2018-04-09

Masterclass: Rigidity and algebraic models in stable homotopy theory: Lecture 2.

Classifying G-Equivariant cohomology theories

G : compact Lie group.

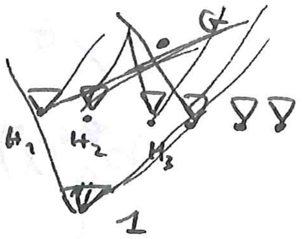
calculate

G-equiv cohom theories ⤳ classifying category — construct conceptual

$\mathcal{C}$ stable monoidal model cat $\simeq$ Adelic model A cat of sheaves

$\downarrow$

$\bar{\mathcal{C}}$ tt triangulated category — $/ \mathfrak{X} = \mathrm{Spec}(\bar{\mathcal{C}})$



+ finiteness & catenary

$\Uparrow$

pieces over $\mathfrak{X}(K)$

$\wr$

G/K

$C^*(BG/K)$

$\wr$

$H^*(BG/K)$ $\Big\}\ \otimes \mathbb{Q}$

1. Examples:

- K_G & many variants
- Ω_G & many variants
- Elliptic cohomology. Many.
- Borel cohomology. X G-space $b_G^*(X) = H^*(EG \times_G X)$
- Bredon cohomology $H_G^*(X;M)$, $H_G^*(G/H;M) = M(G/H)$

(MMackey functor)

- cohomotopy

These are $X \hookrightarrow S^{V+1}$

$(\Sigma^V DX \simeq) S^{V+1} \setminus X$

$$E^G_i(S^{V+1}\setminus X) \cong E^{V-i}_G(X)$$

Hence want $RO(G)$-gradeable.

V orthogonal ~~rep~~ representation of G
S(V) unit sphere
S^V compactification

2. Localization Theorem

G-torus $\quad$ X finite G-CW-complex.

$$\begin{tikzcd} X^G \arrow[r, "i"] & X \\ b^*_G(X^G) \arrow[d, equal] & b^*_G(X) \arrow[l, "i^*"'] \\ H^*BG \otimes H^*(X^G) & \end{tikzcd}$$

Borel: with rational coeff this is an iso if ~~[illegible]~~ you invert all Euler classes

$$H^*BG = \mathbb{Q}[\overset{②}{x_1}, \dots, \overset{②}{x_r}]$$

$$\mathcal{E}_G = \langle e(\alpha) \mid \alpha : G \to \mathbb{T} \rangle = H^2BG \setminus \{0\}$$

Proof when $G = \mathbb{T}$: $\quad X = X^G \cup$ cells with finite isotropy

$$b^*_G(G/C) = H^*(EG \times_G G/C) = H^*(BC) = \mathbb{Q}$$

$$H^*BG = \mathbb{Q}[x].$$

Corollary: (Smith thm): X rational S^n $\Rightarrow$ X^G rational S^n //

commutes with $\underrightarrow{\text{holim}}$

$\Phi^H X \simeq * \quad \forall H \le G \Rightarrow X \underset{G}{\simeq} *$ (Geometric fixed point whitehead theorem)

~~$[X, X' \wedge$~~ "$N\le$" $= \{ H \le G \mid N \le H \}$

$$[X, X' \wedge E\langle N\le\rangle]^G = [\Phi^N X, \Phi^N X']^Q$$

$$(E\langle\mathcal{C}\rangle)^H \simeq \begin{cases} *, & H \notin \mathcal{C} \\ S^0 & H \in \mathcal{C} \end{cases}$$

e.g. $EG_+ = E\langle 1\rangle$

$E\mathcal{F}_+ = E\langle\mathcal{F}\rangle$ $\mathcal{F}$ closed under conj & passage to subgroups.

$$\begin{tikzcd} E\mathcal{F}_+ \arrow[r] & S^0 \arrow[r] & \tilde{E}\mathcal{F} = E\langle\mathcal{F}^c\rangle \arrow[d, equal] \\ & & S^0 * E\mathcal{F} \end{tikzcd}$$

$\mathcal{P}$ = proper subgroups $\quad \tilde{E}\mathcal{P} = E\langle G\rangle$

X G-spectrum

$X = \{X(V)\}_{V \subset \mathcal{U}}$ ↙ complete G-universe

$\sigma: X(V) \wedge S^W \longrightarrow X(V \oplus W)$

A G-space: $(\Sigma^\infty A)(V) = A \wedge S^V$.

Categorical fixed points

$$[Z, X]^G = [Z, X^N]^Q$$

$\uparrow$ inflated from Q.

Whitehead theorem: $X^H \simeq * \ \forall H \Rightarrow X \underset{G}{\simeq} *$

3. Formalities:

(henceforth, all spaces have G-fixed basepoint & have reduced cohom.)

Represented by G-spectra, $E^*_G(X) = [X, E]^*_G$

G-cohomology theories / nat transformations $\simeq Ho(G\text{-spectra})$

res and adjoints: $E^G_i(X) = [S^i, E \wedge X]^G$

$$\begin{tikzcd}
G\text{-spectra} \arrow[r, "res^G_H"] & H\text{-spectra} \arrow[l, bend right=50, "G_+ \wedge_H \bullet"'] \arrow[l, bend left=50, "F_H(G_+, \bullet)"]
\end{tikzcd}$$

Given $E^*_G(\ )$

$$E^*_H(Y) = [Y, E]^*_H = [G_+ \wedge_H Y, E]^*_G = E^*_G(G_+ \wedge_H Y)$$

free infⁿ $H \subseteq G$, $N \trianglelefteq G$, $Q = G/N$

Z is a Q-spectrum X G-spectrum Y H-spectrum

$$[X, Z]^G = [X/N, Z]^Q \quad \text{if } X \text{ is } N\text{-free}$$

infⁿ-free: $[Z, X]^G = [Z, (S^{LN} \wedge X)/N]^Q$ if X is N-free

(Adams iso)

Wirthmüller iso $F_H(G_+, Y) \simeq G_+ \wedge_H (S^{-L(G/H)} \wedge Y)$

Fixed points: Geometric fixed points Φ^N

$$\Phi^N : G\text{-spectra} \xrightarrow{N\text{-fixed point}} Q\text{-spectra}$$

$$\Phi^N(\Sigma^V A) \simeq \Sigma^{V^N}(A^N)$$

$$\Phi^N(X_1 \wedge X_2) \simeq \Phi^N X_1 \wedge \Phi^N X_2$$

Masterclass: Rigidity and algebraic models in stable homotopy theory: Lecture 4.

2018-04-10

Yesterday: G-spectra: $E^*_G(X) = [X, E]^*_G$

$\otimes \mathbb{Q}$

1. Burnside ring

$[A, B]^G = \varinjlim_V [S^V \wedge A, S^V \wedge B]^G_{\text{unst.}}$

A, G based G-spaces, A finite

$[S^0, S^0]^G$ acts on everything.

Prop: (tom Dieck): $[S^0, S^0]^G = C(\mathcal{F}G/G, \mathbb{Q})$

$\mathcal{F}G = \{H \le G \mid |W_G H| < \infty$

$W_G H = N_G H / H$

$(\mathcal{F}G)/G$ compact & totally disconnected

Proof: Burnside ring ✓ $[S^0, S^0]^G = \varinjlim_V [S^V, S^V]^G_{\text{unst}}$

$(f : S^V \to S^V) \mapsto \delta(f) : H \mapsto \deg(f^H)$

$\longrightarrow C\left(\frac{\text{Sub}(G)}{G}, \mathbb{Q}\right)$

By Loc^n theorem if

$K \lhd H$ H/K torus

$\delta(f)(K) = \delta(f)(H)$

$\longrightarrow C\left(\frac{\mathcal{F}G}{G}, \mathbb{Q}\right)$

injective: obs^n thy.

\+ checking surj.

//

e.g. $O(2)$

$SO(2)$

$\cup$

C_n

D_2 D_4 D_6 D_8 ...

G finite: $A(G) = Gr(\text{finite } G\text{-sets})$

is isomorphic to $[S^0, S^0]^G$ (Segal)

$$\begin{tikzcd} S^0 \arrow[r, "D\pi"] \arrow[rr, bend right, "{[G/H]}"'] & G/H_+ \arrow[r, "\pi^G_H"] & G/G_+ = S^0 \end{tikzcd}$$

(G-Mackey functors)

$\overset{\cong}{}$ $\quad$ $\overset{\cong}{}$

<u>Prop:</u> If G is finite $Ho(G\text{-spectra}) \simeq \prod_{(H)} \mathbb{Q}[W_G(H)]$-modules

<u>Proof:</u> $A(G) = [S^0, S^0]^G \xrightarrow{\cong} C(\mathrm{Sub}(G)/_G, \mathbb{Q}) = \prod_{(H)} \mathbb{Q}$

e_H

$\Phi^H(e_H X) \simeq \Phi^H X$

<u>Corollary:</u> G finite

AHSS for every E^*_G collapses

$E^n_G(X) \cong \prod_i H^{n-i}_G(X; \underline{E}^i_G)$

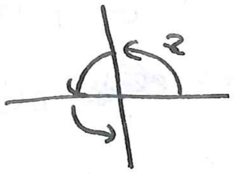
"All cohom theories are ordinary".

<u>Example:</u> (Haeberly)

If G is inf. the AHSS for $K^*_G(X)$ does not collapse for some X

<u>Proof:</u> Suff. to do $G = \mathbb{T} =$ circle

$X = S^{27} = S^0 \cup e^1 \wedge \mathbb{T}_+ \cup e^3 \wedge \mathbb{T}_+$



Hence ⓪ ① ② ③

$H^*_{\mathbb{T}}(X; M) = H^*(M(\mathbb{T}) \to M(1) \to 0 \to M(1))$

i.e. $H^3_{\mathbb{T}}(X; M) = M(1)$

$M = \underline{K}^*_G$. $M(1) = K^* \neq 0$

But Bott;

$K^*_G(S^{2z}) \cong K^*_G$

in even degree,

$S^0 \cup e^1 \wedge \mathbb{T}_+ =$

$\mathbb{T}_+ \longrightarrow S^0 \longrightarrow S^z$

S^z

$\mathbb{T}_+ \wedge S^z \cong \overline{\mathbb{T}_+ \wedge S^z} \longrightarrow S^z \longrightarrow S^{2z}$

What next?

Conjuncture: For any compact Lie group G there is an abelian category $\mathcal{A}(G)$ so that

$$G\text{-spectra}/\mathbb{Q} \underset{Q}{\simeq} d\mathcal{A}(G)$$

& hence $\mathrm{Ho}(G\text{-spectra}/\mathbb{Q}) \underset{\Delta}{\simeq} D(\mathcal{A}(G))$

Furthermore $\mathcal{A}(G)$ has injdim = rank(G)

$\mathcal{A}(G)$ is a cat of sheaves over $\mathrm{Sub}(G)/G$

True for: tori, $O(2)$, $SO(3)$,
free G-spectra, toral G-spectra

Two approaches $[X,Y]^G$:

T = a Δ-Ho category of G-spectra.

Approach 1: Resolution of X ~ cellular approx^n of X ~ Morita

This would show: Say all is built from σ, $T = \mathrm{Loc}\langle\sigma\rangle$

$$\mathrm{Ext}^{**}_{\mathcal{E}_*}([\sigma,X]^G_*, [\sigma,Y]^G_*) \Rightarrow [X,Y]^G$$

$\mathcal{E}_* = [\sigma,\sigma]^G_*$ typically of ∞ hom^l dimension

if $T = \mathrm{Ho}(\tilde{T})$. $\tilde{T} \simeq$ mod-$\mathcal{E}$, $\mathcal{E} = \underline{\mathrm{Hom}}(\sigma,\sigma)$

Approach 2: Resolution of Y ~ Postnikov decomp^n

Choose an exact functor $\pi^{\mathbb{C}}_*: T \longrightarrow \mathbb{C}$

$\hookleftarrow$ graded ab. cat^n

Want an ASS

$$E_2^{**} \quad \mathrm{Ext}_{\mathbb{C}}^{**}(\pi^{\mathbb{C}}_* X, \pi^{\mathbb{C}}_* Y) \Rightarrow [X,Y]^G_*$$

<u>Method:</u> <u>Step 1:</u>

$$0 \longrightarrow \pi_*^{\mathbb{C}} Y \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \quad (\text{inj. res}^{n} \text{ in } \mathbb{C})$$

<u>Step 2:</u> Realize it in T $\qquad$ I inj want ① $\mathbb{I} \in \text{ob}\,T$

$$\pi_*^{\mathbb{C}} \mathbb{I} = I$$

$$② \; [X, \mathbb{I}]^G \xrightarrow[\pi_*^G]{\cong} \text{Hom}_{\mathbb{C}}(\pi_*^{\mathbb{C}} X, I)$$

$$\begin{tikzcd}
Y = Y_0 \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathbb{I}_1 & \Sigma^{-1}\mathbb{I}_1 & \Sigma^{-2}\mathbb{I}_2 &
\end{tikzcd}$$

<u>Step 3:</u> Apply $[X, -]_*^G$ & get the Spectral seq. by 2.2.

<u>Step 4:</u> $\pi_*^{\mathbb{C}}(\underset{\overleftarrow{s}}{\text{holim}}\, Y_s) = 0$ if $\pi_*^{\mathbb{C}}$ preserves $\prod$.

Need $\pi_*^{\mathbb{C}}(Z) = 0 \Rightarrow Z \simeq *$

Take T = free G-spectrum = G-spectra built from G_+

G connected

$\mathbb{C} = H^*BG$-mod $\qquad \pi_*^G :$ Free G-spectra $\longrightarrow H^*BG$-mod

<u>Thm:</u> $\text{Ext}^{**}_{H^*BG}(H_*(X/G), H_*(Y/G))$

$\Rightarrow [X, Y]^G$

$(X, Y \text{ free})$

$X \longmapsto \pi_*^G(X) = \pi_*(S^{LG} \wedge X/G)$

$= \Sigma^g H_*(X/G)$

(torsion module)

<u>Proof:</u> $(H^*BG)^\vee = H_*BG$ injective & enough of this form.

$$\pi_*^G(EG_+) = \Sigma^g \pi_*(BG_+) = \Sigma^g H_*BG$$

$$[X, EG_+]^G \longrightarrow \text{Hom}_{H^*BG}(H_*(X/G), H_*(BG))$$

$$\shortparallel$$

$$\text{Hom}_{\mathbb{Q}}(H_*(X/G), \mathbb{Q})$$

ok for $X = G_+$ Hence any.

<u>Convergence:</u> $H_*(Z/G) = 0 \Rightarrow H_*(Z) = 0$

$$C^*(Z/G) \otimes_{C^*BG} \mathbb{Q} \underset{EM}{\simeq} C^*(Z) \qquad //$$

<u>Next:</u> free-G-spectra $\simeq$ d torsion ~~models~~-H^*BG-mod

Greenlees, Lecture 3

The circle group: $G = \mathbb{T} = SO(2)$ = circle gp,
$H^*(B\mathbb{T}) = \mathbb{Q}[\underset{(-2)}{c}]$

Thm A: There is an abelian category $\mathcal{A}(\mathbb{T})$ of inj. dim. 1 & an ASS

$$\begin{tikzcd}
✓\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}(\mathbb{T})}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \arrow[r] & {[X,Y]^{\mathbb{T}}} \arrow[d] \\
 & & \mathrm{Hom}_{\mathcal{A}(\mathbb{T})}(\pi_*^{\mathcal{A}} X, \pi_*^{\mathcal{A}} Y) \arrow[d] \\
 & & 0
\end{tikzcd}$$

, where $\pi_*^{\mathcal{A}}: \mathbb{T}\text{-spectra} \to \mathcal{A}(\mathbb{T})$ is an exact functor.

Thm B: There is a Quillen ~~[illegible]~~ equiv.

$$\mathbb{T}\text{-spectra} \simeq d\mathcal{A}(\mathbb{T})$$

(+ monoidal + $\underset{\substack{\uparrow \\ \text{"naive" } E_\infty}}{E_\infty^1}$-objects)

Thm C: Given any elliptic curve C over a $\mathbb{Q}$-alg. k there is a $\mathbb{T}$-equivariant cohom. theory $EC_{\mathbb{T}}^*(\cdot)$ so that

$$EC_{\mathbb{T}}^*(S^V) = H^1(C; \mathcal{O}(-D(V))),$$ ~~[illegible]~~

where $V = \sum a_n z^n$, $V^{\mathbb{T}} = 0$ and $D(V) = \sum_n a_n C[n]$

1. The category $\mathcal{A}(\mathbb{T})$

X G-spectrum, $\mathcal{I}(X) = \{H \mid \Phi^H X \not\simeq *\}$

(e.g. X is free $\iff \mathcal{I}(X) \subseteq \{1\}$)

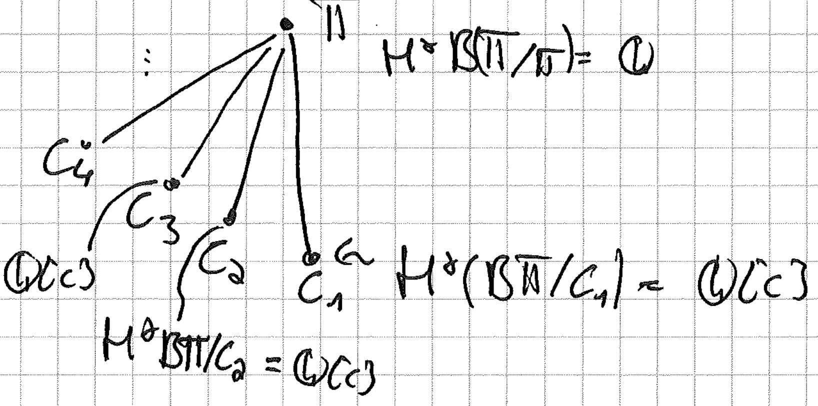
Def. X is semifree if $\mathcal{I}(X) \subseteq \{1, \mathbb{T}\}$

$$\underset{\substack{\uparrow\\ \text{semifree}}}{\mathcal{A}_{sf}}(\mathbb{T}) = \{ \underset{\substack{\circlearrowleft\\ \mathbb{Q}[c]}}{N} \xrightarrow[\substack{\mathbb{Q}[c]\\ \text{linear}}]{\beta} \mathbb{Q}[c,c^{-1}]\otimes \underset{\substack{\circlearrowleft\\ \mathbb{Q}}}{V} \mid \beta \text{ is } \underset{c}{\text{inverting}} \}$$

Morphisms:
$$\begin{tikzcd} \mathbb{Q}[c,c^{-1}]\otimes U \arrow[r, "1\otimes\varphi"] & \mathbb{Q}[c,c^{-1}]\otimes V \\ M \arrow[u] \arrow[r, "\theta"] & N \arrow[u] \end{tikzcd}$$

$$\mathcal{A}(\mathbb{T}) = \{ \underset{\circlearrowleft}{N} \xrightarrow{\beta} t\otimes \underset{\substack{\circlearrowleft\\ \mathbb{Q}}}{V} \mid \beta \text{ is inverting } \mathcal{E}\}$$

$$\mathcal{O}_\mathcal{F} = \prod_{n\geq 1}\mathbb{Q}[c]$$



Here: $\mathcal{E} = \{e(V) \mid V^{\mathbb{T}} = 0\}$.

$$e(V) \in \mathcal{O}_\mathcal{F} = \prod_C H^*(B\mathbb{T}/C)$$

with $e(V)_C = e(V^C) \in H^{|V^C|}(B\mathbb{T}/C)$
(non-homogeneous)

Now:

$$t := \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \varinjlim\left(\mathcal{O}_{\mathcal{F}} \xrightarrow{e(z)} \Sigma^{z}\mathcal{O} \xrightarrow{e(z^2\oplus z^4)} \Sigma^{z\oplus z^2\oplus z^4}\mathcal{O}_{\mathcal{F}} \to \dots\right)$$

$$(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})_n = \begin{cases} \bigoplus \mathbb{C} & \text{if } n \geq 2 \\ \prod \mathbb{C} & \text{if } n \not\leq 0 \end{cases}$$

2. Homological algebra

$N \circlearrowleft \mathcal{O}_{\mathcal{F}} \leftarrow$ has idempotents e_s

$e_s N$ is a module over $e_s\mathcal{O}_{\mathcal{F}} = \mathbb{C}[c]$

Lemma: If $\mathcal{E}^{-1}N = 0$, then $\bigoplus_s e_s N \xrightarrow{\cong} N$ //

$X = (N \longrightarrow t\otimes V)$ an object

Def: If W is a vector space, then

$$e(W) = (t\otimes W \xrightarrow{id} t\otimes W) \in \mathcal{A}(\mathbb{T})$$

$$\mathrm{Hom}_{\mathcal{A}(\mathbb{T})}\left(\begin{matrix} t\otimes V \\ \uparrow \\ N \end{matrix}, \begin{matrix} t\otimes W \\ \uparrow \\ t\otimes W \end{matrix}\right) = \mathrm{Hom}_{\mathbb{C}}(V, W)$$

$\Rightarrow e(W)$ is injective.

If $\mathbb{T}$ is an $\mathcal{E}$-torsion $\mathcal{O}_{\mathcal{F}}$-mod, then

$$f(T) = (T \longrightarrow t\otimes\mathcal{O}) \in \mathcal{A}(\mathbb{T})$$

$$\mathrm{Hom}_{\mathcal{A}(\mathbb{T})}\left(\begin{matrix} t\otimes V \\ \uparrow \\ N \end{matrix}, \begin{matrix} 0 \\ \uparrow \\ T \end{matrix}\right) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(N, T)$$

$\Rightarrow$ If $T = H_*(B\mathbb{T}/C_s)$ (in the sth factor), then $f(T)$ is injective.

Lemma: $\mathcal{A}(\mathbb{T})$ is of inj. dimension 1.

Pf:

$$\begin{tikzcd}
0 \arrow[r] & \begin{pmatrix} \mathbb{Q} \\ \uparrow \\ K \end{pmatrix} \arrow[r] & \begin{pmatrix} t\otimes V \\ \uparrow \\ N \end{pmatrix} \arrow[r] \arrow[d, equal] & \begin{pmatrix} t\otimes V \\ \uparrow \\ t\otimes V \end{pmatrix} \arrow[r] & \begin{pmatrix} \mathbb{Q} \\ \uparrow \\ C \end{pmatrix} \arrow[r] & 0 \\
0 \arrow[r] & f(K) \arrow[r] & X \arrow[r] & e(V) \arrow[r] & f(C) \arrow[r] & 0
\end{tikzcd}$$

exact.

$\uparrow$ inj. dim ≤ 1 (as $\mathbb{Q}[c]$ does)

$\uparrow$ injective, since K divisible

//

3. Elliptic cohomology

C elliptic curve (e.g. $C = \mathbb{C}/\Lambda$, Λ lattice)

$\mathcal{K}$ meromorphic functions with poles at pts of finite order.

Write: $C[n] = \ker(C \xrightarrow{n} C)$

$C\langle n\rangle$ pts of exact order n

e = origin

We further choose a function t_1 vanishing to first order at e: $\operatorname{div}(t_1) = (e) + (Q) - (P) - (R)$ + t_n function vanishing to first order at $C\langle n\rangle$.

~~A~~ Then define

$EC = (NC \xrightarrow{\beta} t\otimes VC)$ with

$VC = \mathcal{K}_p \leftarrow$ made 2-periodic

$NC = \ker\left(t\otimes\mathcal{K}_p \longrightarrow \bigoplus_{n\geq 1} H^1_{C\langle n\rangle}(C)_p\right)$

$c^{v}\otimes f \longmapsto \{\overline{t^{v(n)} f}\}_n$ $\quad v: \mathcal{S}\to\mathbb{Z}$ $\quad \uparrow$ local cohomology

In this model: (exercise) ⑤

$$S^V \longleftrightarrow \left(\begin{tikzcd} t\otimes\mathbb{Q} \\ \Sigma^V\mathcal{O}_{\mathcal{F}} \arrow[u] \end{tikzcd}\right)$$

Exc: Compute $\mathcal{E}C_{\mathbb{T}}^*(S^V)$

4 Connection with topology (semifree case)

$$[E\mathbb{T}_+, E\mathbb{T}_+]^{\mathbb{T}} \cong [E\mathbb{T}_+, S^0]^{\mathbb{T}} = [B\mathbb{T}_+, S^0] = H^*B\mathbb{T}$$

Kedziorek $\rightarrow \|$

talk $\pi_*(DE\mathbb{T}_+)$ $\quad DE\mathbb{T}_+ = F(E\mathbb{T}_+, S^0)$

Note that $\widehat{E\mathcal{F}} = \bigcup_{W^{\mathbb{T}}=0} S^W$

$$\widehat{E\mathcal{F}} \wedge DE\mathbb{T}_+ = \underrightarrow{\mathrm{holim}}\, S^V \wedge DE\mathbb{T}_+$$

$$\pi_*^{\mathbb{T}}(\widehat{E\mathcal{F}} \wedge DE\mathbb{T}_+) = \mathcal{E}^{-1}\pi_*^{\mathbb{T}} DE\mathbb{T}_+ = \mathbb{Q}[c, c^{-1}]$$

Def: $\pi_*^{\mathcal{A}}$: semifree $\mathbb{T}$-spectra $\longleftrightarrow \mathcal{A}_{sf}(X)$

$$\begin{tikzcd} X \arrow[r, mapsto] & (\pi_*^{\mathbb{T}}(DE\mathbb{T}_+\wedge X) \arrow[d] \\ & \pi_*^{\mathbb{T}}(\widehat{E\mathcal{F}}\wedge DE\mathbb{T}_+\wedge X)) \arrow[d, phantom, "\|"] \\ & t\otimes\pi_*(\Phi^{\mathbb{T}}X) \end{tikzcd}$$

$\pi_*^{\mathcal{A}}$ is in $\mathcal{A}(\mathbb{T})$, since $\widetilde{E\mathcal{F}} = \bigcup_{W^{\mathbb{T}}=0} S^W$

Now by ASS ~~techniques~~ technology

$$\mathcal{C}(\mathbb{Q}) = \left(\begin{tikzcd} t\otimes\mathbb{Q} \\ t\otimes\mathbb{Q} \arrow[u] \end{tikzcd}\right) = \pi_*^{\mathcal{A}}(\widetilde{E\mathcal{F}})$$

So realizable

$$[X, \tilde{E}\tilde{\mathfrak{F}}]^{\Pi} \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}(\Pi)}(\pi_*^{\mathcal{A}} X, \pi_*^{\mathcal{A}} \tilde{E}\tilde{\mathfrak{F}})$$

$$\wr$$

$$[\Phi^{\Pi} X, S^0] \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Q}}(\pi_*(\Phi^{\Pi} X), \mathbb{Q})$$

$$f(H_* B\Pi) = \begin{pmatrix} 0 \\ \uparrow \\ H_* B\Pi \end{pmatrix} = \begin{pmatrix} 0 \\ \uparrow \\ \pi_*^{\Pi}(\Sigma^{-1} E\Pi_+) \end{pmatrix} = \pi_*^{\mathcal{A}}(\Sigma^{-1} E\Pi_+)$$

$$[X, \Sigma^{-1} E\Pi_+]^{\Pi} \to \mathrm{Hom}_{\mathcal{A}(\Pi)}(\pi_*^{\mathcal{A}} X, \pi_*^{\mathcal{A}} E\Pi_+)$$

$$\wr$$

$$\mathrm{Hom}_{\mathbb{Q}[c]}(\pi_*^{\Pi}(DE\Pi_+), H_*(?))$$

<u>Convergence</u>: $\pi_*^{\mathcal{A}}(X) = 0 \Rightarrow X \simeq *$

$$\wr$$

$$(\pi_*^{\Pi}(DE\Pi_+ \wedge X) \to t \otimes \pi_*(\Phi^{\Pi} X))$$

If $\pi_*^{\mathcal{A}} X = 0 \Rightarrow \pi_* \Phi^{\Pi} X = 0 \Rightarrow \Phi^{\Pi} X \simeq *$

Hence, X is free: $DE\Pi_+ \wedge X \xleftrightarrow{\cong} S^0 \wedge X$

So $\pi_*^{\Pi}(DE\Pi_+ \wedge X) = \pi_*^{\Pi}(X) = \pi_*(\Sigma X/\Pi)$

$\Rightarrow \pi_*(\Sigma X) = 0$

EMSS

2018-04-12

Masterclass: Rigidity and algebraic models in stable homotopy theory

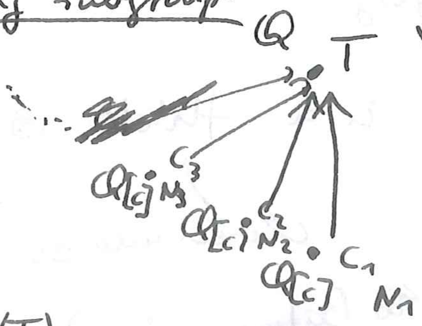
Lecture 8:

1

$\otimes \mathbb{Q}$

1. Collecting subgroups

$$\begin{tikzcd}
\mathbb{Q} & \bullet & T & V \\
\mathcal{O}_{\mathcal{F}} & \bullet\, 1 \arrow[u] & & N
\end{tikzcd}$$



$\mathfrak{X}_c = \mathrm{ConjSub}(T)$

$\mathfrak{X}_a = \mathrm{Sub}(T)$

$$\mathcal{A}(T) = \mathcal{A}_c = \{ N \xrightarrow{\beta} t\otimes V \mid \mathcal{E}^{-1}\beta \text{ is iso} \}$$

Has idempotents e_s

$$\mathcal{O}_{\mathcal{F}} = \prod_{s\geq 1} \mathbb{Q}[c], \quad \mathcal{E} = \mathcal{E}_T = \{ e(V) \mid V^T = 0 \}$$

$$\mathcal{A}_a(T) = \left\{ \underset{s\geq 1}{N_s \xrightarrow{\beta_s} \mathbb{Q}[c,c^{-1}]\otimes V} \;\middle|\; \begin{array}{l} ① \ \beta_s[\tfrac{1}{c}] \text{ iso} \\ ② \ \text{continuity} \end{array} \right\}$$

$$(N \to t\otimes V) \longmapsto \{ N_s = e_s N$$

$$e_s(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}) = \mathbb{Q}[c,c^{-1}]$$

$$\begin{tikzcd}
\boxed{N \to t\otimes V} \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & V \arrow[ll] \arrow[dl, dashed] \arrow[d] & \\
\prod_s N_s \arrow[r] & \mathcal{E}^{-1}\prod_s N_s \arrow[dr] & \prod_s N_s[c^{-1}] \arrow[d, equal] & \{ N_s \to \mathbb{Q}[c,c^{-1}]\otimes V \} \arrow[l, mapsto] \\
& & \prod_s \mathcal{E}^{-1} N_s &
\end{tikzcd}$$

$$\rightsquigarrow \mathcal{A}(T) \simeq \mathcal{A}_a(T)$$

Mostly stick to connected.

2. Why subgroups?

Either tradition or ...

If T a tt-category.

A <u>Balmer prime</u> $\mathcal{P}$ is a thick $\otimes$-ideal which is prime

closed under Δs & retracts — $x \in \mathcal{P}$ — $x \otimes y \in \mathcal{P} \Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P}$.

$$Spc(T) = \{\mathcal{P} \mid \mathcal{P} \text{ is Balmer prime}\}$$

$$Spc(R\text{-mod}^c) \xleftarrow{\cong} Spec(R) \quad (\text{order reversing})$$

Noeth ring

$$\mathcal{P}_b = \{M / M_{\mathcal{P}_a} \simeq 0\} \longleftarrow \mathcal{P}_a$$

Theorem: $Spc(G\text{-spectra}) \xleftarrow{\cong} Sub(G)/G$

compact

$$\mathcal{P}_H = \{X \mid \Phi^H X \simeq 0\} \longleftarrow H$$

$$\mathcal{P}_K \subseteq \mathcal{P}_H \iff \underbrace{K \sim K^g \lhd H}_{\text{cotoral inclusion}}$$

torus

G a torus Zariski top gen by closeness of pts

Proof: Consider inf spectra

$$Loc_{\otimes}(G\text{-spectra}) = \{\underbrace{G\text{-spectra}\langle\mathcal{H}\rangle}_{\text{geometric isotropy} \subseteq \mathcal{H}} \mid \mathcal{H} \subseteq Sub(G)/G\}$$

Ingredients: G-spectra $\langle H \rangle \simeq$ free $W_G(H)$-spectra

$$\simeq \text{tors-}H^*(BW_G^e(H))[W_G^d(H)]\text{-mod}$$

id component — discrete quot

$\underbrace{\text{tors} \cdots\cdots \text{-mod}}$

No proper nontriv
Loc ⊗-ideals
(Ex)

$X \leftsquigarrow X_n \; E\langle H\rangle$

Now go compact & suppose $\mathfrak{P} \in \mathrm{Spc}(G\text{-spectra})$

$$\mathrm{Loc}_{\otimes}(\mathfrak{p}) = G\text{-spectra}\langle \mathcal{H} \rangle$$

If $\mathcal{H}$ omits ~~admits~~ more than one subgp H, K.

Pick Y_H, Y_K. $\quad \Phi^H Y_H \not\simeq 0$

$\Phi^K Y_H \simeq 0$

$Y_H \otimes Y_K \in \mathrm{Loc}(\mathfrak{p}) \;\therefore \in \mathfrak{P}$ $\quad$ Hence $\mathcal{H} = \mathrm{All} \setminus \{H\}$

Hence $\quad \mathfrak{P} = \mathrm{Loc}_{\otimes}(\mathfrak{p}) \cap \text{compact} = \mathfrak{P}_H$

If K is cotoral in H

loc^n theorem $\mathcal{E}_K^{-1} H_H^*(X) \cong \mathcal{E}_K^{-1} H_H^*(\Phi^K X) = \mathcal{E}_H^{-1} H^* BH$

$\otimes H^*(\Phi^K X)$

hence $\mathfrak{P}_K \subseteq \mathfrak{P}_H$ $\quad$ //

3. $\mathcal{A}(G)$ for G a torus of rank $r \geq 0$

Ingredients $\quad \mathcal{O}_{\mathcal{F}/\circ} : \mathcal{K}_c^{op} \longrightarrow \mathrm{Rings}$

connected subgroups

$$K \longmapsto \mathcal{O}_{\mathcal{F}/K} = \prod_{\tilde K} H^*(BG/\tilde K)$$

$$\uparrow$$

$$L \longmapsto \mathcal{O}_{\mathcal{F}/L} = \prod_{\tilde L} H^*(BG/\tilde L)$$

$\mathcal{F}/K = \{\tilde K \mid \tilde K_e = K\}$ $\quad \tilde L \cdot K \quad \tilde K \quad H^*(BG/\tilde K)$

$\uparrow i_K \qquad\qquad\qquad\qquad\qquad \hat{\uparrow} \qquad \updownarrow \qquad \downarrow$

$\mathcal{F}/L = \{\tilde L \mid \tilde L_e = L\}$ $\quad \tilde L \quad (i^*)^{-1}(\tilde K) \quad H^*(BG/\tilde L)$

$\tilde L \in (i^*)^{-1}(\tilde K)$

$$\mathcal{E}_{K/L} = \left\{ e(V) \;\middle|\; \begin{array}{l} V \in \mathrm{Rep}(G/L) \\ V^K = 0 \end{array} \right\} \qquad \mathcal{O}_{\mathcal{F}/L} \longrightarrow H^{\bullet}(BG/L)$$

<u>Def</u>

Defⁿ $\mathbb{R} \quad \mathcal{X}'_c \longrightarrow \text{Rings}$

$$(K_0 > K_1 > \dots > K_s) \longmapsto \mathcal{E}^{-1}_{K_0/K_s} \mathcal{O}_{\mathcal{F}/K_s}$$

(only depends on K_0, K_s)

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{F}/T} = \mathbb{Q} \arrow[d] & & \bullet\, T \arrow[dd, dash] & V \arrow[d] \\
\mathcal{E}^{-1}_{T} \mathcal{O}_{\mathcal{F}} & \bullet & & P \\
\mathcal{O}_{\mathcal{F}} \arrow[u] & & \bullet\, 1 & N \arrow[u]
\end{tikzcd}$$

Now consider $\mathbb{R}$-modules

$\mathbb{M} : \mathcal{X}'_c \longrightarrow \mathbb{Q}$-vector spaces $\quad \mathbb{M}(F)$ is an $\mathbb{R}(F)$-module

$$\begin{tikzcd}
\mathbb{M}(F) & \mathbb{R}(F) & F \\
\mathbb{M}(E) \arrow[u] & \mathbb{R}(E) \arrow[u] & E \arrow[u]
\end{tikzcd}$$

$$(K) \longrightarrow (K > L) \longleftarrow (L)$$

$\mathcal{A}_c(G) = \{ \mathbb{M} \mid \mathbb{M}$ is an $\mathbb{R}$-module s.t. ① values only depend on K_0 & K_s ② The two maps of c and e are iso $\}$

$$\begin{tikzcd}
(K) \arrow[r] & (K > L) & (L) \arrow[l] \\
\mathbb{R}(K) \arrow[r] & \mathbb{R}(K > L) & \mathbb{R}(L) \arrow[l] \\
\mathcal{O}_{\mathcal{F}/K} \arrow[u, equal] \arrow[r, "i = \mathrm{inf}^{n}"] & \mathcal{E}^{-1}_{K/L} \mathcal{O}_{\mathcal{F}/L} & \mathcal{O}_{\mathcal{F}/L} \arrow[u, equal] \arrow[l, "\ell"']
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{E}^{-1}_{K/L} \mathbb{M}(L) = \ell_* \mathbb{M}(L) \arrow[d, "\cong", "\text{①c.}"'] & \\
\mathbb{M}(K) \arrow[r] \arrow[dr, bend right] & \mathbb{M}(K > L) & \mathbb{M}(L) \arrow[l] \arrow[ul, bend right] \\
& i_* \mathbb{M}(K) = \mathcal{E}^{-1}_{K/L} \mathcal{O}_{\mathcal{F}/L} \otimes_{\mathcal{O}_{\mathcal{F}/K}} \mathbb{M}(K) \arrow[u, "\cong"', "\text{ⓔ}"] &
\end{tikzcd}$$

3. The Adams spectral sequence & intrinsic formality of cells

Theorem: There is a functor

$$G\text{-spectra} \longrightarrow \mathcal{A}_c(G)$$

$$X \longmapsto \{(K \supseteq L) \mapsto \pi_*^G(S^{\infty V(K)} \wedge DEF/L_+ \wedge X)$$

$$\text{''}\pi_*^{G/K}\text{''}(S^{\infty V(K/L)} \wedge DEF/L_+ \wedge \Phi^L X)$$

$S^{\infty V(K)} = \bigcup S^V = E\langle K \subseteq \rangle$

$V^K = 0$

$X \wedge S^{\infty V(K)} \simeq \Phi^K X \wedge S^{\infty V(K)}$

so that there is an A Sp. Seq.

$$\mathrm{Ext}^{**}_{\mathcal{A}(G)}(\pi_*^{\mathcal{A}} X, \pi_*^{\mathcal{A}} Y) \Rightarrow [X, Y]^G_*$$

cgt for any X, Y inj dim $\mathcal{A}(G) = r$

$S^{\infty V(K)} = S^{\infty V(K/L)} \wedge S^{\infty V(L)}$

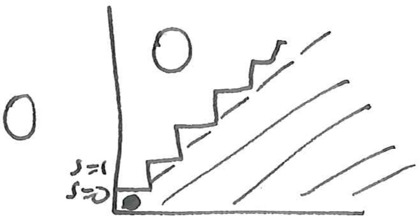
Corollary: If X; $\pi_*^{\mathcal{A}}(X) \cong \pi_*^{\mathcal{A}}(G/K_+)$ then $X \simeq G/K_+$

Proof:

$$\mathrm{Ext}_{\mathcal{A}(G)}(\underbrace{\pi_*^{\mathcal{A}}(X)}_{\pi_*^{\mathcal{A}}(G/K_+)}, \pi_*^{\mathcal{A}}(G/K_+)) \Rightarrow [X, G/K_+]^G$$

$\|$

0 for $t-s<s$



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2018-04-13

Masterclass: Rigidity and algebraic models in stable homotopy theory

Lecture 10:

of rank r

G=Torus
⊗ℚ

1. Geometry brings order out of chaos.

$H^* BG = \mathbb{Q}[x_1, .., x_r] =$ ring of fuctions on $\mathrm{Spec} =: L_{\mathbb{Q}} G$.

$$G_* = \mathrm{Hom}(\mathbb{T}, G) = \mathbb{Z}^r$$

$$\begin{tikzcd} H \arrow[r] & G \arrow[r, "\alpha"] & T \end{tikzcd}$$

$$\begin{tikzcd} H^*(BH) & H^* BG \arrow[l] & & H^* B T \arrow[ll, "\alpha^*"'] \arrow[d, phantom, "\shortparallel"] \\ & \ni & & \mathbb{Q}[c] \\ & e(\alpha) & & c \arrow[ll, mapsto] \end{tikzcd}$$

$L_{\mathbb{Q}} H$

$L_{\mathbb{Q}} G$

$V(e(\alpha))$

$K = \ker(\alpha_1) \cap \dots \cap \ker(\alpha_s)$

$L_{\mathbb{Q}} K = V(e(\alpha_1), \dots, e(\alpha_s))$

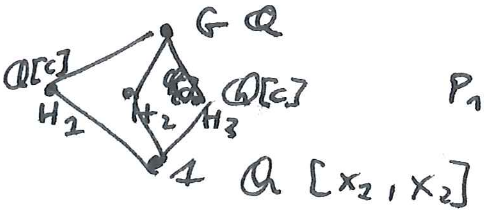
$\mathcal{E}_K = \{ e(V) \mid V^K = 0 \}$

$\mathcal{E}_K^{-1} H^* BG = H^* BG_{(L_{\mathbb{Q}} K)}$

$\mathcal{X}_c^d = d$-din^l conn subgroups $= Gr_d(L_{\mathbb{Q}} G)$

Example 1: rk $G = 1$

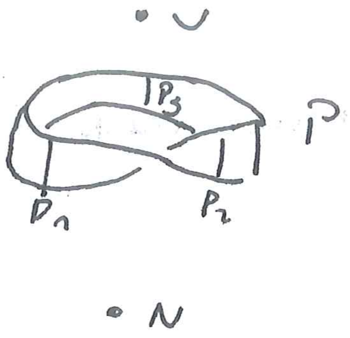
Example 2: rk $G = 2$

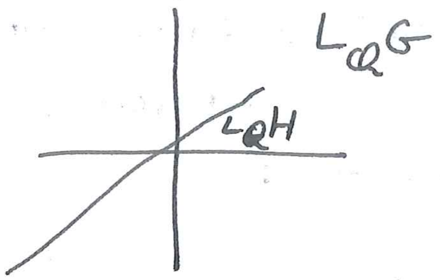


V

P_1 P_2 P_3

N





2. The Main Theorem:

Main Theorem: There is a Quillen equivalence G-spectra $\simeq d\mathcal{A}(G)$

In the circle case
$$\begin{tikzcd}
EF_+ \arrow[r] \arrow[d, "\simeq"] & S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{E}F \arrow[d] \\
EF_+ \wedge DEF_+ \arrow[r] & DEF_+ \arrow[r] & \tilde{E}F \wedge DEF_+
\end{tikzcd}$$

Spherical ~~scheme~~ theorem:

There is a $(r+1)$-cube of ring spectra with S^0 initial vertex. & other terms "essentially non equiv" & "formal".

Cube $= \mathcal{P}(\{0,1,2,\dots,r\})$

$$\mathbb{R}(d_0 > d_1 > \dots > d_s) = \prod_{\dim K_0 = d_0} \Big(S^{\infty V(K_0)} \wedge \prod_{\substack{\dim K_1 = d_1 \\ K_1 \subseteq K_0}} \Big(S^{\infty V(K_1)} \wedge \dots \wedge \prod_{\substack{\dim K_s = d_s \\ K_s \subseteq K_{s-1}}} \Big(S^{\infty V(K_s)} \wedge DEF/K_{s+} \Big) \dots \Big)$$

$$S^{\infty V(K)} = \bigcup_{V^K=0} S^V \simeq E\langle K\subseteq\rangle$$

3

Example 2: $r=2$ $\qquad \mathcal{F}/K = \{\tilde{K} \mid \tilde{K}_e = K\}$

$$\begin{tikzcd}
 & \prod_H S^{\infty V(H)} \wedge DE\mathcal{F}/H_+ \arrow[dd] \arrow[rr] & & S^{\infty V(G)} \wedge \prod_H S^{\infty V(H)} \wedge DE\mathcal{F}/H_+ \arrow[dd] \\
S^0 \arrow[ur] \arrow[rr] \arrow[dd] & & S^{\infty V(G)} \arrow[ur] \arrow[dd] & \\
 & \prod_H S^{\infty V(H)} \wedge DE\mathcal{F}_+ \arrow[rr] & & S^{\infty V(G)} \wedge \prod S^{\infty V(H)} \wedge DE\mathcal{F}_+ \\
DE\mathcal{F}_+ \arrow[ur] \arrow[rr] & & S^{\infty V(G)} \wedge DE\mathcal{F}_+ \arrow[ur] &
\end{tikzcd}$$

Proof: G-spectra $\simeq$ S^0-mod-G-spectra

$(S^0)^{\lrcorner}$ = cube with initial vertex omitted

$\simeq$ cell-$(S^0)^{\lrcorner}$-mod-G-spectra

↑ Spherical Schone theorem + cell. principle

$\simeq$ cell-$((S^0)^{\lrcorner})^G$-mod-spectra ⊛

← infl: $R(\underline{d})^G \to R(\underline{d})$ fixed point module adjunction + cell prin.

$\simeq$ cell-$\theta(((S^0)^{\lrcorner})^G)$-mod-$\mathbb{Q}$-mod

← Shiplification ← diagram of comm. DGA's.

$\simeq$ cell-$\pi_*^G((S^0)^{\lrcorner})$-mod-$\mathbb{Q}$-mod

← formality

$r=2$:

$$\begin{tikzcd}
 & \prod_{\dim H=1} H^*BG/H \arrow[dd] \arrow[rr] & & \mathcal{E}_G^{-1} \prod_H H^*BG/H \arrow[dd] \\
\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & \mathbb{Q} \arrow[ur] \arrow[dd] & \\
 & \prod_H \mathcal{E}_H^{-1} \prod_F H^*BG/F \arrow[rr] & & \mathcal{E}_G^{-1} \prod_H \mathcal{E}_H^{-1} \prod_F H^*BG/F \\
\prod_F H^*BG/F \arrow[ur] \arrow[rr] & & \mathcal{E}_G^{-1} \prod_F H^*(BG/F) \arrow[ur] &
\end{tikzcd}$$

~~[illegible]~~

circle case:

$(S^0)^{\lrcorner}$:

$$\begin{tikzcd}
 & \tilde{E}\mathcal{F} \arrow[d] \\
DE\mathcal{F}_+ \arrow[r] & \tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+
\end{tikzcd}$$

$\downarrow \pi_*^G$

$$\begin{tikzcd}
 & \mathbb{Q} \arrow[d] \\
\prod_F \mathbb{Q}[c]_N \arrow[r] & \mathcal{E}_G^{-1} \prod \mathbb{Q}(c)_P
\end{tikzcd}$$

✓

$\pi_*^G((S^0)^{\lrcorner})$

$\simeq d\mathcal{A}(G)$ ($\mathcal{A}(G)$ = diagram of modules with all maps extension of scalars)

$\uparrow$ use torsion functor Γ.

//

Fixed point module adjunction

R is a G-ring spectrum: $\mathrm{inf}\, R^G \longrightarrow R$

Gives $R\text{-mod-}G\text{-spectra} \underset{\Phi^G}{\overset{\longleftarrow}{\longrightarrow}} R^G\text{-mod-spectra}$

$$R \otimes_{\mathrm{inf}R^G} \mathrm{inf}\, N \longleftarrow N$$

Derived unit $\simeq$ on R

Hence $R\text{-cell-}\overbrace{R\text{-mod}\to G\text{-spectra}}^{\{R\wedge G/K_+\}}$

$\simeq R^G\text{- cell-}R^G\text{-mod-spectra}$

$= R^G\text{-mod-spectra}.$

Example 1: DEG_+

DEG_+-mod are generated by DEG_+

G = circle: $|K| = n$

$G/K_+ = S(z^n)_+ \longrightarrow S^0 \longrightarrow S^{z^n}$

$G/K_+ \wedge DEG_+ \longrightarrow DEG_+ \wedge S^0 \longrightarrow DEG_+ \wedge S^{z^n}$

✓ $\quad \cong$ Thom iso.

$DEG_+ \wedge S^2$

//

Example 2: $R = \widetilde{E}\mathcal{P}$ $R^G = S^0$

$R \wedge G/H_+ \simeq R \wedge \Phi^G G/H_+ \simeq *$ if $H = G$

$E\langle H \subseteq \rangle$-mod G spectra $\simeq S^0$-mod - G/H-spectra

$\shortparallel$

$S^{\infty V(H)}$

Adelic cohomology:

$\mathcal{X} = \mathcal{H}_c$, family of localizations. A

$1 \longrightarrow A_H$

$K < H$

$$\begin{tikzcd}
1 \arrow[r, "\eta"] \arrow[d, "\eta"'] & A_K \arrow[d, "\eta"] \\
A_H \arrow[r, "A_H \eta"'] & A_H A_K
\end{tikzcd}$$

$1 \xrightarrow{\eta} A_H$

$+ \; M : \mathcal{H}^{op} \to \mathbb{C}$

Can form $C^s_{ad}(\mathcal{H}, A_\bullet, M) = C^{d_0 > \dots > d_s}$

$$C^{d_0 > \dots > d_s} = \prod_{\substack{\dim K_0 = d_0}} A_{K_0} \prod_{\substack{\dim K_1 = d_1 \\ K_1 < K_0}} \cdots \prod_{\substack{\dim K_s = d_s \\ K_s < K_{s-1}}} A_{K_s} M(K_s))$$

eg. $A_H = \mathcal{E}_H^{-1}$ $M(H) = DEF/H_+$

$$H^*_{ad} = \begin{cases} \mathbb{Q} & * = 0 \\ \bigoplus_{\dim K = *} H_* BG/K_* , & \text{in general} \end{cases}$$

eg. $\mathcal{H} = Spec(R)$ ← comm. Noeth ring

$A_{\mathfrak{p}} = \underset{\text{localization at } \mathfrak{p}}{L_{\mathfrak{p}}} \Lambda_{\mathfrak{p}}$ ← completion at $\mathfrak{p}$. $M = R$ Beilinson-Parshin

~~[illegible]~~ Parshin

Proof of Spherical scheme Theorem:

$$\begin{tikzcd}
S^0 & \tilde{E}\mathcal{F} & \tilde{E}\mathcal{P} = S^{\infty V(G)} \\
E\langle \dim \geq 0\rangle \arrow[u, equal] \arrow[r] & E\langle \dim \geq 1\rangle \arrow[u, equal] \arrow[r] & E\langle \dim \geq 2\rangle \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
& \prod S^{\infty V(H)} \wedge DEF/H_+ \arrow[rr, "\cong"] \arrow[dd] & & \tilde{E}\mathcal{F} \wedge \prod S^{\infty V(H)} \wedge DEF/H_+ \arrow[rr] \arrow[dd] & & S^{\infty V(G)} \wedge \prod_H S^{\infty V(H)} \wedge DEF/H_+ \arrow[dd] \\
S^0 \arrow[ur] \arrow[rr] \arrow[dd] & & \tilde{E}\mathcal{F} \arrow[ur] \arrow[rr] \arrow[dd] & & S^{\infty V(G)} \arrow[ur] \arrow[dd] & \\
& \prod S^{\infty V(H)} \wedge DEF_+ \arrow[rr, "\cong"] & & \tilde{E}\mathcal{F} \wedge \Pi \arrow[rr] & & \cdot \\
DEF_+ \arrow[ur] \arrow[rr] & & \tilde{E}\mathcal{F} \wedge DEF_+ \arrow[ur] \arrow[rr] & & S^{\infty V(G)} \wedge DEF_+ \arrow[ur] &
\end{tikzcd}$$

front square: Take

pullback?

$$\begin{tikzcd}
E\langle 1\dim \text{ subgps}\rangle \arrow[r] \arrow[d, "\cong"] & \tilde{E}\mathcal{F} \arrow[r] \arrow[d] & \tilde{E}\mathcal{P} \arrow[d] \\
E\langle 1\dim 1\rangle \wedge \Pi \arrow[r] & \tilde{E}\mathcal{F} \wedge \prod S^{\infty V(H)} \wedge DEF/H_+ \arrow[r] & E\mathcal{P} \wedge \prod S^{\infty V(H)} \wedge DEF/H_+
\end{tikzcd}$$