Masterclass: Rigidity and algebraic models in stable homotopy theory. 2018-04-09

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$K_{(2)}$ 2-local complex top. K-theory

everything is 2-local

$L_1 Sp$: spectra with weak equivalences $K_{(2)*}$-isos. cofibrations as before

Filtrations: what they have to be

$$K_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2im} E(1)$$

$\leftarrow$ Adams summand

$\rightsquigarrow L_{K_{(2)}} = L_{E(1)} =: L_1.$

Goal: Let $\mathcal{C}$ be a stable model cat and $\Phi : Ho(L_1 Sp) \xrightarrow{\Delta} Ho\,\mathcal{C}$, then $L_1 Sp$ and $\mathcal{C}$ are Quillen equivalent.

Strategy for $Ho(Sp)$:

(1) Quillen functor $X \wedge - : Sp \longrightarrow \mathcal{C} : Hom(X,-)$
$X := \Phi(S^0)^{cf}$

[Schwede-Shipley, Lenhardt]

$\rightsquigarrow$ exact ~~functor~~ endofunctor

$$Ho(Sp) \xrightarrow[X\wedge-]{} Ho(\mathcal{C}) \xrightarrow{\Phi^{-1}} Ho(Sp)$$
$$S^0 \longmapsto S^0$$

(2) Use: S^0 is a compact generator: use this to reduce question to asking if

$$F: [S^0, S^0]_* \longrightarrow [S^0, S^0]_* \text{ is an iso.}$$

If yes, then F is an equivalence

($\Rightarrow X\wedge^L -$ equivalence

$\Rightarrow X\wedge -$ Quillen equivalence)

<u>K-locally:</u> $\Phi : Ho(L_1 Sp) \xrightarrow{\Delta} Ho(\mathcal{C})$,

$X = \Phi(L_1 S^0)^{cf}$

$$\begin{tikzcd}
Sp \arrow[r, shift left, "X\wedge -"] \arrow[d, "id"'] & \mathcal{C} \arrow[l, shift left, "RHom(X,-)"] \\
L_1 Sp \arrow[ur, dashed] &
\end{tikzcd}$$

$\rightsquigarrow$ this is a Quillen functor

$\Leftrightarrow$ $RHom(X,Y)$ $E(1)$-local. $\forall Y$.

How does one check this?

$M = M(\mathbb{Z}/2)$ mod-2 Moore spectrum.

$\Rightarrow$ M has "v_1-self map" $v_1^4 : \Sigma^8 M \to M$ which is a $E(1)_*$-iso

[Adams] [Mahowald, et al] Z $E(1)$-local $\Leftrightarrow$ $[M, Z]_* \xrightarrow{(v_1^4)^*} [M, Z]_{*+8}$ iso

In our case:

$[M, RHom(X,Y)]_* \xrightarrow{v^*} [M, RHom(X,Y)]_{*+8}$ iso $\forall Y$?

$\Leftrightarrow$ $[X\wedge M, Y]_* \xrightarrow{v^*} [X\wedge M, Y]_{*+8}$ iso $\forall Y$

$\Leftrightarrow$ $[M, \Phi^{-1}(Y)]^{L_1 Sp}_* \xrightarrow{\Phi^{-1}(v)^*} [M, \Phi^{-1}(Y)]^{L_1 Sp}_{*+8}$ iso $\forall Y$

identify possible $\Phi^{-1}(X\wedge v_1^4)^* \in [M, M]^{L_1 Sp}_8$

$(\ldots) = v_1^4 + \underset{\text{irrelevant}}{\underline{T}}$ (all induce isos as desired)

$\Rightarrow X \wedge - : L_1 Sp \rightleftarrows \mathcal{C} : \mathrm{Hom}(X,-)$ Quillen adj.

(2) Use: $L_1 S^0$ is a compact generator for $\mathrm{Ho}(L_1 Sp)$

$\rightsquigarrow$ exact endofunctor $F: \mathrm{Ho}(L_1 Sp) \xrightarrow{X \wedge^L -} \mathrm{Ho}\,\mathcal{C}$

$L_1 S^0 \mapsto$

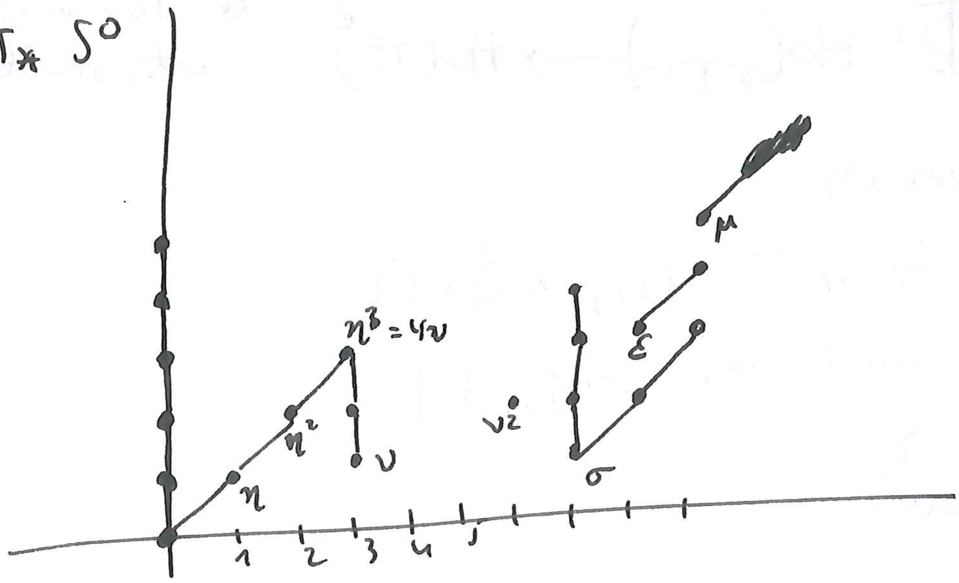
$\xrightarrow{\Phi^{-1}} \mathrm{Ho}(L_1 Sp)$

$\longrightarrow L_1 S^0$

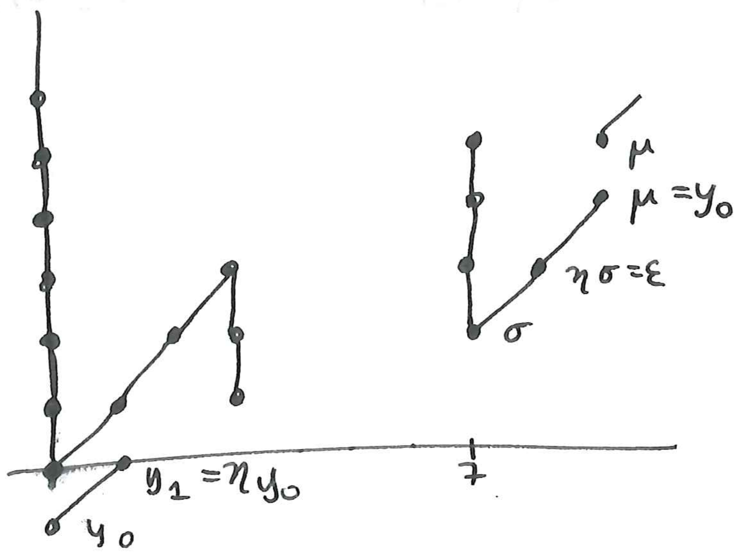
Would like to show:

$F: [L_1 S^0, L_1 S^0]_* \longrightarrow [L_1 S^0, L_1 S^0]_*$ is an iso

$\pi_* S^0$



$\pi_* L_1 S^0$



Non-local: $F : \pi_* S^0 \circlearrowleft$

- induction on Adams filtration $\downarrow$
- left to prove: $\underline{F(\eta)}$, $F(\nu)$, $F(\sigma)$

Locally: reduction in this $\Leftarrow$ direction

Both: $F(\eta)$

p>2: only elt in 1-line of $\pi_* S^0$ is

$$\alpha_1 \in \pi_{2p-3}(S^0_{(p)}) = \mathbb{Z}/p.$$

$$F : Ho(Sp) \to Ho(Sp)$$

Equivariantly: $\Phi : Ho(L_1 Sp_G) \longrightarrow Ho(\mathcal{C})$ G-top model cat, stable

- triangulated eq.
- $\Phi(\Sigma^\infty_+ G/H) \cong G/H_+ \wedge \Phi(S^0)$
- natural w.r.t. res. conj, trf

$\Rightarrow L_1 Sp_G \underset{G\text{-Quillen}}{\simeq} \mathcal{C}$

Localize Sp_G w.r.t. $\{ \Sigma^\infty_+ G/H \wedge \Sigma^8 M \xrightarrow{1 \wedge v_1^4} \Sigma^\infty_+ G/H \wedge M \mid H \leq G \}$