Masterclass: Rigidity and algebraic models in stable homotopy theory: Lecture 1 — 2018-04-09 —

"The stable homotopy category has an essentially unique model"

* Model:
  - Model category (Quillen)
  - infinity category (quasi-category)
  - derivators
  - cofibration categories
  - ⋮

<u>Thm:</u> (original version): Let $\mathcal{C}$ be a stable model category. If $Ho(\mathcal{C})$ is equivalent to the ~~stha~~ stable homotopy category as triangulated categories, then there is a Quillen equivalence

$$\text{Spectra} \xrightarrow{\sim} \mathcal{C}$$

<u>Rigidity Thm (modern version):</u> Let $\mathcal{C}$ be a stable ∞-category. If $Ho(\mathcal{C})$ is equivalent to the homotopy category of finite spectra as triangulated categories, then $\mathcal{C}$ is equivalent to $\text{Spectra}^{\text{fin}}$

<u>Ex:</u> G finite group, k field.
$\mathcal{C} =$ ~~[illegible]~~ kG-modules
cofibrations = monomorphisms
fibrations = epimorphism.
Two morphisms $f, g: M \longrightarrow N$ in kG-modules are <u>homotopic</u> if $f-g: M \longrightarrow N$ factors through

a projective module.

weak equivalences = homotopy equivalences

$\mathrm{Ho}(\mathcal{C})$ = stable module category = kG-Stmod

= kG-$\underline{\mathrm{Mod}}$.

Let $\mathcal{C}$ be a model category

The <u>homotopy category</u> is an initial example of a functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}\,\mathcal{C}$ that takes weak equivalences to isomorphisms.

<u>Construction:</u> $\mathrm{Ob}(\mathrm{Ho}\,\mathcal{C}) = \mathrm{Ob}\,\mathcal{C}$

$$\mathrm{Ho}(\mathcal{C})(X, Y) = \mathcal{C}(X^c, Y^f)/\sim$$

$X^c \xrightarrow{\sim} X$ cofibrant replacement

$Y \xrightarrow{\sim} Y^f$ fibrant replacement

Every model category $\mathcal{C}$ has an associated quasi-category $\mathcal{C}[W^{-1}]$ the quasi-category localization s.t.

$$\mathrm{Ho}(\mathcal{C}[W^{-1}]) \simeq \mathrm{Ho}(\mathcal{C})$$

$\uparrow$ ∞-category sense $\quad$ $\uparrow$ model category sense

Universal property: There is a functor

$N\mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ initial among morphisms that invert weak equivalences

Possible constructions:

- If $\mathcal{C}$ is a simplicial model category, can take

$$\mathcal{C}[W^{-1}] = N^{\text{cordier}}\left(\begin{array}{c}\text{full simplicially}\\ \text{enriched subcategory}\\ \text{on bifibrant objects}\end{array}\right)$$

- If $\mathcal{C}$ is a cofibration category
  Szumito $\rightsquigarrow$ quasi-category of frames.
  $\mathcal{C}$ model category structure $\Rightarrow \mathcal{C}[W^{-1}]$ $= \text{Frames}^{\text{Szumito}}(\mathcal{C}^{\text{cof}})$

Stable model categories

Def: A model category is pointed if it has a zero object.

Construction: $\mathcal{C}$ pointed model category with zero object $*$.

$X = ob(\mathcal{C})$, choose cofibration $X \rightarrowtail CX \xrightarrow{\sim} *$

The suspension is a pushout

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & CX \simeq * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

Dually: Choose fibration $* \simeq PX \twoheadrightarrow X$
choose pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[d, two heads] \\ * \arrow[r] & X \end{tikzcd}$$

loop object.

<u>Prop</u>: These constructions "derived" to an adjoint functor pair

$$\mathrm{Ho}\,\mathcal{C} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{C}$$

<u>Def</u>: A <u>stable model category</u> is a pointed model category such that (Σ, Ω) are adjoint equivalences of $\mathrm{Ho}(\mathcal{C})$

<u>Non-ex</u>: based spaces

<u>Ex</u>: $\mathcal{C} = Ch_A$ = $\mathbb{Z}$-graded chain complexes of A-modules ~~[illegible]~~ w.eq. = quasi-isomorphisms.

<u>Exercise</u>: Show that (kG-mod, st. equivalences) is stable

Key Hint: proj & injectives are the same.

<u>Remark</u>: If $\mathcal{C}$ is a stable model category, then $\mathcal{C}[W^{-1}]$ is a stable quasi-category.

<u>The universal stable model category</u>: Spectra

~~A Z Spectra~~

A <u>spectrum</u> consists of based simplicial sets ~~spaces~~ $\{X_n\}_{n\geq 0}$ and ~~continuous~~ based maps σ_n: $X_n \wedge S^1 \longrightarrow X_{n+1}$.

A <u>morphism</u> of spectra $f: X \to Y$ consists of based continuous maps $f_n: X_n \to Y_n$ s.t.

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "f_n \wedge S^1"] \arrow[d, "\sigma_n"'] & Y_n \wedge S^1 \arrow[d, "\sigma_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1} \end{tikzcd}$$

The __k-th stable homotopy group__, $k \in \mathbb{Z}$ is

$$\pi_k(X) = \operatorname{colim}_{n \gg 0} \pi_{k+n}(X_n, *) \quad , \text{ along}$$

$$\pi_{k+n}(X_n,*) \xrightarrow{-\wedge S^1} \pi_{k+n+1}(X_n \wedge S^1, *) \xrightarrow{(\sigma_n)_*} \pi_{k+n+1}(X_{n+1}, *)$$

- A morphism $f: X \to Y$ is a stable equivalence if $\pi_k(f)$ is an isomorphism for all $k \in \mathbb{Z}$.
- $f: X \to Y$ is a __cofibration__ if $f_0: X_0 \to Y_0$ and

$$f_{n+1} \cup \sigma_n : X_{n+1} \cup_{X_n \wedge S^1} Y_n \wedge S^1 \longrightarrow Y_{n+1}$$ are all cofibrations. (i.e. monomorphisms)

- $f: X \to Y$ is a __stable fibration__ if $f_n : X_n \to Y_n$ are fibrations (i.e. Kan fibrations) and

$$\begin{tikzcd} X_n \arrow[r, "\tilde{\sigma}_n"] \arrow[d, "f_n"'] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \\ Y_n \arrow[r, "\tilde{\sigma}_n"'] & \Omega Y_{n+1} \end{tikzcd}$$ is homotopy cartesian.

__Theorem:__ The stable equivalences, cofibrations and stable fibrations ~~form~~ form a stable model structure of the category of spectra.

2018-04-10

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Lecture 3

A ~~spectrum~~ spectrum consists of based simplicial sets X_n, $n \geq 0$ and morphisms $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$.

A morphism $f: X \longrightarrow Y$ is a stable equivalence if $\pi_k f$ is iso, $k \in \mathbb{Z}$

Thm: [BF] The stable equivalence are part of a stable model structure on the category of spectra.

Other model categories of spectra

- symmetric spectra
- orthogonal spectra
- Kan ~~sets~~ semisimplicial spectra
- S-modules

The ∞-category of spectra can be obtained

- take quasi-categorical localitation of some category of spectra
- start with ∞-category of based spaces. and then stabilize.

$$\Sigma^\infty_+ : \mathrm{Spaces}^{\mathrm{fin}} \longrightarrow \mathrm{Spectra}^{\mathrm{fin}}$$

initial example among left exact functors to stable ∞-categories.

Def: The stable homotopy category is $SHC = Ho(\mathrm{Spectra})$

Thm: (folklore, Quillen, Hovey, Lurie):
The homotopy category of a stable model category / stable ∞-category comes with a preferred ~~…~~ triangulated structure.

- The suspension/shift $\Sigma: Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{C})$ is the (derived) suspension functor, an equivalence by stability
- Recall the ~~…~~ functor $\gamma: \mathcal{C} \to Ho\,\mathcal{C}$.

Let $f: A \to B$ be any morph. between cofibrant objects. Let $i: A \rightarrowtail CA \simeq *$ be the chosen cone.
Then the images in $Ho\,\mathcal{C}$ of the diagrams are exact triangles

$$A \xrightarrow{f} B \xrightarrow{i} CA \cup_A B = Cf \longrightarrow CA\cup_A B / B \cong \Sigma A$$

+ close up under isomorphism.

~~Thm~~

Additivity: For every fibrant X, there is a "concatenation of loops" $\Omega X \times \Omega X \longrightarrow \Omega X$, making ΩX into a group object in $Ho(\mathcal{C})$
Since $\mathcal{C}$ is stable, any $Y \in \mathcal{C}$ is isomorphic in $Ho\,\mathcal{C}$ to $\Omega(\Sigma Y)$, so all objects are naturally group objects, double ~~…~~ loop objects, ..

<u>Def:</u> A <u>Quillen functor pair</u> ~~that~~ between model categories is an adjoint pair

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that the following equivalent conditions hold:

- F preserves cofibrations and acyclic cofibrations
- G preserves fibrations and acyclic fibrations
- F preserves cofibrations and G preserves fibrations.

<u>Prop:</u> A Quillen pair (F, G) has adjoint pair of functors on the homotopy categories.

$$\mathrm{Ho}\,\mathcal{C} \underset{RG}{\overset{LF}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{D}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{D} \arrow[d] \\ \mathrm{Ho}\,\mathcal{C} \arrow[r, "LF"'] & \mathrm{Ho}\,\mathcal{D} \end{tikzcd}$$

<u>Construction</u> $(LF)(A) = F(A^{cof})$

<u>Remark:</u> If $\mathcal{C}$ and $\mathcal{D}$ are stable, then LF and RG are canonically exact functors.

An <u>exact functor</u> between triangulated categories is a pair (F, τ) where $F: \mathcal{T} \to \mathcal{S}$, additive

$\tau: \Sigma F \cong F \Sigma$ such that for every exact triangle in $\mathcal{T}$

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

the following is exact

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{Fh} F\Sigma A \underset{\tau}{\cong} \Sigma FA$$

A Quillen pair (F, G) is a Quillen equivalence if the following equivalent conditions hold:

- for all cofibrant $A \in \mathcal{C}$, fibrant $X \in \mathcal{D}$ and morphism $f: A \to GX$, f is a weak equivalence in $\mathcal{C}$ $\Leftrightarrow$ $f^{\#}: FA \to X$ is a weak equivalence in $\mathcal{D}$.
- The derived functors $LF: Ho\,\mathcal{C} \to Ho\,\mathcal{D}$ and $RG: Ho(\mathcal{D}) \to Ho(\mathcal{C})$ are equivalences.

<u>Remark:</u> A Quillen equivalence also induces an equivalence of ∞-categories $\mathcal{C}[W^{-1}] \simeq \mathcal{D}[W^{-1}]$

<u>Ex:</u>

- $Top \underset{S}{\overset{|-|}{\leftrightarrows}}$ simpl. sets is a Quillen equivalence
- A ring, HA associated symmetric ring spectrum (EM-spectrum)

  $Ch_A \underset{Q}{\simeq} HA\text{-Mod}$
- spectra $\underset{Q}{\simeq}$ symmetric spectra $\underset{Q}{\simeq}$ orthogonal spectra

  $\underset{Q}{\simeq}$ semi-simplicial spectra $\underset{Q}{\simeq}$ S-modules.

<u>Thm:</u> (Rickard) Let A, B be two rings s.t. $D(A) \simeq D(B)$ as Δed categories.

Then $Ch_A \underset{Q}{\simeq} Ch_B$ "tilting theory".

Some exotic derived equivalences:

- $K(n)$ n-th Morava K-theory spectrum at prime p, admits an A_∞-ring spectrum structure

$$\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$$

$$Ho(K(n)\text{-mod}) \underset{\Delta cd}{\simeq} D(\mathbb{F}_p[v_n^{\pm 1}])$$

- Similarly: $Ho(KU\text{-mod}) \simeq D(\mathbb{Z}[u, u^{-1}]\text{-mod})$ as categories

<u>Ex:</u> $\mathcal{C} = \mathbb{F}_2[\varepsilon]\text{mod}$ $\quad \varepsilon^2 = 0$

$\mathcal{D} = \mathbb{Z}/4\text{-mod}$

$Ho\,\mathcal{C} = \mathbb{F}_2\text{-vectorspaces} \cong Ho\,\mathcal{D}$.

But $\mathcal{C} \underset{\text{Quillen}}{\not\simeq} \mathcal{D}$

<u>Thm:</u> (Universal property of spectra)
Let $\mathcal{C}$ be a stable model category and X a bifibrant object. Then there is a Quillen pair

$$Sp \underset{Hom(X,-)}{\overset{-\wedge X}{\rightleftarrows}} \mathcal{C} \quad \text{such that} \quad \underset{(-\wedge X)(\mathbb{S})}{\mathbb{S} \wedge X} \cong X$$

Construction if $\mathcal{C}$ is <u>simplicial model category</u>

<u>Remark:</u> If $\mathcal{C}$ is simplicial, the suspension and loop are modeled functorially by $\mathcal{C} \underset{\Omega = map_*(S^1, -)}{\overset{\wedge S^1}{\rightleftarrows}} \mathcal{C}$ $\quad S^1 = \Delta[1]/\partial$

We construct bifibrant objects $\omega^n X$, $n \geq 0$ with $\omega^0 X = X$

$$\varphi_n : \underset{\substack{\uparrow \\ \text{cofibrant}}}{\omega^n X} \xrightarrow{\ \simeq\ } \Omega\, \omega^{n-1} X$$

For $Y \in \mathcal{C}$, define

$$\mathrm{Hom}(X,Y)_n = \mathrm{map}_*(\omega^n X, Y)$$

with structure maps

$$\mathrm{map}_*(\omega^{n-1}X, Y) \wedge S^1 \xrightarrow{\text{adjunction}} \mathrm{map}_*(\Omega\, \omega^{n-1}X, Y) \xrightarrow{\ \varphi_n^*\ } \mathrm{map}_*(\omega^n X, Y)$$

Observe: $\mathrm{Hom}(X,-)$ preserves limits, and it has an adjoint.

<u>Thm:</u> (Lurie) Let $\mathcal{C}$ be a stable ∞-category. Then the evaluation at $\mathbb{S}$ is an equivalence

$$\begin{aligned} \mathrm{Func}^{l.e.}(\mathrm{Spectra}^{fin}, \mathcal{C}) &\xrightarrow{\ \simeq\ } \mathcal{C} \\ F &\longmapsto F(\mathbb{S}) \end{aligned}$$

Let $\mathcal{C}$ be a presentable stable ∞-category. Then

$$\mathrm{Funct}^{l.e.}(\mathrm{Spectra}, \mathcal{C}) \longrightarrow \mathcal{C}, \quad F \mapsto F(\mathbb{S})$$

is an equivalence.

$$\Phi : \mathrm{Ho}(\mathrm{Spectra}) \cong \mathrm{Ho}\,\mathcal{C} \qquad \Big\| \qquad \mathrm{Spectra} \longrightarrow \mathcal{C}$$

$$\mathbb{S} \longmapsto X \qquad\qquad\qquad \mathbb{S} \longmapsto X$$

$$\begin{tikzcd} \mathrm{Ho}(\mathrm{Spectra}) \arrow[r, "- \wedge^L X"] \arrow[rr, bend left, "F"] & \mathrm{Ho}\,\mathcal{C} \arrow[r, "\Phi^{-1}", "\simeq"'] & \mathrm{Ho}(\mathcal{C}) \\ \mathbb{S} \arrow[r, mapsto] & X \arrow[r, mapsto] & \mathbb{S} \end{tikzcd}$$

Schwede: Lecture 3

Let $\mathcal{C}$ be a stable $\left\{\begin{array}{l}\text{model category}\\ \infty\text{-category}\\ \text{derivator}\end{array}\right\}$

and $\Phi: \mathrm{Ho}(\mathrm{Spectra}) \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{C})$ a triangulated equivalence. $\mathbb{S} \longmapsto \Phi(\mathbb{S}) =: X$

The universal property of spectra produces a map of $\left\{\begin{array}{l}\text{model categories}\\ \infty\text{-categories}\\ \text{derivators}\end{array}\right.$ $\mathrm{Spectra} \xrightarrow{\wedge X} \mathcal{C}$, $\mathbb{S} \longmapsto \mathbb{S} \wedge X = X$

$\rightsquigarrow \; - \wedge^L X : \mathrm{Ho}(\mathrm{Spectra}) \to \mathrm{Ho}(\mathcal{C})$

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Ho}(\mathrm{Spectra}) \arrow[r, "-\wedge^L X"] \arrow[rr, bend right, "F"'] & \mathrm{Ho}(\mathcal{C}) \arrow[r, "\simeq", "\Phi"'] & \mathrm{Ho}(\mathrm{Spectra}) \end{tikzcd}$$

If we can show that F is an endoequivalence, then $-\wedge^L X$ is an equivalence, and we are done.

Thm: Let p be a prime, and $F: \mathrm{Ho}(Sp_{(p)})\circlearrowleft$ an exact endofunctor s.th. $F(\mathbb{S}_{(p)}) \simeq \mathbb{S}_{(p)}$

i) If $p=2$, F is an equivalence.

ii) If $p \geq 3$ and $F(\alpha_1 : \Sigma^{2p-3}\mathbb{S} \to \mathbb{S}) \neq 0$, then F is an equivalence.

($\alpha_1 \in \pi_{2p-3}\mathbb{S}$ generates p-torsion)

Proof: Later.

Slogan: "Stable homotopy of spheres is generated by Adams filtration 1 elements and Toda brackets"

Careful: $F(\alpha_1 : \mathbb{S}^{2p-3} \to \mathbb{S}) \overset{?}{\longleftrightarrow} \Sigma^{2p-3} F(\mathbb{S}^{2p-3} \to F(\mathbb{S}))$

$$\Sigma^{2p-3}\mathbb{S} \xrightarrow[\alpha_1]{\not\simeq} \mathbb{S}$$

The p-order

p prime, $\mathcal{T}$ triangulated category, $K \in Ob(\mathcal{T})$.
Write K/p for any object that is part of an exact Δ

$$K \xrightarrow{\cdot p} K \xrightarrow{\pi} K/p \longrightarrow \Sigma K$$

An extension of a morphism $f : K \to Y$ is a morphism $\bar{f} : K/p \to Y$ s.th. $\bar{f} \circ \pi = f$

$\bar{f}$ exists $\Leftrightarrow$ $pf = 0$

Def: We define the p-order $p\text{-}ord(K) \in \mathbb{N} \cup \{\infty\}$ by induction:

- $p\text{-}ord(Y) \geq 0$ for all $Y \in ob(\mathcal{T})$
- $p\text{-}ord(Y) \geq k+1 :\Leftrightarrow$ for every K of $\mathcal{T}$ and every $f : K \to Y$ there is an extension $\bar{f} : K/p \to Y$ such that some (hence any) cone of $\bar{f}$ has p-order $\geq k$

$p\text{-}ord(Y) = \max\{k : p\text{-}ord(Y) \geq k\}$

Ex: $p\text{-ord}(Y) \geq 1 \iff p \cdot \mathrm{Id}_Y = 0$

Thm. A: Let $\mathbb{S}/p \in \mathcal{SHC}$ denote the mod p Moore spectrum

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \to \mathbb{S}/p \to \Sigma \mathbb{S}$$

Then $p\text{-ord}(\mathbb{S}/p) \leq p-2$

Thm B: Let $\mathcal{C}$ be a stable model category / ∞-cat, $X \in \mathrm{Ho}(\mathcal{C})$. Then:

a) $p\text{-ord}(X/p) \geq p-2$

b) If $\alpha_1 \wedge X = 0$, then $p\text{-ord}(X/p) \geq p-1$ (where $\alpha_1 = \eta$ for $p=2$)

~~Proof of Thm A~~

$p=2$: $2 \cdot \mathrm{id}_{\mathbb{S}/2} \neq 0$ (classical)

p odd: Recall: X space/spectrum

$$\beta : H^*(X, \mathbb{F}_p) \to H^{*+1}(X, \mathbb{F}_p) \quad \text{Bockstein}$$

$$\beta \circ \beta = 0$$

Steenrod operations:

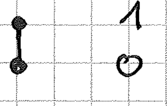
$i \geq 1$: $P^i : H^*(X, \mathbb{F}_p) \longrightarrow H^{*+i \cdot q}(X; \mathbb{F}_p)$

$q = 2p-2$

- β and P^i, $i \geq 1$, generate the algebra of all stable cohomology operations
- Satisfy Adem relations

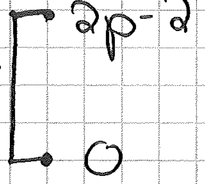
Ex: • β detects $p \in \pi_0 \mathbb{S}$:

$$\beta : H^0(\mathbb{S}/p, \mathbb{F}_p) \xrightarrow{\cong} H^1(\mathbb{S}/p, \mathbb{F}_p)$$

In pictures: 

• P^1 detects $\alpha_1 \in \pi_{2p-3} \mathbb{S}$:

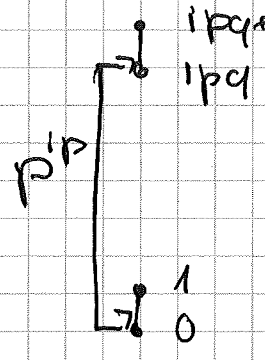
$$P^1 : H^0(C(\alpha_1), \mathbb{F}_p) \xrightarrow{\cong} H^{2p-2}(C(\alpha_1), \mathbb{F}_p)$$

In pictures: 

Proof of Thm A: Let X be a finite p-local spectrum. Consider the following condition on X: $i \geq 0$

(C_i) The mod p cohomology of X is 1-dimensional in degrees jpq and $jpq+1$ for $j = 0, \dots, i$, connected by a β, and

$$P^{ip} : H^0(X, \mathbb{F}_p) \longrightarrow H^{ipq}(X, \mathbb{F}_p) \text{ is non-zero.}$$

Picture: 

← many more relations forced by the conditions and Adem relations

Note: $\mathbb{S}/p$ satisfies C_0

Claim 1: Suppose X satisfies C_i for some $i \in \{0, \dots, p-1\}$. Then there is a morphism $f: S^{(i+1)pq-1} \to X$ detected by P^p, i.e., this:

$$P^p: H^{ipq}(Cf, \mathbb{F}_p) \to H^{(i+1)pq}(Cf, \mathbb{F}_p) \text{ is non-zero}$$

Proof: $X/\underset{\text{skeleton}}{Sk^{(ipq-1)}}X \simeq \Sigma^{ipq} \mathbb{S}/p$

Start with $\tilde{\beta}_1: S^{(i+1)pq-1} \to X/Sk^{(ipq-1)}X$ where $\beta_1 \in \pi_{pq-2}\mathbb{S}$, $p \cdot \beta_1 = 0$

This is detected by P^p

Fact: $\pi_{jpq-2}\, \mathbb{S}/p = 0$ for $j = 2, \dots, p$

Proof: Adams-Novikov-spectral sequence

$\Rightarrow$ There are no obstructions to extending $\tilde{\beta}_1$ to have image in X. $\square$

Claim 2: If X satisfies $C(i)$, then $p\text{-ord}(X) \leq p-2-i$

[$\mathbb{S}/p$ satisfies C_0, so $p\text{-ord}(\mathbb{S}/p) \leq p-2$], $i \in \{0, \dots, p-1\}$

Proof: By downward induction on i:

Start: $i = p-1 \iff p\text{-ord}(X) \leq -1$, i.e., there does not exist any X satisfying C_{p-1}.

Suppose X existed: Let $f: S^{p^2q-1} \to X$ be as in Claim 1, detected by P^p on $H^*(C(f), \mathbb{F}_p)$, we would have $P^p \circ P^{(p-1)p}: H^q(Cf, \mathbb{F}_p) \to H^{p^2q}(Cf, \mathbb{F}_p)$ $\neq 0$

Adem relation:

$$P^p \circ P^{(p-1)p} \underset{\text{Adem}}{=} P^{p^2-1} P^1 \underset{\text{dimensional}}{=} 0$$

Informally: $\underbrace{\langle p, \beta_1, \dots, p, \beta_1, p \rangle}_{2p-1} \neq 0$

Toda bracket

Inductive step: Suppose X satisfies (C_i) and Claim 2 is known for $i+1$. By Claim 1, there is $f: S^{(i+1)pq-1} \to X$ detected by P^p.

~~Choose an exte~~

Let $\bar{f}: S/p^{(i+1)pq-1} \to X$ be any extension and $C\bar{f}$ its cone. Then $C\bar{f}$ satisfies C_{i+1}, since

$$P^p \circ P^{ip} = (i+1) \cdot P^{(i+1)p} + \underbrace{P^{(i+1)p-1} \circ P^1}_{=0 \text{ (dimension reasons)}}$$

$\neq 0$ $\Rightarrow$ non-zero

We know that $p\text{-ord}(C\bar{f}) \leq p-3-i$ by induction. By the definition of p-order:

$$\Rightarrow p\text{-ord}(X) \leq p-2-i$$

Aside: Let R be a p-local E_∞-ring spectrum

$\rightsquigarrow$ homotopy power operation

$$\beta P^{k+1}: \pi_{2k} R \to \pi_{2pk+2p-3} R$$

If $x \in \pi_{2k} R$, $y \in \pi_* R$ even degree and $xy = 0$

Then $\beta P^{k+1}(x) \circ y^{p-1} \in \langle \underbrace{x, y, x, y, \dots, y, x}_{2p-1} \rangle$

Special cases:

$r = \vartheta(p),\ x = p\,e^{\bar{n}}u\vartheta,\ y = \beta_1,\ \beta P'(p) = a_1.$

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Lecture 7:

p prime, X object of $\mathcal{T}$ (Δ-ed category), ~~p-ord(X) ∈ N~~,
$p\text{-ord}(Y) \in \mathbb{N} \cup \{\infty\}$

<u>Def:</u> $p\text{-ord}(Y) \geq 0$

$p\text{-ord}(Y) \geq k+1 :\Leftrightarrow$ for all $f: K \to Y$ there is an extension $\bar{f}: K/p \to Y$ s.t. $p\text{-ord}(C\bar{f}) \geq k$

<u>Thm A:</u> $p\text{-ord}(\mathbb{S}/p) \leq p-2$ in SHC

<u>Thm B:</u> Let $\mathcal{C}$ be a stable model, $Y \in \mathrm{Ho}\,\mathcal{C}$

a) $p\text{-ord}(Y/p) \geq p-2$

b) If $\alpha_1 \wedge Y = 0$, then $p\text{-ord}(Y/p) \geq p-1$.

$\alpha \in \pi_{2p-3}\mathbb{S}$

If $\mathcal{T}$ is algebraic ~~($\simeq_\Delta H^0$(pre-Δ-ed dg cat))~~ ($\simeq_\Delta H^0$(pre Δ-ed dg category)) then $p\text{-ord}(X/p) = \infty$ for all $X \in \mathcal{T}$

<u>Coherent actions of Moore spaces</u>

The <u>n-th extended power</u> of a based simplicial set X is

$$E\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n} = D_n X$$

$$\Sigma_i \times \Sigma_j \hookrightarrow \Sigma_{i+j} \quad \rightsquigarrow \quad \mu_{i,j}: D_i X \wedge D_j X \to D_{i+j} X$$

<u>Associative:</u> (...)

Let M be a $\overset{\text{mod } p}{\vee}$ simplicial Moore space, ie a finite simplicial set s.t. $\tilde{H}_*(M, \mathbb{Z}) = \begin{cases} \mathbb{F}_p, & * = 2 \\ 0 & * \text{ else} \end{cases}$

Fix a morphism $\iota : S^2 \longrightarrow M$ that induces an iso in $H_2(-, \mathbb{F}_p)$.

<u>Def:</u> $1 \leq k \leq p$. A <u>k-coherent M-module</u> in a pointed simplicial model category $\mathcal{C}$ consists of:

- cofibrant objects $X_{(1)}, X_{(2)}, \dots, X_{(k)}$
- morphisms $D_i M \wedge X_{(j)} \longrightarrow X_{(i+j)}$ $\quad i+j \leq k$

s.t. •
$$\begin{tikzcd}
D_i M \wedge D_j M \wedge X_{(l)} \arrow[r, "\mathrm{id} \wedge \mu_{j,l}"] \arrow[d, "\mu_{i,j} \wedge \mathrm{id}"'] & D_i M \wedge X_{(j+l)} \arrow[d, "\mu_{i,j+l}"] \\
D_{i+j} M \wedge X_{(l)} \arrow[r, "\mu_{i+j,l}"'] & X_{(i+j+l)}
\end{tikzcd}$$
$\circlearrowleft$ $\quad \forall\, i+j+l \leq k$

- The composites
$$\begin{tikzcd}
S^2 \wedge X_{(j-1)} \arrow[r, "\iota \wedge X_{(j-1)}"] & M \wedge X_{j-1} \arrow[d, "\mu_{1,j-1}"] \\
 & X_{(j)}
\end{tikzcd}$$
is a weak equivalence for all $2 \leq j \leq k$

"k-coherent module $\iff$ A_k-action of the mod-p Moore spectrum on $X_{(1)}$"

- 1-coherent M-module = cofibrant object
- 2-coherent M-module = $\{X_{(1)}, X_{(2)},\ M \wedge X_{(1)} \to X_{(2)}\}$

$$\begin{tikzcd}
S^2 \wedge X_{(1)} \arrow[r, "\iota \wedge \mathrm{id}"] \arrow[rr, bend right, "\simeq"'] & M \wedge X_{(1)} \arrow[r, "\mu_{1,1}"] & X_{(2)}
\end{tikzcd}$$

$$\begin{tikzcd} S^2 \wedge X_{(1)} \arrow[r, "p \wedge X_{(1)}"] & S^2 \wedge X_{(1)} \arrow[r, "\iota \wedge X_{(1)}"] \arrow[dr, bend right, "\simeq"'] & M \wedge X_{(1)} \arrow[d, "\mu_{1,M}"] \\ & & X_{(2)} \end{tikzcd}$$

stable.

$\Leftrightarrow \quad p \cdot \mathrm{Id}_{X_{(1)}} = 0$

3-coherent = homotopy associative action

$\boxed{\mathcal{C} \text{ stable}}$

Observation: $M \wedge Y \simeq Y/p$ in $\mathrm{Ho}\,\mathcal{C}$

Claim 1: For every cofibrant object $Y \in \mathcal{C}$, $M \wedge Y$ has a tautological $(p-1)$-coherent M-action

Claim 2: Let $\bar{\alpha}_1 : S^{2p} \to S^3$ be a map that represents $\alpha_1 \in \pi_{2p-3}\mathbb{S}$

If $\bar{\alpha}_1 \wedge Y = 0$ in $\mathrm{Ho}\,\mathcal{C}$, then the $(p-1)$-coherent action of Claim 1 extends to a p-coherent action.

Claim 3: Let $X_{(1)}$ underly a k-coherent M-module. Then $p\text{-ord}(X_{(1)}) \geq k-1$

Proof of Claim 1:

$$H^*(\Sigma_i, \bar{H}^*(X;\mathbb{F}_p)^{\otimes i}) \Rightarrow \bar{H}^*(D_i X; \mathbb{F}_p)$$

For $i<p$, no higher group cohomology, so this collapses.

$X = M$

$$\bar{H}^*(D_i M; \mathbb{F}_p) = (\bar{H}^*(M; \mathbb{F}_p)^{\otimes i})^{\Sigma_i}$$

$$\bar{H}^*(M, \mathbb{F}_p) = \mathbb{F}_p\{\underset{2}{x}, \underset{3}{y}\} \qquad = \mathbb{F}_p\{x \otimes \cdots x, \sum_{k=1}^{i} x \otimes \cdots \otimes y \otimes \cdots \otimes x\}$$

$\Rightarrow$ $D_i M \simeq S^{2i-2} \wedge M$

$$\begin{tikzcd}[column sep=large] S^2 \wedge D_{i-1} M \arrow[r, "\iota \wedge id"] \arrow[rr, bend right, "\simeq"'] & M \wedge D_{i-1} M \arrow[r, "\mu_{1,i-1}"] & D_i M \end{tikzcd}$$

Tautological $(p-1)$-coherent M-module $\underline{M \wedge Y}$

$\underline{M \wedge Y}_{(i)} = D_i M \wedge Y$, $\quad \mu_{i+j} = \mu^M_{ij} \wedge id_Y$

□

<u>Proof of claim 3:</u> By induction on k.

$k = 1$ nothing to show.

$k \geq 2$: W.l.o.g. $X_{(1)}$ is fibrant

Let $f : K \to X_{(1)}$ be any morphism in $Ho(\mathcal{C})$. W.l.o.g. K cofibrant. Let $\varphi : K \to X_{(1)}$ be a $\mathcal{C}$-morphism representing f.

Let shX be the shifted $(k-1)$-coherent M-module

$(shX)_{(i)} = X_{(i+1)}$, $\quad \mu^{shX}_{i,j} = \mu^X_{i,j+1}$

We define a morphism of $(k-1)$-coherent M-modules

$\tilde{\varphi} : \underline{M \wedge K} \longrightarrow shX$

$(\underline{M \wedge K})_{(i)} = D_i M \wedge K \xrightarrow{D_i M \wedge \varphi} D_i M \wedge X_{(1)} \xrightarrow{\mu_{i,1}} X_{(i+1)} = (shX)_{(i)}$

$\mathrm{cone}(\tilde{\varphi})$: the $(k-1)$-coherent M-module with

$\mathrm{cone}(\tilde{\varphi})_{(1)} = \mathrm{cone}(\tilde{\varphi}_{(1)})$ with underlying object

$$\begin{tikzcd} \mathrm{cone}(\tilde{\varphi})_{(1)} = \mathrm{cone}\big(M \wedge K \arrow[r] \arrow[d, phantom, "\simeq" rotate=90] & M \wedge X_{(1)} \arrow[r] & X_{(2)} \big) \arrow[d, phantom, "\simeq" rotate=90] \\ \Sigma^2 K/p \arrow[rr, "\Sigma^2 \bar{f}"'] & & \Sigma^2 X_{(1)} \end{tikzcd}$$

Induction: $p\text{-ord}(\mathrm{cone}(\tilde{\varphi})) \geq k-2$

$\Rightarrow$ $p\text{-ord}(X_{(1)}) \geq k-1$

<u>Proof of claim 2:</u> If $\bar{\alpha}_2 \wedge Y = 0$ in $\mathrm{Ho}\mathcal{C}$, then $\underline{M \wedge Y}$ admits a p-coherent action

The extended powers of M can be assembled into an n-cube

$n=1$: $M \wedge M \xrightarrow{\mu_{1,1}} D_2 M$

$n=2$:

$$\begin{tikzcd}
M \wedge M \wedge M \arrow[r, "M \wedge \mu_{1,1}"] \arrow[d, "\mu_{1,1} \wedge M"'] & M \wedge D_2 M \arrow[d, "\mu_{1,2}"] \\
D_2 M \wedge M \arrow[r, "\mu_{2,1}"'] & D_3 M
\end{tikzcd}$$

$C : P\{1,\dots,n\} \to sSets$

$n=3$

$$\begin{tikzcd}
M \wedge M \wedge M \wedge M \arrow[rr] \arrow[dr] \arrow[dd] & & M \wedge M \wedge D_2 M \arrow[dr] \arrow[dd] & \\
& M \wedge D_2 M \wedge M \arrow[rr] \arrow[dd] & & M \wedge D_3 M \arrow[dd] \\
D_2 M \wedge M \wedge M \arrow[rr] \arrow[dr] & & D_2 M \wedge D_2 M \arrow[dr] & \\
& D_3 M \wedge M \arrow[rr] & & D_4 M
\end{tikzcd}$$

$\boxed{n=p}$

- Let $\mathfrak{X} = \operatorname{colim}_{S \subsetneq \{1,\dots,p\}} C(S) \quad \in sSet$

<u>Prop:</u> The composite
$$S^2 \wedge D_{p-1} M \xrightarrow{c \wedge id} M \wedge D_{p-1} M \longrightarrow \mathfrak{X}$$ is injective with cokernel a mod p Moore space with bottom cell in degree $4p-3$.

Moreover $P^1 : H^{2p}(\mathfrak{X}; \mathbb{F}_p) \to H^{4p-2}(\mathfrak{X}, \mathbb{F}_p)$ is non-zero.

$$P^1 \left[\begin{array}{ll} i & 4p-2 \\ & 4p-3 \\ & \\ j & 2p+1 \\ & 2p \end{array} \right.$$

$\boxed{P^1 \text{ detects } \alpha_1}$

$\Rightarrow$ exact Δ

$$\begin{tikzcd}
\Sigma^{4p-4} \mathbb{S}/p \arrow[r, "f"] \arrow[d, "\text{pinch to top cell}"'] & \Sigma^{2p} \mathbb{S}/p \arrow[r] & \mathcal{X} \\
\mathbb{S}^{4p-3} \arrow[r] & \mathbb{S}^{2p} \arrow[u, "\text{incl of bottom cell}"'] &
\end{tikzcd}$$

~~The tautological $(p-1)$ coherent M-action on $\underline{M \wedge Y}$ provide a map~~

~~$\overline{\mathcal{X}}$~~

Extending the $(p-1)$-coherent action to a p-coherent action requires:

- $\underline{(M \wedge Y)}_{(p)}$
- map
  $$\mu_{i,p-i} : D_i M \wedge D_{p-i} M \wedge Y \longrightarrow \underline{(M \wedge Y)}_{((p))}$$

$\Longleftrightarrow$

$$\begin{tikzcd}
\mathcal{X} \wedge Y \arrow[r] \arrow[ddd, dashed, bend right, "\sigma"'] & \underline{M \wedge Y}_{(p)} \\
M \wedge D_{p-1} M & \\
& \\
S^2 \wedge D_{p-1} M \arrow[u] \arrow[uuur, "\sim"'] & \arrow[l, "=0"'] \\
\Sigma^{2p} \mathbb{S}/p \wedge Y \arrow[u, phantom, "\cong"] & \\
\Sigma^{4p-2} \mathbb{S}/p \wedge Y \arrow[u, "f \wedge Y"'] &
\end{tikzcd}$$

$$\alpha_1 \wedge Y = 0$$
$$\Downarrow$$
$$f \wedge Y = 0$$

choose a section $\sigma : \mathcal{X} \wedge Y \to S^2 \wedge D_{p-1} Y$

represent this in $\mathrm{Ho}\,\mathcal{C}$ by a morphism

$\mathcal{X} \wedge Y \longrightarrow Z^{\text{fibrant}}$

$\quad\quad\quad\quad\quad\quad\quad\quad \| \; (M \wedge Y)_{(p)}$

Masterclass: Rigidity and algebraic models in stable homotopy theory. 2018-04

Lecture 9

<u>Proof of rigidity theorem.</u>

Let $\mathcal{C}$ be a stable model ~~cat~~ category, and

$\Phi : \mathrm{Ho}(\mathrm{Spectra}) \xrightarrow{\cong} \mathrm{Ho}(\mathcal{C})$ Δed equivalence

$X = \Phi(\mathbb{S})$

Universal property: $\mathrm{Spectra} \underset{\mathrm{Hom}(X,-)}{\overset{-\wedge X}{\rightleftarrows}} \mathcal{C}$ $\mathbb{S} \wedge X \cong X$

~~derive~~ derive: $\mathrm{Ho}(\mathrm{Spectra}) \xrightarrow{-\wedge^L X} \mathrm{Ho}\,\mathcal{C}$ exact, $\mathbb{S} \mapsto X$

$F := \Phi^{-1} \circ (- \wedge^L X) : \mathrm{Ho}(\mathrm{Spectra}) \circlearrowleft$

$F(\mathbb{S}) = \mathbb{S}$

$F : [\mathbb{S}, \mathbb{S}]_* \longrightarrow [\mathbb{S}, \mathbb{S}]_*$

$\quad \shortparallel$

$\pi_*^s(\mathbb{S})$

~~homo to~~

homomorphism of graded rings

Claim: This is an isomorphism.

"An endomorphism of $\pi_*^s(\mathbb{S})$ that preserves Toda brackets is an isomorphism"

<u>Proof:</u> • F is an isomorphism in ~~a~~ negative degrees

• F is an iso in deg 0 $(F : \mathbb{Z} \longrightarrow \mathbb{Z})$

~~Assume~~ Argue by contradiction, suppose F is not an iso. Let $n \geq 1$ be minimal s.t. ~~F~~

$F : [\mathbb{S}, \mathbb{S}]_n \longrightarrow [\mathbb{S}, \mathbb{S}]_n$ is not epi.

For a finite spectrum $K \neq 0$, choose a minimal stable CW structure.

$T(K)$ = dimension of top cell

$\beta(K)$ = —"— bottom cell

(A) Let K & L be finite spectra. Then
$F : [K, L] \longrightarrow [FK, FL]$ is

$\begin{cases} \text{bijective if } & T(K) - \beta(L) < n-1 \\ \text{surjective if } & T(K) - \beta(L) = n-1 \end{cases}$

Proof: Induction of cell structure & 5-lemma.

(B) Let K be a finite spectrum with $T(K) - \beta(K) \leq n$. Then there is a finite spectrum K' s.t. $K \cong F(K')$ and $T(K') \leq T(K)$, $\beta(K') \geq \beta(K)$

Proof: Induction over $T(K) - \beta(K)$

For $T(K) - \beta(K) = 0$, then $K \cong \bigvee_{i \in I} S^m$, ok because $F(\mathbb{S}) \cong \mathbb{S}$.

For $T(K) - \beta(K) \geq 1$. Write K as an exact triangle

$$\bigvee_{I} \mathbb{S}^{T(K)-1} \longrightarrow M \longrightarrow K \longrightarrow \bigvee_{I} S^{T(K)}$$

$\beta(M) = \beta(K)$

$T(M) < T(K)$

By induction, $M \cong F(M')$ for some finite spectrum M'

$$\begin{tikzcd}
F\left(\bigvee_I S^{T(K)-1}\right) \arrow[r, "F(\alpha')"] \arrow[d, phantom, "\cong"] & F(M') \arrow[d, phantom, "\cong"] \\
\bigvee_I S^{T(K)-1} \arrow[r, "\alpha"'] & M
\end{tikzcd}$$

By claim A $\alpha = F(\alpha')$

$$\bigvee_I S^{T(K)-1} \xrightarrow{\alpha'} M' \longrightarrow K' \longrightarrow$$

$$\Rightarrow F(K') = F(\mathrm{cone}(\alpha')) \cong \mathrm{cone}(F(\alpha'))$$

$$\cong \mathrm{cone}(\alpha) = K$$

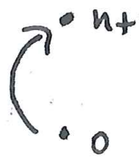
(C) The map $[S,S]_n \longrightarrow [S,S]_n$ is surjective

Let p be the smallest prime s.t. F is not surjective on ~~the~~ $[S,S]_n \otimes \mathbb{Z}_{(p)}$

Let $x \in [S,S]_n$ be p-primary, not in the image

Def: $\mathrm{filt}(x) \geq 1$ if $H^*(x, \mathbb{F}_p) : H^*(S^n, \mathbb{F}_p) \to H^*(S^0, \mathbb{F}_p)$ is zero

$\mathrm{filt}(x) > 1$ if $\mathrm{filt}(x) \geq 1$ and $H^*(C(x), \mathbb{F}_p)$ decomposes as a module over $\mathcal{A}_p^*$



<u>Claim:</u> If $\mathrm{filt}(x) \geq 2$, then there is a finite spectrum K with $\tau(K) \leq n-1$, $\beta(K) \geq 1$ and a factorization

$$\begin{tikzcd} \mathbb{S}^n \arrow[r] \arrow[rr, bend left, "x"] & K \arrow[r] & \mathbb{S}^0 \end{tikzcd}$$

<u>Proof:</u> Let $\overline{H\mathbb{F}_p}$ and $\overline{H\mathbb{Z}_{(p)}}$ be the fibers of

$$\begin{tikzcd} \overline{H\mathbb{Z}_{(p)}} \arrow[r] \arrow[d] & \mathbb{S} \arrow[r] \arrow[d, equal] & H\mathbb{Z}_{(p)} \arrow[d] \\ \overline{H\mathbb{F}_p} \arrow[r] & \mathbb{S} \arrow[r] & H\mathbb{F}_p \\ & \mathbb{S}^n \arrow[u, "x"'] \arrow[ul, dashed, "\tilde{x}"] & \end{tikzcd}$$

Because $\mathrm{filt}(x) \geq 1$, there is a lift

$$\begin{tikzcd} & \mathbb{S}^n \arrow[dl, dashed, bend right, "\tilde{\tilde{x}}"'] \arrow[d, dashed, "\tilde{x}"] \arrow[dr, bend left, "x"] & \\ \overline{H\mathbb{Z}_{(p)}} \arrow[r] & \overline{H\mathbb{F}_p} \arrow[r] & \mathbb{S}^0 \end{tikzcd}$$

$\mathrm{filt}(x) \geq 2 \Leftrightarrow \tilde{x}$ is the trivial map on $H^*(-, \mathbb{F}_p)$.

The map $\overline{H\mathbb{Z}_{(p)}} \longrightarrow \overline{H\mathbb{F}_p}$ is iso on π_* in positive dimension, so x lifts to a map

$$\begin{tikzcd} \mathbb{S}^n \arrow[d, "\tilde{\tilde{x}}"] \\ \overline{H\mathbb{Z}_{(p)}} \end{tikzcd}$$

Since $\overline{H\mathbb{Z}_{(p)}} \longrightarrow \overline{H\mathbb{F}_p}$ is surjective in $H^*(-, \mathbb{F}_p)$
so $\tilde{\tilde{x}}$ is zero on $H^*(-, \mathbb{F}_p)$
Since $p \cdot H_*(\overline{H\mathbb{Z}_{(p)}}, \mathbb{Z}) = 0$.

$$H\mathbb{Z}_{(p)} \xrightarrow{p} H\mathbb{Z}_{(p)} \longrightarrow H\mathbb{F}_p$$

$\Rightarrow$ $H_*(\overline{H\mathbb{Z}_{(p)}}, \mathbb{Z}) \longrightarrow H_*(\overline{H\mathbb{Z}_{(p)}}, \mathbb{F}_p)$ is injective

$\Rightarrow$ $\tilde{\tilde{x}}$ induces the trivial map in $H_n(-,\mathbb{Z})$

$$\begin{tikzcd}
 & \mathbb{S}^n \arrow[dl, dashed] \arrow[d, "\tilde{\tilde{x}}"] \arrow[dd, bend left=60, "x"] \\
\left(\overline{H\mathbb{Z}_{(p)}}\right)^{(n-1)} \arrow[r, hook] \arrow[dr, bend right] & \overline{H\mathbb{Z}_{(p)}} \arrow[d] \\
 & \mathbb{S}^0
\end{tikzcd}$$

$\left(\overline{H\mathbb{Z}_{(p)}}\right)^{(n-1)} = K$

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r] \arrow[rr, bend left, "x"] \arrow[d, "\cong"] & K \arrow[r] \arrow[d, "H\mathbb{Z}\text{-class } 2"] & \mathbb{S}^0 \arrow[d, "\cong"] \\
F(\mathbb{S}^n) \arrow[r, "F(\alpha)"'] & F(K') \arrow[r, "F(\beta)"'] & F(\mathbb{S}^0)
\end{tikzcd}$$

$\tau(K) \le n-1$

$\beta(K) \ge 1$

$\Rightarrow$ $x = F(\beta\alpha)$

So x can not have $\mathrm{filt}(x) \ge 2$.

After subtracting a class of filtration ≥ 2, we can assume that $p=2$, $x = \eta, \nu, \sigma$ or $p \ge 3$, $x = \alpha_1$.

<u>Case p odd, $x = \alpha_1$: or $p=2$, $x = \eta$</u>

$F: \mathrm{Ho}(Sp) \xrightarrow{-\wedge X} \mathrm{Ho}\,\mathcal{C} \underset{\cong}{\xrightarrow{\Phi^{-1}}} \mathrm{Ho}Sp$

If $F(\alpha_1) = 0$, then $\alpha_1 \wedge X = 0 \Rightarrow p\text{-ord}(X/p) \ge p-1$

$\cancel{F(\mathbb{S}) \xrightarrow{p} F(\mathbb{S}) \longrightarrow F(\mathbb{S}/p) \longrightarrow F(\Sigma\mathbb{S})}$

$\mathbb{S} \xrightarrow{p} \mathbb{S}$

$$\begin{array}{ccccccc}
\mathbb{S}\wedge X & \xrightarrow{p} & \mathbb{S}\wedge X & \longrightarrow & \mathbb{S}/_p \wedge X & \longrightarrow & \Sigma \mathbb{S}\wedge X \\
\| & & \| & & \| & & \\
X & \xrightarrow[p]{} & X & \longrightarrow & X/p & \longrightarrow & \Sigma X
\end{array}$$

$(-\hat{\wedge} X)\,(\mathbb{S}/p) \cong X/p$

$\Phi^{-1}(X/p) \cong \mathbb{S}/p \qquad p\text{-ord}(\mathbb{S}/p) \geq p-1$

$p=2, \quad \eta \checkmark$

$p=2, \; x=\nu: \qquad \pi_1^s \mathbb{S} = \mathbb{Z}/2\{\eta\} \quad F(\eta)=\eta.$

$\cancel{\mathbb{Z}_{(2)}} \quad \mathbb{Z}_{(2)} \otimes \pi_3^s \mathbb{S} = \mathbb{Z}/8\{\nu\} \quad \cancel{4} \quad 4\nu = \eta^3$

$$4F(\nu) = F(4\nu) = F(\eta^3) = \eta^3 = 4\nu.$$

$$\mathbb{Z}/8 \xrightarrow{F} \mathbb{Z}/8$$

$\Rightarrow \quad F(\nu) = u\cdot\nu \quad$ for $\; u \in \mathbb{Z}/8^\times$

$p=2, \, x=\sigma \qquad \pi_7^s(\mathbb{S})\otimes \mathbb{Z}_{(2)} \cong \mathbb{Z}/16\{\sigma\}$

$8\sigma \in \langle \nu, 8, \nu\rangle \qquad \langle h_2, h_0^3, h_2\rangle \ni h_0^3 h_3$

$$8F(\sigma) = F(8\sigma) = F(\langle \nu,8,\nu\rangle) = \langle F(\nu), 8, F(\nu)\rangle$$

$$= \langle u\cdot\nu, 8, u\cdot\nu\rangle = u^2\langle \nu, 8, \nu\rangle = 8u^2\sigma$$

$$\mathbb{Z}/16 \xrightarrow[\cong]{F} \mathbb{Z}/16$$

Conclusion: $F : [\mathbb{S}, \mathbb{S}]_* \longrightarrow [\mathbb{S}, \mathbb{S}]_*$ is an isomorphism

$\Rightarrow \quad F : [K, L] \xrightarrow{\cong} [FK, FL]$

K, L finite complexes

$\Rightarrow$ F fully faithful on finite spectra

$+$ F is dense on finite spectra.

$\text{Spectra} \simeq \text{Ind}(\text{Spectra}^{\text{finite}}) \rightsquigarrow F$ is an equivalence on all spectra.

$\square$