Masterclass: Rigidity and algebraic models in stable homotopy theory 2018-04-12

Talk: Irakli Patchkoria : Equivariant Rigidity

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There ~~are~~ are many models for G-equivariant stable homotopy theory:

- Sp_G - orthogonal G-spectra.
- $G\text{-}Sp_{LMS}$ Lewis-May G-spectra
- $\mathcal{M}_{S_G}$ - S_G-modules (Mandell-May)
- Sp_G^{Σ} - symmetric G-spectra (Mandell, Hausmann)

All these are Quillen equivalent

Properties of these models $\mathcal{C} \in \{Sp_G, \mathcal{M}_{S_G}, \dots\}$

(i) • $X, Y \in \mathcal{C}$ $\quad Map(X,Y) \in G\text{-}Top_*$

(ii) • $K \in G\text{-}Top_*$, $\quad K \underset{\mathcal{C}}{\wedge} X, \; X^K, \; \forall X \in \mathcal{C}$.

(iii) • These are homotopically well-behaved

(iv) • These models are G-stable: $\forall V \in G\text{-}Rep^{O}$

$$S^V \wedge - : Ho\,\mathcal{C} \xrightarrow{\simeq} Ho\,\mathcal{C}$$

Def: A G-equivariant G-stable model category $\mathcal{C}$ is a cofibrantly gen model category with (i) - (iv)

Thm: Let $\mathcal{C}$ be a G-equivariant G-stable model category. (G finite)

A) (P.) If $\psi : Ho(\cancel{Sp_G}\; Sp_{G,(2)}) \underset{\Delta}{\xrightarrow{\sim}} Ho(\mathcal{C})$

(*) such that $\psi(\Sigma^\infty G/H_+) \simeq G/H_+ \wedge \psi(\mathbb{S}) \quad \forall H \leq G$

natural w.r.t. ~~res~~ res, tr, conj

Then $\mathcal{C}$ and $Sp_{G,(2)}$ are G-Quillen equivalent.

B) p is odd $\quad p \nmid |G|$

$\psi : Ho(Sp_{G,(p)}) \xrightarrow[\Delta]{\sim} Ho\mathcal{C}$ s.t. $(*)$ is satisfied

Then $Sp_{G,(p)}$ is G-Quillen equiv to $\mathcal{C}$.

$\boxed{\text{A) + B) } \Rightarrow Sp_G \text{ for any finite 2-group } G \text{ is equivariantly rigid.}}$

C) (jt. with Roitzheim) If $\psi : Ho(L_1 Sp_{G,(2)}) \xrightarrow[\Delta]{\sim} Ho(\mathcal{C})$ s.t. ψ satisfies $(*)$, then $L_1 Sp_{G,(2)} \underset{G\text{-Quillen}}{\simeq} \mathcal{C}$

D) If $p \geq 5$, $p \nmid |G|$, then $L_1 Sp_{G,(p)}$ has an exotic model.

<u>Rationally:</u> Schwede-Shipley $\Rightarrow Sp_{G,\mathbb{Q}}$ is rigid.

Shipley (Barnes-Roitzheim) $\Rightarrow Sp_{S^1,\mathbb{Q}}$ is rigid

- W.l.o.g. assume $\mathcal{C}$ is G-spectral, $\forall X, Y$: $map(X,Y) \in Sp_G$ $\quad (\mathcal{C} \underset{\mathbb{Q}\text{-Quillen}}{\simeq} Sp^O(\mathcal{C}))$ ← internal orth. spectra

- $G/H_+ \wedge X \qquad X = \psi(\mathbb{S})$

- $[G/H_+, G/K_+]^G_0$ in $Ho(Sp^G)$

$$\cong \bigoplus_{[g] \in H \backslash G / K} A(H \cap {}^gK)$$

$$\begin{tikzcd} \Sigma^\infty G/H_+ \arrow[r, "res^H_K"] \arrow[d, "(-)\cdot g"', "conjugation"] & \Sigma^\infty G/K_+ \arrow[l, bend left, "tr^H_K"] \\ \Sigma^\infty G/{}^gH_+ & \end{tikzcd}$$

$H \leq K$ (PT constr.)

$$\psi(G/H_+) \simeq G/H_+ \wedge \psi(\mathbb{S})$$

$L_1 Sp_G$ - $K_{(2)}$-localization of Sp_G i.e.

$$X \xrightarrow{\sim} Y \iff K_*(X^H) \otimes \mathbb{Z}_{(2)} \xrightarrow{\cong} K_*(Y^H) \otimes \mathbb{Z}_{(2)} \quad \forall H \leq G$$

Proof: A, B: $\mathcal{C}$ is spectral, $\Psi : Ho(Sp_G) \xrightarrow{\sim} Ho(\mathcal{C})$

$$\mathbb{S} \longmapsto X := \Psi(\mathbb{S})$$

$$Sp_G \underset{F_n(X,-)}{\overset{X \wedge -}{\rightleftarrows}} \mathcal{C}$$

$$\begin{tikzcd} Ho(Sp_G) \arrow[r, "X \wedge^L -"] \arrow[rr, bend right, "F"'] & Ho(\mathcal{C}) \arrow[r, "\Psi^{-1}"] & Ho(Sp_G) \end{tikzcd}$$

To prove A) & B) it is enough to show F is an equivalence.

- $F(\Sigma^\infty G/H_+) = \Sigma^\infty G/H_+$, $F(res) = res$ $F(tr) = tr$
  $F(conj) = conj$
- Since $\{G/H_+ \mid H \leq G\}$ are compact gen. of $Ho(Sp_G)$ it is enough to show $F : [G/H_+, G/K_+]^G_* \longrightarrow [G/H_+, G/K_+]^G_*$ is an iso $\forall * \in \mathbb{Z}$.

If $* = 0$, then F is an iso.

$$[G/H_+, G/K_+]^G_* \cong \bigoplus_{[g] \in H\backslash G/K} \bigoplus_{(L) \leq H \cap {}^gK} \pi_*(BW_{H\cap {}^gK}(L)_+)$$

are finite if $* > 0$

It is enough to show that F is injective.

- We can reduce to the case $H = K$;

$$\begin{tikzcd}
{[G/H_+, G/K_+]^G_*} \arrow[r, "\mathrm{res}^*"] \arrow[d, "F"] \arrow[drr, bend left] & {[G/H\cap {}^gK_+, G/K_+]^G_*} \arrow[r, "\mathrm{conj}_g^*"'] & {[G/H\cap {}^gK_+, G/{}^gK_+]^G_*} \arrow[d, "\mathrm{tr}_*"] \\
{[G/H_+, G/K_+]^G_*} \arrow[drr, bend left] & & \bigoplus_{[g]\in H\backslash G/K} [G/H\cap {}^gK_+, G/H\cap {}^gK_+]^G_* \arrow[d, "\oplus F"] \\
 & & \bigoplus_{[g]\in H\backslash G/K} [G/H\cap {}^gK_+, G/H\cap {}^gK_+]^G_*
\end{tikzcd}$$

$$F : [G/H_+, G/H_+]^G_* \longrightarrow [G/H_+, G/H_+]^G_* \quad ?$$

Induction on $|H|$.

- $H = 1$:

$$\begin{tikzcd}
F : {[G_+, G_+]^G_*} \arrow[r, "(\cong ?)"] \arrow[d, phantom, "\cong" rotate=90] & {[G_+, G_+]^G_*} \arrow[d, phantom, "\cong" rotate=90] \\
\pi_* \mathbb{S}_{(2)}[G] \arrow[r] & \pi_* \mathbb{S}_{(2)}[G]
\end{tikzcd}$$

<u>$p=2$:</u> $* \leq 7$ is enough, $F(\eta \cdot 1) = \eta \cdot 1$.

$F(g) = g = 1 \cdot g$

$$F(\nu) = m \cdot \nu + \sum_{g \in G\setminus\{1\}} n_g \nu \cdot g$$

$$\begin{tikzcd}
F : \pi_3 \mathbb{S}_{(2)}[G] \arrow[r] \arrow[d, equal] & \pi_3 \mathbb{S}_{(2)}[G] \arrow[d, equal] \\
\mathbb{Z}/8[G] \arrow[r] & \mathbb{Z}/8[G]
\end{tikzcd}$$

$$F(\eta\nu) = F(\eta^3) = \eta^3 = \eta\nu$$

$$\| $$

$$\eta m \nu + \eta \sum_{g \in G\setminus\{1\}} n_g g \nu$$

$\underline{p \text{ odd}}$: We need to show

$$F : \pi_{2p-3} \mathbb{S}_{(p)}[G] \xrightarrow{(\cong ?)} \pi_{2p-3}\mathbb{S}[G]$$

$$\mathbb{Z}/p[G] \qquad\qquad \mathbb{Z}/p[G]$$

$p \nmid |G|$

Mashke's theorem:

$\mathbb{Z}/p[G] \cong A_1 \oplus A_2 \oplus \cdots \oplus A_s$

irr. decomp as $G\times G^{op}$-modules.

$a_1, \dots, a_s$ - orth. idempotents $\quad 1 = \sum a_i$

$F(\alpha_1 a_i) \neq 0$.

$a_i \in \mathbb{Z}_p[G] \qquad \overline{a_i} \in \mathbb{Z}_{(p)}[G]$

$G_+ \xrightarrow{\overline{a_i}} G_+ \xrightarrow{\overline{a_i}} G_+ \xrightarrow{\overline{a_i}}$

$\operatorname{hocolim}(\overline{a_i})$

Suppose $\quad F(\alpha_1 \alpha_i) = 0. \quad \alpha_1 a_i \wedge X = 0.$

$\Rightarrow \quad \alpha_1 \wedge \operatorname{hocolim}(\overline{a_i}) \wedge X = 0$

Stefan $\Rightarrow \quad p\text{-ord}(\operatorname{hocolim}(\overline{a_i}) \wedge X/p) \geq p-1$

$H^*_G(-, A_i) \qquad p\text{-ord}(\operatorname{hocolim}(\overline{a_i}) \wedge \mathbb{S}/p) \leq p-2$

$\Rightarrow p\text{-ord}(\operatorname{hocolim}(\overline{a_i}) \wedge \cancel{X/p}) \leq p-2$ $\quad \square$

$\mathbb{S}/p \wedge X$

$H = 1$.

$$\begin{tikzcd}
0 \arrow[r] & {[G/H_+, G \ltimes_H EP(H)_+]^G_*} \arrow[r] \arrow[d, "F", "\cong"'] & {[G/H_+, G/H_+]^G_*} \arrow[r, shift left, "\Phi^H"] \arrow[d, "F", "\cong"'] & {[W(H)_+, W(H)_+]^{W(H)}_*} \arrow[l, shift left, "\varepsilon^*"] \arrow[r] \arrow[d, "\Phi^H \circ \varepsilon^*", "\cong"'] & 0 \\
& {[G/H_+, G \ltimes_H EP(H)_+]^G_*} \arrow[r] & {[G/H_+, G/H_+]^G_h} \arrow[r, "\Phi^H"] & {[W(H)_+, W(H)_+]^{W(H)}} \arrow[r] & 0
\end{tikzcd}$$

$F(G \ltimes_H EP(H)_+) \simeq G \ltimes_H EP(H)_+$

<u>1^{st} open case:</u> $\boxed{Sp_{C_3,(3)}}$

<u>$L_1 Sp_G$:</u>

$$\begin{tikzcd}
Sp_G \arrow[r, shift left, "X \wedge -"] \arrow[d] & \mathcal{C} \arrow[l, shift left, "Fun(X,-)"] \\
L_1 Sp_G &
\end{tikzcd}$$

$Ho\,\mathcal{C} \simeq_{\Delta} Ho(L_1 Sp_G)$

$RFun(X, Y) \in Sp_G$ is local.

$$Ho(Sp_G) \xrightarrow{X \wedge^L -} Ho\,\mathcal{C} \xrightarrow{\varphi^{-1}} Ho(L_1 Sp_G)$$

$F(v_1^4 \wedge G/H_+)$ is an iso ✓.

Main difference: $[L_1 G/H_+, L_1 G/K_+]^G_*$ are not finite when ~~$* = 2, 0$!~~ $* = 2, 0$

but

$[L_1 G/H_+ \wedge \mathbb{S}/2, L_1 G/K_+]^G_*$

finite because $\underset{\text{finite}}{K(1)_*(B\Gamma)}$ is finite $\forall *$

<u>2^{nd} open case:</u> $L_1 Sp_{C_5,(5)}$