp}}(Z \times Z)^G$

**The Trace**:

Define the $G$-functional $\tau: \Psi^{\infty}_{inv}(Z) \to C$,
\[
\tau(K) = \int_Z K(x,x) \sigma(x) \, dx
\]

Lemma: when $G$ is unimodular, $\tau$ is a trace.

**Proof**:\n\[
\tau[K_1, K_2] = \int_Z \sigma(x_1) \left( K_1(x_1,x_2) K_2(x_1,x_2) - K_2(x_1,x_2) K_1(x_1,x_2) \right) \, dx_1 \, dx_2
\]

\[
= \int_G \varphi(g) \, dg
\]

$\varphi(g) = \int_Z \int_Z \sigma(x_1) \sigma(x_2) [K_1, K_2] \, dx_1 \, dx_2$
Now,
\[ \varphi(q^{-1}) = \iint c(x_1)c(x_2)(K_1(x_1, x_2, g) K_2(x_1, g, x_1) - K_2(x_1, x_2, g) K_1(x_1, g, x_1)) \, dx_1 \, dx_2. \]
\[ = \iint c(x_1) c(x_2) (K_1(x_1, g', x_2) K_2(x_2, x_1, g'^{-1}) - \ldots \]
\[ = \iint c(x_1, g) c(x_2) (K_1(\ldots \]
\[ = -\varphi(q) \quad \text{unimodularity} \Rightarrow dq = dq' \]
\[ \Rightarrow [K_1, K_2] = 0 \rightarrow I \text{ is a trace. } \square \]

The Index Pairing

**Elliptic, \( \mathfrak{F} \) parameters,** \( P \) s.t. \( \frac{1}{S_0} PD, \frac{1}{S_1} DP \in \psi_{\text{inv}}^{-\infty}(Z) \).

Consider the **aff. operator matrix:**

\[ L = \begin{pmatrix} S_0 & -(P+S_0) \vphantom{S_1} \\ \underline{D} & S_1 \end{pmatrix} \in M_2(\psi_{\text{inv}}^{-\infty}(Z)) \]

\[ L^{-1} = \begin{pmatrix} S_0 & (1+S_0)P \\ -D & S_1 \end{pmatrix} \]

\[ \det R_0 := \frac{L \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} L^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}{\pi^2 = \pi} = \begin{pmatrix} S_0^2 & S_0(1+S_0)P \\ S_1D & -S_1^2 \end{pmatrix} \in M_2(\psi_{\text{inv}}^{-\infty}(Z)) \]
Remark:
Suppose $G$ is the trivial group, this means that $Z$ is compact.
and $c=1$. Then $\mathfrak{T}(R_0) = \text{Tr}_{\mu}(S^2) - \text{Trac}(S^2) = \text{index}(D)$.

Higher Index
Recall that any $a \in \text{H}^k_{\text{eff}}(G)$ can be represented by a function
$\varphi_a \in C^\infty_{\text{mc}}(Z^{k+1})$.

\[ C^\infty_m(Z^2) \longrightarrow C^\infty(Z^2 \times G) \longrightarrow \]
\[ \delta \]
\[ C^\infty_m(Z) \longrightarrow C^\infty(Z^2 \times G) \longrightarrow \]

\[(\delta \varphi)(Z_0, \ldots, Z_k) = \sum_{\ell=1}^D \varphi(Z_0, \ldots, \hat{Z}_\ell, \ldots, Z_k)\]
acyclic
\[ C^\infty(G) \longrightarrow \]

\[ C^\infty(G) \longrightarrow \]

Definition:
Let $K_0, \ldots, K_n \in \text{H}^\infty_{\text{mc}}(Z)$ and $\varphi \in C^\infty_m(Z^{n+1})$

\[
\langle \varphi, K_0 \otimes \cdots \otimes K_n \rangle = \int_{Z^{n+1}} C(x) \varphi(x_0, \ldots, x_n) K_0(x_0, x_1) K_1(x_1, x_2) \cdots K_n(x_n, x_0) \, dx_1 \cdots dx_n
\]

Lemma:
\[
\langle \varphi, b(K_0 \otimes \cdots \otimes K_n) \rangle = \langle \delta \varphi, K_0 \otimes \cdots \otimes K_n \rangle
\]

Define $\text{Ind}_n(D) := \langle \varphi_a, \text{ch}(R_0) \rangle$.

With the lemma above, this construction defines a map
\[ \text{H}^k_{\text{eff}}(G) \longrightarrow \text{H}^*_{\text{mc}}(Z^2) \]

For the trivial class, $[1] \in \text{H}^0(G)$, one gets the trace $I$. 

Remark on $G$-invariant cohomology

Remark on $G$-invariant cohomology

$\Omega^m_G(Z) \subseteq \Omega^m(Z)$, $\Omega^m_G(Z) = \{ \alpha \in \Omega^m(Z) \mid g^*\alpha = \alpha, \forall g \in G \}$

d $\alpha$ restricts to $\Omega^m_G(Z)$.

$H^m_G(Z) = H^m(\Omega^m_G(Z), d_\alpha)$.

**Lemma:** The integral

$$\int_Z c \wedge \alpha \in \Omega_{top}^m(Z)$$

vanishes on exact forms.

**Proof:**

$$\int_Z c d\beta = -\int_Z d c \wedge \beta = \int \int_{Z \times G} (g^{-1})^* c \wedge \beta \wedge de = \int \int_{Z \times G} c \wedge g^* d\beta = 0$$

This follows from differentiation of the identity $1 = \int_Z c(xg)$. □

**Asymptotic formulas**

$O_p^G(a) = O_p(a(x, t \xi))$

We get a product on symbols by setting

$$a \times b = \begin{cases} a \circ (O_p(a)O_p(b)) & H > 0 \\ a \cdot b & H = 0 \end{cases}$$

$$O_p^G = H \cdot \sigma, \quad l_H(a(x, \xi)) = a(x, H \xi)$$

defines an associative product

where get expansion at $H = 0$

defines a $*$-product on $T^*Z$.
The package

\[ \text{Op}_h(a) \text{ Op}_h(b) \sim \text{Op}_h(a+b) \text{ as } h \to 0 \]

trace \( T(k) = \int c(x) K(x,x) dx \), \( K_{op}(a)(x,y) = \int_{T^*_M} e^{i(x,y)} a(x, \xi) d\xi \)

\[ \Rightarrow T(K_{op}(a)) = \int_{T^*_M} e(x) a(x, \xi) dx d\xi \]

induces an *-deformed algebra a trace.

Compute \( \alpha \in H^*_H(\mathfrak{g}) \)

\[ \text{Ind}_\alpha(D) = \langle \Phi_\alpha, R_\alpha \otimes \ldots \otimes R_\alpha \rangle \]

\[ = T\left( \Phi_\alpha \otimes \Phi_\alpha \ldots \otimes \Phi_\alpha \right) \]

\[ = T\left( \Phi_\alpha \text{ Op}(a) \Phi_\alpha \ldots \otimes \text{ Op}(a) \Phi_\alpha \right) \]

\[ \left[ R_D = \text{ Op}(a) \right] \]

\[ = \lim_{h \to 0} T\left( \Phi_\alpha \text{ Op}_h(a) \Phi_\alpha \ldots \otimes \text{ Op}_h(a) \Phi_\alpha \right) \]

\[ = \lim_{h \to 0} T\left( \text{ Op}_h (\Phi_\alpha^* a) \ldots \text{ Op}_h (\Phi_\alpha^* a) \right) \]

\[ = \lim_{h \to 0} T\left( \text{ Op}_h (\Phi_\alpha + a * \Phi_\alpha + \ldots + a * \Phi_\alpha) \right) \]

\[ = T_\alpha (\Phi_\alpha + \ldots + a) \]

Example

1. \( G = \mathfrak{g} \mathfrak{l}_3 \) \( a * a - a \)

Fedorov; Nest-Tsygan

Index Theorem:

\[ \text{Ind}(D) = \int_{T^*_M} Td(T^*_M) \text{ ch}(a_0) \]

2. \( G = \mathfrak{g} \mathfrak{l}_3, \alpha = [1, 1] \)

The *-product induced on \( T^*_M \) is \( G \)-equivariant.

\[ (A_{T^*_M})^G; \text{ HC}^m(A_{T^*_M}) \xrightarrow{\alpha} \text{ H}^m_{\text{min}}(Z) \]
This gives us \( \langle \xi, R_D \rangle = \int_{T^*Z} c Td(T^*Z) ch(\sigma(D)) \)

**Theorem (General case)**

\[
\text{Ind}_\phi(D) = \int_{T^*Z} \Phi_\phi(\alpha) \cdot \text{van Est map}
\]

**Theorem (Pflaum-Posthuma-Tang)**

\[
\text{Ind}_\alpha(D) = \int_{T^*Z} \Phi_\alpha(\alpha) Td(T^*Z) ch(\sigma(D))
\]

**Remark:**

When \( G \) is not unimodular, there is no trace but there are higher cyclic cocycles:

\[
H_{\text{diff}}(G; \Lambda^* G) \longrightarrow H^*(\Psi^\infty_m(Z))
\]

**Theorem:** Let \( G \rightarrow M \overset{\pi}{\rightarrow} N \) properly, \( \mu \) surjective submersion. Assume \( G \) to be unimodular. Then for any \( \alpha \in H^2(\theta) \) and \( D = \Sigma D_x \) a smooth \( G \)-invariant family of elliptic differential operators \( D_x \) on \( \mu^{-1}(x) \), there is a similar definition of the higher index \( \text{Ind}_\alpha(D) \) and

\[
\text{Ind}_\alpha(D) = \int_{T^*Z} \Phi_\alpha(\alpha) Td(T^*Z) ch(\sigma(D))
\]

**Remark:**

The proof is the same. The relevant Poisson manifold is \( T^*Z \) a regular Poisson co-tangent machinery applies.

**Corollary:** Let \( G \rightarrow M \overset{S}{\rightarrow} N \), \( D \subset \mathcal{U}(A) \) be elliptic then

\[
\text{Ind}_\alpha(D) = \int_{A^*} \Phi_\alpha(\alpha) Td(A^*) ch(\sigma(D))
\]

Wu-theorem - Xu map.
Example:

1. Pair groupoid: \( M \times M \to M, \) \( M \) compact. Proper, \( \Rightarrow \) \( H^\text{eff}_M (M \times M) = 0, \) \( \cdot > 0. \)

The trivial cocycle gives Atiyah-Singer index.

2. Connes-Morava (81):

\[ \tilde{M} \to M \]

universal cover

\[ \Gamma = \pi_1 (M, x_0) \]

\[ \tilde{M} \times \tilde{M} \to \tilde{M} \] a Lie group.

Its Lie algebroid is \( TM \to M. \)

Remark:

1. \( \tilde{M} \times \tilde{M} \to \Gamma \)

2. Groupoid cohomology is invariant under \( \sim \) [claim].

3. \( H^\text{eff}_M (\tilde{M} \times \tilde{M}) \xrightarrow{\sim} H^*_{\text{gr}} (\Gamma) \xrightarrow{\sim} H^* (BP) \)

\[
\text{Ind}_x (D) = \int_{T^*M} \phi_\xi (\alpha) \text{Td} \, ch (\sigma (D))
\]

\[ \Phi_\xi : H^* (BP) \xrightarrow{\psi^*} H^* (M) \]

\[ \begin{CD}
\tilde{M} @>>> M @>>> \psi(BP)
\end{CD} \]