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Masterclass: Rigidity and algebraic models in stable homotopy theory

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The story for a finite group

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orthogonal $G\text{Sp}$

$A(G)_{\mathbb{Q}} = [\mathbb{S}, \mathbb{S}]^G_* \ni e$ idempotent

- E - obj^{cof} in $G\text{Sp}$. $L_E\, G\text{Sp}$ with cof = cof

  f new w.e. = $f \wedge E$ is an old w.e. $\pi^H_*(f \wedge H)$ iso $\forall H \leq G$

- $\mathbb{S}_{\mathbb{Q}}$ $\quad L_{\mathbb{S}_{\mathbb{Q}}} G\text{Sp}$

  $\pi^H_*(X) \otimes \mathbb{Q} = \pi^H_*(X \wedge \mathbb{S}_{\mathbb{Q}})$

  models rational coh. theories

- $e\mathbb{S}_{\mathbb{Q}} = \underrightarrow{\text{holim}} \left(\mathbb{S}_{\mathbb{Q}} \xrightarrow{e} \mathbb{S}_{\mathbb{Q}} \xrightarrow{e} \mathbb{S}_{\mathbb{Q}} \to \cdots \right)$

  $L_{e\mathbb{S}_{\mathbb{Q}}} G\text{Sp}$

<u>Thm:</u> [Barnes] $\quad e \in A(G)_{\mathbb{Q}}$ $\quad G$ - compact Lie group

$$L_{\mathbb{S}_{\mathbb{Q}}} G\text{Sp} \underset{QE}{\simeq} L_{e\mathbb{S}_{\mathbb{Q}}} G\text{Sp} \times L_{(1-e)\mathbb{S}_{\mathbb{Q}}} G\text{Sp}$$

$$\underset{\prod}{\overset{\Delta}{\rightleftarrows}} \quad \perp$$

$$1 = \sum_{\substack{(H) \\ H \leq G}} e_{(H)}$$

The aim: to prove $\quad L_{\mathbb{S}_{\mathbb{Q}}} G\text{Sp} \underset{QE}{\simeq} \prod_{\substack{(H) \\ H \leq G}} Ch(\mathbb{Q}[W_G H]\text{-mod})$

H and G finite gp.

Notice: $H \overset{i}{\hookrightarrow} G$ it induces $A(G)_{\mathbb{Q}} \xrightarrow{i^*} A(H)_{\mathbb{Q}}$

$\mathcal{F}G/_G \qquad \mathcal{F}H/_H$

$\mathcal{F}G := \{K \mid K \le G, |N_G K/_K| < \infty\}$

$K \le H \le G$ in general $C_2 \le C_4 \le SO(2)$

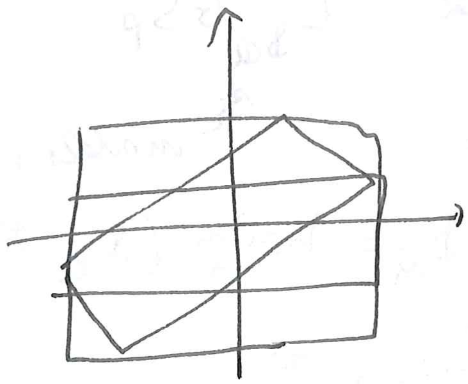
$i^*(e_{(K)_G}) = e_{((K)_H \cup (K')_H)}$

~~K or K'~~

$K \underset{G}{\sim} K'$ but not $K \underset{H}{\sim} K'$

$D_4 \le D_8 \le D_{16}$

$(D_4)_{D_8} \cup (D_4')_{D_8} = (D_4)_{D_{16}}$



Change of group functors and idempotents

$H \overset{i}{\hookrightarrow} G \qquad G_+ \wedge_H -$

$$\begin{tikzcd} GSp \arrow[r, "res = i^*"] & HSp \arrow[l, bend right, "G_+\wedge_H -"'] \arrow[l, bend left, "F_H(G_+{,}-)"] \end{tikzcd}$$

both QP w.r.t. stable model structures after loc. at $\mathbb{S}_{\mathbb{Q}}$ too.

Lemma: $e \in A(G)_{\mathbb{Q}}$

$$L_{e\mathbb{S}_{\mathbb{Q}}} GSp \underset{\perp}{\overset{res}{\rightleftarrows}} L_{i^*(e\mathbb{S}_{\mathbb{Q}})} HSp \text{ is a QP.}$$

$$i^*(e\mathbb{S}_{\mathbb{Q}}) = i^*(e)\mathbb{S}_{\mathbb{Q}}$$

Remark: $e_{(H)} \in A(G)_{\mathbb{Q}}$

$$L_{e_{(H)_G}\mathbb{S}_{\mathbb{Q}}} GSp \underset{res}{\overset{\perp}{\rightleftarrows}} L_{e_{(H)_N}\mathbb{S}_{\mathbb{Q}}} NSp \qquad N = N_G H$$

Sometimes a QP and sometimes not

$i^*(e_{(H)_G}) = e_{(H)_N}$

$i^*(e_{(H)_G}) \neq e_{(H)_N}$

Ex: $G = D_{16}$
$N = D_8$
$H = D_4$

Proof that if $i^*(e_{(H)_G}) = e_{(H)_N} \cup (H')_N$ then not a QP

take $G_+ \underset{H}{\wedge} -$ and show it doesn't preserve ac cofibr.

$N/H_+ \longrightarrow N/H_+ \vee N/H'_+$ an H-equiv in N

$G/H_+ \longrightarrow G/H_+ \vee \underbrace{G/H'_+}_{= G/H_+}$ not an H-equiv in G.

Lemma: $e_{(H)_G} \in A(G)_{\mathbb{Q}}$

$L_{e_{(H)_G}S\mathbb{Q}}\, G\mathrm{Sp} \underset{\perp}{\overset{res = i^*}{\rightleftarrows}} L_{e_{(H)_N}S\mathbb{Q}}\, N\mathrm{Sp}$ a QP

Algebraic model:

- $G = e$

Thm [Shipley] $\mathrm{Sp}_{\mathbb{Q}} \simeq_{QE} Ch(\mathbb{Q}\text{-mod})$

symm. monoidal

ϑ - derived

$X \in \mathrm{Sp}_{\mathbb{Q}}$ $H_*(\vartheta X) \cong \pi_* X$

<u>Cor:</u> H finite group

$$Sp_{\mathbb{Q}}[W] \underset{QE}{\simeq} Ch(\mathbb{Q}\text{-mod})[W] \underset{QE}{\simeq} Ch(\mathbb{Q}[W]\text{-mod})$$

W-objects in Sp

<u>Idea:</u> for finite G:

- split into simple pieces
- apply $(-)^H$ ⇝ "equivariance on the outside"
- apply Shipley's thm
- recognize / simplify

<u>Thm:</u> [Schwede-Shipley, Barnes, K.]

$$L_{S_{\mathbb{Q}}} GSp \simeq_{QE} \prod_{\substack{(H)\\ H \leq G}} Ch(\mathbb{Q}[\underbrace{W_G H}_{N_G H/H}]\text{-mod})$$

symmetric monoidal

<u>Proof:</u>

$$L_{S_{\mathbb{Q}}} GSp \overset{[\text{Barnes}]}{\simeq_{QE}} \prod_{\substack{(H)\\ H\leq G}} L_{e_{(H)_G} S_{\mathbb{Q}}} GSp$$

$W = N_G H/H$

$$L_{e_{(H)_G} S_{\mathbb{Q}}} GSp \underset{QE}{\overset{res}{\underset{\perp}{\rightleftarrows}}} L_{e_{(H)_N} S_{\mathbb{Q}}} NSp \underset{QE}{\overset{\perp}{\underset{(-)^H}{\rightleftarrows}}} L_{e_1 S_{\mathbb{Q}}} WSp$$

$$\underset{QE}{\simeq} Sp_{\mathbb{Q}}[W] \overset{[\text{Shipley}]}{\simeq} Ch(\mathbb{Q}[W]\text{-mod})$$

Cellularization:

$\mathcal{C}$ - stable model structure

K - set of ~~compact~~ objects in $\mathcal{C}$

Define K-cell-$\mathcal{C}$: new fib = fib

new w.e. = $[k, f]^{\mathcal{C}}_*$ is an iso $\forall k \in K$

Thm: [Greenlees–Shipley] Cellularization Principle

$\mathcal{C}, \mathcal{D}$ - stable model str.

$F: \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : R$ QP

K - set of compact obj in $\mathcal{C}$

$F: K\text{-cell-}\mathcal{C} \underset{\perp}{\rightleftarrows} F(\hat{c}K)\text{-cell-}\mathcal{D}$; K-cell-$\mathcal{C}$, $F(\hat{c}K)$-cell-$\mathcal{D}$ exist and $F(\hat{c}K)$ are compact in $\mathcal{D}$.

QP and if $k \to R\hat{f}(F\hat{c}k)$ is a w.e. then QE.

Free GSp: G connected

Thm: [Greenlees - Shipley]:

Free $GSp \underset{QE}{\simeq}$ torsion-H^*BG-mod in $Ch(\mathbb{Q}\text{-mod})$

$\|$

~~$GSp_{\mathbb{Q}}$~~

$G_+\text{-cell-}GSp_{\mathbb{Q}} \underset{\perp}{\overset{DEG_+\wedge -}{\rightleftarrows}} (DEG_+ \wedge G_+)\text{-cell-}DEG_+\text{-mod}$ (in $GSp_{\mathbb{Q}}$)

QE.

Proof.

$G_+ \to DEG_+ \wedge G_+$ w.e. in G_+-cell-$GSp_{\mathbb{Q}}$.

cell. principle.

$$(DEG_+ \wedge G_+)\text{-cell-}DEG_+\text{-mod} \underset{(-)^G}{\overset{\perp}{\rightleftarrows}} (DEG_+ \wedge G_+)^G\text{-cell-}\underbrace{(EG_+)^G}_{DBG_+}\text{-mod}$$

(in $GSp_{\mathbb{Q}}$) QE (in $Sp_{\mathbb{Q}}$)

$$\underset{QE}{\overset{[\text{Shipley}]}{\longleftrightarrow}} \Theta((G_+ \wedge DEG_+)^G)\text{-cell-}\Theta(DEG_+)\text{-mod in } Ch(\mathbb{Q}\text{-mod})$$

$H_*(DBG_+)$ = polynomial even deg. gen. formal

$$H_*((G_+ \wedge DEG_+)^G) = \pi_*((G_+ \wedge DEG_+)^G) = \Sigma^g \mathbb{Q}$$

$$\underset{QE}{\simeq} \mathbb{Q}\text{-cell-}H^*(BG_+)\text{-mod (in } Ch(\mathbb{Q}\text{-mod}))$$

$$\underset{QE}{\simeq} \text{tors-}H^*(BG_+)\text{-mod (in } Ch(\mathbb{Q}\text{-mod}))$$