K-local rigidity

$L_1 \mathcal{S}p$: spectra with w.eq. = $K_{(2)}$-equivalences

$$K_{(2)} = \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1), \quad E(1)_* = \mathbb{Z}_{(2)}[v_1, v_1^{-1}]$$

$$\rightsquigarrow L_{K_{(2)}} = L_{E(1)} =: L_1$$

EVERYTHING IS 2-LOCAL

<u>Goal</u>: If $\Phi : Ho(L_1 \mathcal{S}p) \longrightarrow Ho(\mathcal{C})$ is a triangulated equivalence ($\mathcal{C}$ stable), then $L_1 \mathcal{S}p \underset{\text{Quillen}}{\simeq} \mathcal{C}$.

<u>Strategy</u> for $Ho(\mathcal{S}p)$

① Quillen functor $X \wedge - : \mathcal{S}p \rightleftarrows \mathcal{C} : Hom(X, -)$

$\underset{\text{fib.+cof.}}{X} := \Phi(S^0)^{cf}$ [Lenhardt]

$\rightsquigarrow$ exact endofunctor

$$F : Ho(\mathcal{S}p) \xrightarrow{X \wedge -} Ho(\mathcal{C}) \xrightarrow{\Phi^{-1}} Ho(\mathcal{S}p)$$

$$S^0 \longmapsto S^0$$

② show: F is an equivalence ($\Rightarrow X \wedge -$ is a Quillen eq.) by reducing to showing that

$$F : [S^0, S^0]_* \longrightarrow [S^0, S^0]_*$$

is an iso.

This works because S^0 is a compact generator of $Ho(\mathcal{S}p)$.

$\underline{K\text{-locally}}$: $\quad \Phi : \mathrm{Ho}(L_1 Sp) \longrightarrow \mathrm{Ho}(\mathcal{C})$, $X := \Phi(S^0)^{cf}$

$$\begin{tikzcd} Sp \arrow[r, "X\wedge -"] \arrow[d, "id"'] & \mathcal{C} \\ L_1 Sp \arrow[ur, dashed, "X\wedge -"'] \arrow[u, phantom, "?"] & \end{tikzcd}$$

$\rightarrow$ Quillen functor
$\Leftrightarrow$ $R\mathrm{Hom}(X, Y)$ $E(1)$-local $\forall Y$.
How does one check this?

$M = M(\mathbb{Z}/2)$ has $M_{\mathbb{Q}} = *$.

$\Rightarrow$ M has "v_1-self map" $\quad v_1^4 : \Sigma^8 M \longrightarrow M$
which is an $E(1)_*$-iso

[Adams] $\quad Z$ $E(1)$-local $\Leftrightarrow$ $[M, Z]_* \xrightarrow{(v_1^4)^*} [M, Z]_{*+8}$ iso.

$\underline{\text{in our case}}$: $\quad [M, R\mathrm{Hom}(X,Y)]_* \xrightarrow{v^*} [M, R\mathrm{Hom}(X,Y)]_{*+8}$

$\Leftrightarrow$ $[X\wedge M, Y]^{\mathcal{C}}_* \xrightarrow{v^*} [X\wedge M, Y]^{\mathcal{C}}_{*+8}$ $\quad$ iso $\forall Y$

$\Leftrightarrow$ $[M, \Phi^{-1}(Y)]^{L_1 Sp} \xrightarrow[\Phi^{-1}(X\wedge v)^*]{} [M, \Phi^{-1}]^{L_1 Sp}_{*+8}$

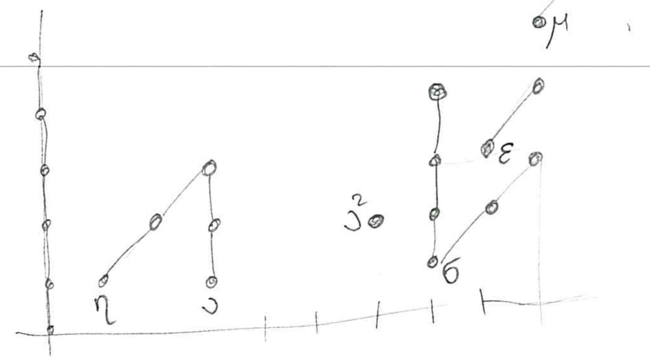
identify possible $\underset{= v_1^4 + T}{\Phi^{-1}(X\wedge v_1^4)} \in [M, M]^{L_1 Sp}_8$ and see that they all induce isos.

$\rightsquigarrow$ $F : \mathrm{Ho}(L_1 Sp) \xrightarrow{X\wedge^L -} \mathrm{Ho}(\mathcal{C}) \xrightarrow{\Phi^{-1}} \mathrm{Ho}(L_1 Sp)$

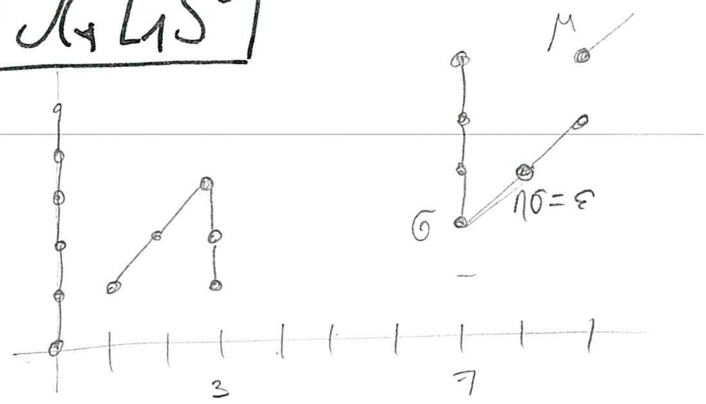
$\qquad L_1 S^0 \longmapsto L_1 S^0$

WLTS: $\quad [L_1 S^0, L_1 S^0]_* \xrightarrow{F} [L_1 S^0, L_1 S^0]_*$ iso.

$\boxed{\pi_* S^0}$



$\boxed{\pi_* L_1 S^0}$



<u>non-local</u>: induction in direction ↓, i.e. Adams filtration.
left to prove manually: $\underline{\underline{F(\eta)}}$, $F(\nu)$, $F(\sigma)$

<u>K-local</u>: show: $F: [M, S^0]_n^{L_1 Sp} \longrightarrow [M, S^0]_{n+8}^{L_1 Sp}$
iso for $n = 0, \dots, 9$
This is 8-periodic, because $v_1^4 : \Sigma^8 M \to M$
is an $E(1)$-equiv.
5-lemma ⇒ claim.
⇝ built on $\underline{\underline{F(\eta) = \eta}}$.

<u>Equivariantly</u>: $\Phi: Ho(Sp_G) \longrightarrow Ho(\mathcal{C})$ cof. proper G-equivariant stable → G-Top model cat
- $\Phi(\Sigma_+^\infty G/H) \cong G/H_+ \wedge \Phi(S^0)$
- natural w.r.t. res, conj, trf.

$\Rightarrow \; Sp_G \underset{G\text{-Quillen}}{\simeq} \mathcal{C}$

① Quillen functor $Sp_G \longrightarrow \mathcal{C}$
⇝ endofunctor $F: Ho(Sp_G) \longrightarrow Ho(\mathcal{C}) \longrightarrow Ho(Sp_G)$
$\Sigma_+^\infty G/H \longmapsto \Sigma_+^\infty G/H$

3. reduction to $F : [\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K]_* \circlearrowleft \cong$

$F : [\Sigma^\infty_+ G, \Sigma^\infty_+ G]_* \circlearrowleft \cong$
$\cong \pi_* S^0 \otimes \mathbb{Z}[G]$.

K-local equivariant: $L_1 Sp_G$ localisation at
$\{ \Sigma^\infty_+ G/H \wedge \Sigma^8 M \xrightarrow{v_1^4} \Sigma^\infty_+ G/H \wedge M \mid H \leq G \}$

G-spectrum X is "v_1-local"
$\iff$ X^H is v_1-local $=$ $E(1)$-local $\forall H$
$\iff$ $X^{\Phi(H)}$ — " —

This localisation is smashing
$\Rightarrow$ Ho $(L_1 Sp_G)$ has compact generators $\{\Sigma^\infty_+ G/H\}$

1. Getting a Quillen functor is tricky. (Inductive argument!)

$$\begin{tikzcd} Sp_G \arrow[r, "X \wedge -"] \arrow[d] & \mathcal{C} \\ L_1 Sp_G \arrow[ur, dashed] & \end{tikzcd}$$

2. $F : \mathrm{Ho}(L_1 Sp_G) \longrightarrow \mathrm{Ho}(L_1 Sp_G)$

$F : [\Sigma^\infty_+ G, \Sigma^\infty_+ G]^{L_1}_* \longrightarrow [\Sigma^\infty_+ G, \Sigma^\infty_+ G]^{L_1}_* = \pi_* L_1 S^0 \otimes \mathbb{Z}[G]$

$F(\eta) = \qquad\qquad = \eta + (\text{other stuff})$

THIS WORKS!