Quantized Functional Analysis, Tensor Norms and the Grothendieck Program

Magdalena Musat
University of Copenhagen

YWC*A
Department of Mathematics
University of Copenhagen
August 5 and 6, 2017

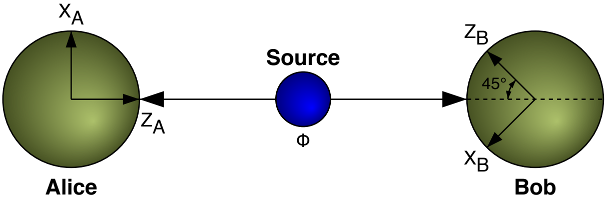
Lecture I

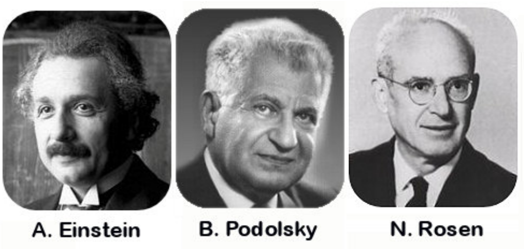
Classical and Quantum Correlations, Operator Algebras, and an Asymptotic Behaviour of Quantum Channels

August 5, 2017

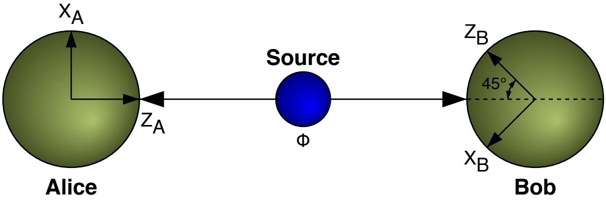
Outline

The Einstein–Podolsky–Rosen paradox





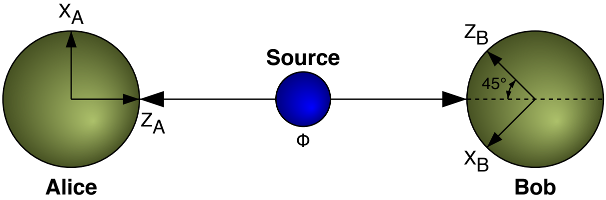
Alice and Bob, residing in spatially separated labs, each receives a quantum system on which they can perform measurements.



When Alice measures an observable on her part of the system, the wave function representing the combined system changes, and this affects subsequent measurements made by Bob.

EPR (1935) claim that this can only have two explanations: Either there is some interaction between the particles (even though they are separated)—but this violates *locality*; or the information about the outcome of all possible measurements is already present in both particles (hidden variables)—*this has been refuted by experiments!*

Alice and Bob, residing in spatially separated labs, each receives a quantum system on which they can perform measurements.



When Alice measures an observable on her part of the system, the wave function representing the combined system changes, and this affects subsequent measurements made by Bob.

EPR (1935) claim that this can only have two explanations: Either there is some interaction between the particles (even though they are separated)—but this violates *locality*; or the information about the outcome of all possible measurements is already present in both particles (hidden variables)—*this has been refuted by experiments!*

Let's say that Alice and Bob can measure any one of k possible observables each with m possible outcomes. Let

$$P(a, b \mid x, y)$$

be the probability that Alice gets outcome a and Bob outcome b, when Alice measures observable x and Bob measures observable y.

Mathematical model (classical): There is a probability space (Ω, μ) of "hidden variables" and measurable partitions $\{X_a^x\}_a$ and $\{Y_b^y\}_b$ of Ω (one for each x and y) such that

$$P(a, b \mid x, y) = \mu(X_a^x \cap Y_b^y).$$

Let $\mathcal{C} = \mathcal{C}(m, k) \subseteq M_{mk}(\mathbb{R})$ be the set of all matrices of the form

$$\left[\mu(X_a^x \cap Y_b^y)\right]_{(a,x),(b,y)}$$

where (Ω, μ) ranges over all probability spaces and $\{X_a^x\}_a$, $\{Y_b^y\}_b$ range over all partitions of Ω.

Let's say that Alice and Bob can measure any one of k possible observables each with m possible outcomes. Let

$$P(a, b \mid x, y)$$

be the probability that Alice gets outcome a and Bob outcome b, when Alice measures observable x and Bob measures observable y.

Mathematical model (classical): There is a probability space (Ω, μ) of "hidden variables" and measurable partitions $\{X_a^x\}_a$ and $\{Y_b^y\}_b$ of Ω (one for each x and y) such that

$$P(a, b \mid x, y) = \mu(X_a^x \cap Y_b^y).$$

Let $\mathcal{C} = \mathcal{C}(m, k) \subseteq M_{mk}(\mathbb{R})$ be the set of all matrices of the form

$$\left[\mu(X_a^x \cap Y_b^y)\right]_{(a,x),(b,y)}$$

where (Ω, μ) ranges over all probability spaces and $\{X_a^x\}_a$, $\{Y_b^y\}_b$ range over all partitions of Ω.

Definition: A PVM (projection valued measure) is an m-tuple $P_1, P_2, \ldots, P_m$ of projections on some Hilbert space H such that $\sum_{j=1}^{m} P_j = I$.

Mathematical model (quantum case, I): For each pair of observables x, y, Alice has a PVM $\{P_a^x\}_a$ on a Hilbert space H_A, and Bob has a PVM $\{Q_b^y\}_b$ on a Hilbert space H_B such that

$$P(a, b \mid x, y) = \langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle,$$

for some unit vector $\psi \in H_A \otimes H_B$, representing the *quantum state*. Let $\mathcal{Q}_s = \mathcal{Q}_s(m, k) \subseteq M_{mk}(\mathbb{R})$ be the **closure** the set of matrices

$$\left[\langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \right]_{(a,x),(b,y)}$$

where $\{P_a^x\}_a$, $\{Q_b^y\}_b$ range over all PVM's in Hilbert spaces H_A, H_B, and ψ ranges over all unit vectors in $H_A \otimes H_B$.

Definition: A PVM (projection valued measure) is an m-tuple $P_1, P_2, \ldots, P_m$ of projections on some Hilbert space H such that $\sum_{j=1}^m P_j = I$.

Mathematical model (quantum case, I): For each pair of observables x, y, Alice has a PVM $\{P_a^x\}_a$ on a Hilbert space H_A, and Bob has a PVM $\{Q_b^y\}_b$ on a Hilbert space H_B such that

$$P(a, b \mid x, y) = \langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle,$$

for some unit vector $\psi \in H_A \otimes H_B$, representing the *quantum state*. Let $\mathcal{Q}_s = \mathcal{Q}_s(m, k) \subseteq M_{mk}(\mathbb{R})$ be the **closure** the set of matrices

$$\left[\langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \right]_{(a,x),(b,y)}$$

where $\{P_a^x\}_a$, $\{Q_b^y\}_b$ range over all PVM's in Hilbert spaces H_A, H_B, and ψ ranges over all unit vectors in $H_A \otimes H_B$.

The axiom of spatially separated systems asserts *commutativity* of observables measured by Alice and Bob, leading to the following:

Mathematical model (quantum case, II): For each pair of observables x and y, Alice has a PVM $\{P_a^x\}_x$ and Bob has a PVM $\{Q_b^y\}_y$ on the *same* Hilbert space H, satisfying $P_a^x Q_b^y = Q_b^y P_a^x$, for all a, b, x, y, such that

$$P(a, b \mid x, y) = \langle (P_a^x Q_b^y)\psi, \psi \rangle,$$

for some unit vector $\psi \in H$, which represents the *quantum state*. Let $\mathcal{Q}_c = \mathcal{Q}_c(m, k) \subseteq M_{mk}(\mathbb{R})$ be the set of all matrices

$$\Big[\langle (P_a^x Q_b^y)\psi, \psi \rangle\Big]_{(a,x),(b,y)}$$

where $\{P_a^x\}_a$, $\{Q_b^y\}_b$ range over all PVM's in a Hilbert space H, and ψ ranges over all unit vectors in H.

The probability matrix $\left[P(a,b \mid x,y)\right]$ belongs to one of the three closed convex sets: $\mathcal{C} \subseteq \mathcal{Q}_s \subseteq \mathcal{Q}_c \subseteq M_{mk}(\mathbb{R})$ accoding to which of the three mathematical models we choose. Recall:

$$\mathcal{C} = \left\{\left[\mu(X_a^x \cap Y_b^y)\right] : \{X_a^x\}_x, \{Y_b^y\}_y \text{ partitions in } (\Omega,\mu)\right\}$$

$$\mathcal{Q}_s = \mathrm{cl}\left\{\left[\langle (P_a^x \otimes Q_b^y)\psi, \psi\rangle\right] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVM's}, \psi \in H_A \otimes H_B\right\}$$

$$\mathcal{Q}_c = \left\{\left[\langle (P_a^x Q_b^y)\psi, \psi\rangle\right] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVM's}, [P_a^x, Q_b^y] = 0, \psi \in H\right\}$$

- $\mathcal{C} \neq \mathcal{Q}_s$ by Bell's inequalities (verified experimentally to be violated starting in the early 1980's)
- $\mathcal{C} \neq \mathcal{Q}_s$ also follows from the fact that the constant K_G in Grothendieck's inequality is > 1!

Conjecture (Tsirelson): $\mathcal{Q}_s = \mathcal{Q}_c$.

The probability matrix $\left[P(a,b \mid x,y)\right]$ belongs to one of the three closed convex sets: $\mathcal{C} \subseteq \mathcal{Q}_s \subseteq \mathcal{Q}_c \subseteq M_{mk}(\mathbb{R})$ accoding to which of the three mathematical models we choose. Recall:

$$\begin{aligned}
\mathcal{C} &= \left\{\left[\mu(X_a^x \cap Y_b^y)\right] : \{X_a^x\}_x, \{Y_b^y\}_y \text{ partitions in } (\Omega,\mu)\right\} \\
\mathcal{Q}_s &= \mathrm{cl}\left\{\left[\langle (P_a^x \otimes Q_b^y)\psi,\psi\rangle\right] : \{P_a^x\}_a, \{Q_b^y\}_b \,\text{PVM's},\, \psi \in H_A \otimes H_B\right\} \\
\mathcal{Q}_c &= \left\{\left[\langle (P_a^x Q_b^y)\psi,\psi\rangle\right] : \{P_a^x\}_a, \{Q_b^y\}_b \,\text{PVM's}, [P_a^x, Q_b^y] = 0, \psi \in H\right\}
\end{aligned}$$

• $\mathcal{C} \neq \mathcal{Q}_s$ by Bell's inequalities (verified experimentally to be violated starting in the early 1980's)

• $\mathcal{C} \neq \mathcal{Q}_s$ also follows from the fact that the constant K_G in Grothendieck's inequality is > 1!

Conjecture (Tsirelson): $\mathcal{Q}_s = \mathcal{Q}_c$.

The probability matrix $\left[P(a,b\mid x,y)\right]$ belongs to one of the three closed convex sets: $\mathcal{C}\subseteq\mathcal{Q}_s\subseteq\mathcal{Q}_c\subseteq M_{mk}(\mathbb{R})$ accoding to which of the three mathematical models we choose. Recall:

$$\begin{aligned}
\mathcal{C} &= \left\{\left[\mu(X_a^x\cap Y_b^y)\right] : \{X_a^x\}_x, \{Y_b^y\}_y \text{ partitions in } (\Omega,\mu)\right\}\\
\mathcal{Q}_s &= \mathrm{cl}\left\{\left[\langle(P_a^x\otimes Q_b^y)\psi,\psi\rangle\right] : \{P_a^x\}_a, \{Q_b^y\}_b \,\text{PVM's}, \psi\in H_A\otimes H_B\right\}\\
\mathcal{Q}_c &= \left\{\left[\langle(P_a^xQ_b^y)\psi,\psi\rangle\right] : \{P_a^x\}_a, \{Q_b^y\}_b \,\text{PVM's}, [P_a^x,Q_b^y]=0, \psi\in H\right\}
\end{aligned}$$

• $\mathcal{C}\neq\mathcal{Q}_s$ by Bell's inequalities (verified experimentally to be

violated starting in the early 1980's)

• $\mathcal{C}\neq\mathcal{Q}_s$ also follows from the fact that the constant K_G in Grothendieck's inequality is >1!

Conjecture (Tsirelson): $\mathcal{Q}_s=\mathcal{Q}_c$.

Outline

C^*-algebras and groups can be used to study quantum correlations.

Given C^*-algebras A, B, consider the *algebraic* tensor prod $A \odot B$.

Minimal (spatial) C^*-norm: For $x \in A \odot B$,

$$\|x\|_{\min} = \|(\pi_A \otimes \pi_B)(x)\|_{B(H_A \otimes H_B)},$$

where $\pi_A \colon A \to B(H_A)$ and $\pi_B \colon B \to B(H_B)$ are faithful *-rep's.

Maximal C^*-norm: For $x \in A \odot B$,

$$\|x\|_{\max} = \sup \|(\pi_A \times \pi_B)(x)\|_{B(H)},$$

sup being taken over all *-rep's $\pi_A \colon A \to B(H)$, $\pi_B \colon B \to B(H)$ with $[\pi_A(a), \pi_B(b)] = 0$, $a \in A, b \in B$.

- $A \otimes_{\min} B$ and $A \otimes_{\max} B$ are the completions of $A \odot B$ wrt the min, resp., the max tensor norm. In general, $A \otimes_{\min} B \neq A \otimes_{\max} B$.

- $C^*(\mathbb{F}_n)$ = full group C^*-algebra associated with $\mathbb{F}_n$ $(2 \le n \le \infty)$.

Kirchberg (1993) asked the following:

(a) $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \stackrel{?}{=} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$

(b) $B(H) \otimes_{\max} B(H) \stackrel{?}{=} B(H) \otimes_{\min} B(H)$

He **proved** that

(c) $C^*(\mathbb{F}_n) \otimes_{\max} B(H) = C^*(\mathbb{F}_n) \otimes_{\min} B(H)$

and that *(a)* is equivalent to the *Connes embedding problem* !

▶ That there is a unique C^*-norm on $C^*(\mathbb{F}_n) \odot C^*(\mathbb{F}_n)$ is also equivalent to $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ being RFD (residually finite dimens). **Brown-Ozawa**: $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ is QD (quasidiagonal).

▶ (b) was answered in the negative by **Junge–Pisier (1995)**. The proof uses a noncommutative version of Grothendieck's inequality! **Pisier–Ozawa (2014)**: There are at least $c = 2^{\aleph_0}$ non-equivalent norms on $B(H) \otimes B(H)$.

- $C^*(\mathbb{F}_n)$ = full group C^*-algebra associated with $\mathbb{F}_n$ $(2 \leq n \leq \infty)$.

Kirchberg (1993) asked the following:

(a) $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \stackrel{?}{=} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$

(b) $B(H) \otimes_{\max} B(H) \stackrel{?}{=} B(H) \otimes_{\min} B(H)$

He **proved** that

(c) $C^*(\mathbb{F}_n) \otimes_{\max} B(H) = C^*(\mathbb{F}_n) \otimes_{\min} B(H)$

and that *(a)* is equivalent to the *Connes embedding problem* !

▶ That there is a unique C^*-norm on $C^*(\mathbb{F}_n) \odot C^*(\mathbb{F}_n)$ is also equivalent to $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ being RFD (residually finite dimens).
Brown-Ozawa: $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ is QD (quasidiagonal).

▶ (b) was answered in the negative by **Junge–Pisier (1995)**. The proof uses a noncommutative version of Grothendieck's inequality!
Pisier–Ozawa (2014): There are at least $c = 2^{\aleph_0}$ non-equivalent norms on $B(H) \otimes B(H)$.

- $C^*(\mathbb{F}_n)$ = full group C^*-algebra associated with $\mathbb{F}_n$ $(2 \leq n \leq \infty)$.

Kirchberg (1993) asked the following:

(a) $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \stackrel{?}{=} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$

(b) $B(H) \otimes_{\max} B(H) \stackrel{?}{=} B(H) \otimes_{\min} B(H)$

He **proved** that

(c) $C^*(\mathbb{F}_n) \otimes_{\max} B(H) = C^*(\mathbb{F}_n) \otimes_{\min} B(H)$

and that *(a)* is equivalent to the *Connes embedding problem* !

▶ That there is a unique C^*-norm on $C^*(\mathbb{F}_n) \odot C^*(\mathbb{F}_n)$ is also equivalent to $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ being RFD (residually finite dimens). **Brown-Ozawa**: $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ is QD (quasidiagonal).

▶ (b) was answered in the negative by **Junge–Pisier (1995)**. The proof uses a noncommutative version of Grothendieck's inequality! **Pisier–Ozawa (2014)**: There are at least $c = 2^{\aleph_0}$ non-equivalent norms on $B(H) \otimes B(H)$.

- $C^*(\mathbb{F}_n)$ = full group C^*-algebra associated with $\mathbb{F}_n$ $(2 \le n \le \infty)$.

Kirchberg (1993) asked the following:

(a) $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \overset{?}{=} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$

(b) $B(H) \otimes_{\max} B(H) \overset{?}{=} B(H) \otimes_{\min} B(H)$

He **proved** that

(c) $C^*(\mathbb{F}_n) \otimes_{\max} B(H) = C^*(\mathbb{F}_n) \otimes_{\min} B(H)$

and that *(a)* is equivalent to the *Connes embedding problem* !

▶ That there is a unique C^*-norm on $C^*(\mathbb{F}_n) \odot C^*(\mathbb{F}_n)$ is also equivalent to $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ being RFD (residually finite dimens). **Brown-Ozawa**: $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ is QD (quasidiagonal).

▶ (b) was answered in the negative by **Junge–Pisier (1995)**. The proof uses a noncommutative version of Grothendieck's inequality! **Pisier–Ozawa (2014)**: There are at least $c = 2^{\aleph_0}$ non-equivalent norms on $B(H) \otimes B(H)$.

- $C^*(\mathbb{F}_n)$ = full group C^*-algebra associated with $\mathbb{F}_n$ $(2 \le n \le \infty)$.

Kirchberg (1993) asked the following:

(a) $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \stackrel{?}{=} C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n)$

(b) $B(H) \otimes_{\max} B(H) \stackrel{?}{=} B(H) \otimes_{\min} B(H)$

He **proved** that

(c) $C^*(\mathbb{F}_n) \otimes_{\max} B(H) = C^*(\mathbb{F}_n) \otimes_{\min} B(H)$

and that *(a)* is equivalent to the *Connes embedding problem* !

▶ That there is a unique C^*-norm on $C^*(\mathbb{F}_n) \odot C^*(\mathbb{F}_n)$ is also equivalent to $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ being RFD (residually finite dimens). **Brown-Ozawa**: $C^*(\mathbb{F}_n \times \mathbb{F}_n)$ is QD (quasidiagonal).

▶ (b) was answered in the negative by **Junge–Pisier (1995)**. The proof uses a noncommutative version of Grothendieck's inequality! **Pisier–Ozawa (2014)**: There are at least $c = 2^{\aleph_0}$ non-equivalent norms on $B(H) \otimes B(H)$.

- $\Gamma = \mathbb{Z}_m * \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ (k free factors).
  - $C^*(\Gamma) \simeq C^*(\mathbb{Z}_m) *_1 C^*(\mathbb{Z}_m) *_1 \cdots *_1 C^*(\mathbb{Z}_m)$.
  - $C^*(\mathbb{Z}_m) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_m$, where e_j are proj'n and $\sum_j e_j = 1$.
- Let $e_a^x \in C^*(\Gamma)$ be the projection e_a in the xth free factor above.

Theorem (Fritz, 2009).

- $\mathcal{Q}_s = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{Q}_c = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

Idea of proof:

- If $\{P_a^x\}_a \subseteq B(H)$ is a PVM, $\exists$ *-hom $\Phi\colon C^*(\Gamma) \to B(H)$ st $\Phi(e_a^x) = P_a^x, \forall a$.
- If $\{P_a^x\}_a, \{Q_b^y\}_b \subseteq B(H)$ are commuting PVM's, $\exists$ *-hom $\Psi\colon C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \to B(H)$ st $\Psi(e_a^x \otimes e_b^y) = P_a^x Q_b^y, \forall a, x, b, y$.

- $\Gamma = \mathbb{Z}_m * \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ (k free factors).
  - $C^*(\Gamma) \simeq C^*(\mathbb{Z}_m) *_1 C^*(\mathbb{Z}_m) *_1 \cdots *_1 C^*(\mathbb{Z}_m)$.
  - $C^*(\mathbb{Z}_m) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_m$, where e_j are proj'n and $\sum_j e_j = 1$.
- Let $e_a^x \in C^*(\Gamma)$ be the projection e_a in the xth free factor above.

Theorem (Fritz, 2009).

- $\mathcal{Q}_s = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{Q}_c = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

Idea of proof:

- If $\{P_a^x\}_a \subseteq B(H)$ is a PVM, $\exists$ *-hom $\Phi \colon C^*(\Gamma) \to B(H)$ st $\Phi(e_a^x) = P_a^x, \forall a$.
- If $\{P_a^x\}_a, \{Q_b^y\}_b \subseteq B(H)$ are commuting PVM's, $\exists$ *-hom $\Psi \colon C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \to B(H)$ st $\Psi(e_a^x \otimes e_b^y) = P_a^x Q_b^y, \forall a, x, b, y$.

Outline

Theorem (Kirchberg '93, Fritz '09, Ozawa '12): *TFAE:*

1 $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$, *for all* $m, k \geq 2$,

2 $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,

3 *Tsirelson's conjecture is true, i.e.,* $\mathcal{Q}_s = \mathcal{Q}_c$,

4 *The Connes embedding problem has positive answer.*

Recall that

$$\mathcal{Q}_s = \mathrm{cl}(\mathcal{Q}_s^o), \quad \text{where}$$

$$\mathcal{Q}_s^o = \left\{ \left[\langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \right] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVM's}, \psi \in H_A \otimes H_B \right\},$$

$$\mathcal{Q}_c = \left\{ \left[\langle (P_a^x Q_b^y)\psi, \psi \rangle \right] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVM's}, [P_a^x, Q_b^y] = 0, \psi \in H \right\}.$$

Slofstra (June, 2016) proved that $\mathcal{Q}_s^o \neq \mathcal{Q}_c$.

Theorem (Kirchberg '93, Fritz '09, Ozawa '12): *TFAE:*

1. $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$, *for all* $m, k \geq 2$,
2. $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,
3. *Tsirelson's conjecture is true, i.e.,* $\mathcal{Q}_s = \mathcal{Q}_c$,
4. *The Connes embedding problem has positive answer.*

Recall that

$$\begin{aligned}
\mathcal{Q}_s &= \mathrm{cl}(\mathcal{Q}_s^o), \quad \text{where} \\
\mathcal{Q}_s^o &= \left\{ \left[\langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \right] : \{P_a^x\}_a, \{Q_b^y\}_b \,\text{PVM's}, \psi \in H_A \otimes H_B \right\}, \\
\mathcal{Q}_c &= \left\{ \left[\langle (P_a^x Q_b^y)\psi, \psi \rangle \right] : \{P_a^x\}_a, \{Q_b^y\}_b \,\text{PVM's}, [P_a^x, Q_b^y] = 0, \psi \in H \right\}.
\end{aligned}$$

Slofstra (June, 2016) proved that $\mathcal{Q}_s^o \neq \mathcal{Q}_c$.

Theorem (Kirchberg '93, Fritz '09, Ozawa '12): *TFAE:*

1. $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$, *for all* $m, k \geq 2$,
2. $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,
3. *Tsirelson's conjecture is true, i.e.,* $\mathcal{Q}_s = \mathcal{Q}_c$,
4. *The Connes embedding problem has positive answer.*

Recall that

$$\begin{aligned}
\mathcal{Q}_s &= \mathrm{cl}(\mathcal{Q}_s^o), \quad \text{where} \\
\mathcal{Q}_s^o &= \left\{ \left[\langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \right] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVM's}, \psi \in H_A \otimes H_B \right\}, \\
\mathcal{Q}_c &= \left\{ \left[\langle (P_a^x Q_b^y)\psi, \psi \rangle \right] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVM's}, [P_a^x, Q_b^y] = 0, \psi \in H \right\}.
\end{aligned}$$

Slofstra (June, 2016) proved that $\mathcal{Q}_s^o \neq \mathcal{Q}_c$.

The celebrated *Connes embedding problem* asks whether every II_1-factor (on a separable Hilbert space) embeds in an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$. [**A. Connes**; *Classification of injective factors*, Ann. of Math. 104 **(1976)**]

Following **Dykema–Juschenko (2009)**, for $n \geq 1$, let

$$\mathcal{F}_n = \mathrm{cl} \bigcup_{k \geq 1} \left\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_k(u_i u_j^*)\,, u_j \in \mathcal{U}(M_k(\mathbb{C})) \right\}$$

$$\mathcal{G}_n = \Big\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_M(u_i u_j^*)\,, u_j \in \mathcal{U}(M)\,, \text{ for some vN algebra } (M, \tau_M) \text{ with n.f. tracial state} \Big\}$$

- By a refinement of **Kirchberg (1993)**: The Connes embedding problem has a positive answer iff $\mathcal{F}_n = \mathcal{G}_n$, for all $n \geq 1$.

The celebrated *Connes embedding problem* asks whether every II_1-factor (on a separable Hilbert space) embeds in an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$. [**A. Connes**; *Classification of injective factors*, Ann. of Math. 104 **(1976)**]

Following **Dykema–Juschenko (2009)**, for $n \geq 1$, let

$$\mathcal{F}_n = \mathrm{cl}\bigcup_{k\geq 1}\left\{B=(b_{ij})\in M_n(\mathbb{C}) : b_{ij}=\tau_k(u_i u_j^*)\,,\, u_j\in\mathcal{U}(M_k(\mathbb{C}))\right\}$$

$$\mathcal{G}_n = \Big\{B=(b_{ij})\in M_n(\mathbb{C}) : b_{ij}=\tau_M(u_i u_j^*)\,,\, u_j\in\mathcal{U}(M)\,,\ \text{for some vN algebra } (M,\tau_M) \text{ with n.f. tracial state}\Big\}$$

- By a refinement of **Kirchberg (1993)**: The Connes embedding problem has a positive answer iff $\mathcal{F}_n = \mathcal{G}_n$, for all $n \geq 1$.

The celebrated *Connes embedding problem* asks whether every II_1-factor (on a separable Hilbert space) embeds in an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$. [**A. Connes**; *Classification of injective factors*, Ann. of Math. 104 **(1976)**]

Following **Dykema–Juschenko (2009)**, for $n \geq 1$, let

$$\mathcal{F}_n = \mathrm{cl} \bigcup_{k \geq 1} \left\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_k(u_i u_j^*) \, , u_j \in \mathcal{U}(M_k(\mathbb{C})) \right\}$$

$$\mathcal{G}_n = \Big\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_M(u_i u_j^*) \, , u_j \in \mathcal{U}(M) \, , \text{ for some vN algebra } (M, \tau_M) \text{ with n.f. tracial state} \Big\}$$

- By a refinement of **Kirchberg (1993)**: The Connes embedding problem has a positive answer iff $\mathcal{F}_n = \mathcal{G}_n$, for all $n \geq 1$.

Outline

▶ **Choi (1973)**: $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear map is completely positive (CP) iff

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ can be chosen linearly indep., in which case $d =$ Choi-rank(T). This is the *Choi canonical form* of T.

A quantum channel in dimension n is a CP *trace-preserving* linear map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, and write $T \in \mathrm{CPT}(n)$. If, moreover, T is unital, write $T \in \mathrm{UCPT}(n)$.

- **Choi (1973)**: $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear map is completely positive (CP) iff

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ can be chosen linearly indep., in which case $d =$ Choi-rank(T). This is the *Choi canonical form* of T.

A quantum channel in dimension n is a CP *trace-preserving* linear map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, and write $T \in \mathrm{CPT}(n)$. If, moreover, T is unital, write $T \in \mathrm{UCPT}(n)$.

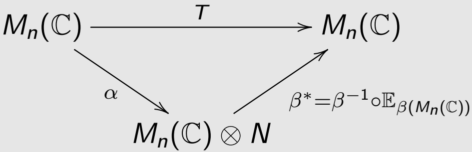
Definition (Anantharaman-Delaroche '05): *A UCPT(n) map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called* factorizable *if $\exists$ vN algebra N with n.f.tracial state ϕ and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$ s.t.* $T = \beta^* \circ \alpha$.

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow{\quad T \quad} & M_n(\mathbb{C}) \\ & \searrow^{\alpha} \qquad \nearrow_{\beta^* = \beta^{-1} \circ \mathbb{E}_{\beta(M_n(\mathbb{C}))}} & \\ & M_n(\mathbb{C}) \otimes N & \end{array}$$

Problem (Anantharaman-Delaroche): *Is every unital quantum channel factorizable?*

- Automorphisms of $M_n(\mathbb{C})$ are factorizable (with $N = \mathbb{C}$). The set $\mathcal{F}(M_n(\mathbb{C}))$ of factorizable UCPT(n) maps is convex, hence $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subseteq \mathcal{F}(M_n(\mathbb{C}))$. It is also closed, thus compact.

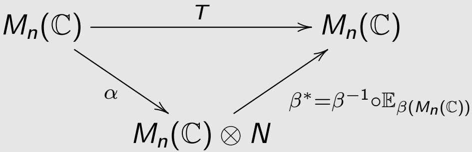
Definition (Anantharaman-Delaroche '05): *A UCPT(n) map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called factorizable if $\exists$ vN algebra N with n.f.tracial state ϕ and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$ s.t.* $T = \beta^* \circ \alpha$.



Problem (Anantharaman-Delaroche): *Is every unital quantum channel factorizable?*

- Automorphisms of $M_n(\mathbb{C})$ are factorizable (with $N = \mathbb{C}$). The set $\mathcal{F}(M_n(\mathbb{C}))$ of factorizable UCPT(n) maps is convex, hence $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subseteq \mathcal{F}(M_n(\mathbb{C}))$. It is also closed, thus compact.

Definition (Anantharaman-Delaroche '05): *A UCPT(n) map $T: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called factorizable if $\exists$ vN algebra N with n.f.tracial state ϕ and unital $*$-homs $\alpha, \beta: M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$ s.t. $T = \beta^* \circ \alpha$.*



Problem (Anantharaman-Delaroche): *Is every unital quantum channel factorizable?*

- Automorphisms of $M_n(\mathbb{C})$ are factorizable (with $N = \mathbb{C}$). The set $\mathcal{F}(M_n(\mathbb{C}))$ of factorizable UCPT(n) maps is convex, hence $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subseteq \mathcal{F}(M_n(\mathbb{C}))$. It is also closed, thus compact.

Theorem (Haagerup-M, 2011): *Let* $T \in UCPT(n)$, $n \geq 3$.

Then T *is factorizable iff* $\exists$ *vN algebra* N *with normal faithful tracial state* τ_N *and a unitary* $u \in M_n(\mathbb{C}) \otimes N$ *such that*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)\,, \quad x \in M_n(\mathbb{C})\,.$$

We say that T *exactly factors through* $M_n(\mathbb{C}) \otimes N$.

- **Haagerup-M** : $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ iff T has an exact factorization through $M_n(\mathbb{C}) \otimes L^\infty([0,1])$.

Theorem (Haagerup-M, 2011): *Let* $T \in UCPT(n)$, $n \geq 3$.

Then T *is factorizable iff* $\exists$ *vN algebra* N *with normal faithful tracial state* τ_N *and a unitary* $u \in M_n(\mathbb{C}) \otimes N$ *such that*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

We say that T *exactly factors through* $M_n(\mathbb{C}) \otimes N$.

- **Haagerup-M** : $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ iff T has an exact factorization through $M_n(\mathbb{C}) \otimes L^\infty([0,1])$.

Proposition: *Let $T \in UCPT(n)$, with Choi canonical form $Tx = \sum_{i=1}^{d} a_i^* x a_i$, $x \in M_n(\mathbb{C})$. If $d \geq 2$ and the set*

$$\{a_i^* a_j : 1 \leq i, j \leq d\}$$

is linearly independent, then T is not factorizable.

Example

With $a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we obtained the first example of a non-factorizable map. It is the Holevo-Werner channel W_3^-.

Proposition: *Let $T \in UCPT(n)$, with Choi canonical form $Tx = \sum_{i=1}^{d} a_i^* x a_i$, $x \in M_n(\mathbb{C})$. If $d \geq 2$ and the set*

$$\{a_i^* a_j : 1 \leq i, j \leq d\}$$

is linearly independent, then T is not factorizable.

Example

With $a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$,
$a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we obtained the first example of a non-factorizable map. It is the Holevo-Werner channel W_3^-.

Proposition: *Let $T \in UCPT(n)$, with Choi canonical form $Tx = \sum_{i=1}^{d} a_i^* x a_i$, $x \in M_n(\mathbb{C})$. If $d \geq 2$ and the set*

$$\{a_i^* a_j : 1 \leq i, j \leq d\}$$

is linearly independent, then T is not factorizable.

Example

With $a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$,
$a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we obtained the first example of a non-factorizable map. It is the Holevo-Werner channel W_3^-.

Outline

Theorem (Haagerup-M, 2015): *Let $T \in \mathcal{F}(M_n(\mathbb{C}))$. TFAE:*

(1) *T exactly factors through a vN algebra embeddable into $\mathcal{R}^\omega$, i.e., there exist $(N, \tau_N) \hookrightarrow \mathcal{R}^\omega$ and $u \in \mathcal{U}(M_n(N))$ s.t.*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)\,, \quad x \in M_n(\mathbb{C})\,.$$

(2) *T approximately factors through matrix algebras.*

(3) $\lim_{k\to\infty} d_{cb}\Big(T \otimes S_k\,, conv\big(Aut(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)\Big) = 0,$ *where $S_k(y) = \tau_k(y)1_k$, for all $y \in M_k(\mathbb{C})$.*

Theorem (Haagerup-M): *The Connes embedding problem has a positive answer iff every map $T \in \mathcal{F}(M_n(\mathbb{C}))$ satisfies one of the equivalent conditions in above theorem, for all $n \geq 3$.*

Theorem (Haagerup-M, 2015): *Let* $T \in \mathcal{F}(M_n(\mathbb{C}))$. *TFAE:*

(1) T *exactly factors through a vN algebra embeddable into* $\mathcal{R}^\omega$, *i.e., there exist* $(N, \tau_N) \hookrightarrow \mathcal{R}^\omega$ *and* $u \in \mathcal{U}(M_n(N))$ *s.t.*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

(2) T *approximately factors through matrix algebras.*

(3) $\lim_{k\to\infty} d_{cb}\Big(T \otimes S_k, conv\big(Aut(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)\Big) = 0,$ *where* $S_k(y) = \tau_k(y)1_k$, *for all* $y \in M_k(\mathbb{C})$.

Theorem (Haagerup-M): *The Connes embedding problem has a positive answer iff every map* $T \in \mathcal{F}(M_n(\mathbb{C}))$ *satisfies one of the equivalent conditions in above theorem, for all* $n \geq 3$.

Outline

Theorem (M): *For each $n \geq 3$, $\exists T \in conv(Aut(M_n(\mathbb{C})))$ which does not admit an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, for any $k \geq 1$.*

Corollary: *For $n \geq 3$, the set of maps in $\mathcal{F}(M_n(\mathbb{C}))$ which have an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, for any $k \geq 1$, is neither convex, nor closed.*

- The set of maps in $\mathcal{F}(M_n(\mathbb{C}))$ which exactly factor through $M_n(\mathbb{C}) \otimes A$, where A is a finite dimensional C^*-algebra is convex, for all $n \geq 3$. We do not know if this set is closed.

Theorem (M): *For each $n \geq 3$, $\exists T \in conv(Aut(M_n(\mathbb{C})))$ which does not admit an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, for any $k \geq 1$.*

Corollary: *For $n \geq 3$, the set of maps in $\mathcal{F}(M_n(\mathbb{C}))$ which have an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, for any $k \geq 1$, is neither convex, nor closed.*

- The set of maps in $\mathcal{F}(M_n(\mathbb{C}))$ which exactly factor through $M_n(\mathbb{C}) \otimes A$, where A is a finite dimensional C^*-algebra is convex, for all $n \geq 3$. We do not know if this set is closed.

Theorem (M): *For each $n \geq 3$, $\exists T \in conv(Aut(M_n(\mathbb{C})))$ which does not admit an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, for any $k \geq 1$.*

Corollary: *For $n \geq 3$, the set of maps in $\mathcal{F}(M_n(\mathbb{C}))$ which have an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, for any $k \geq 1$, is neither convex, nor closed.*

- The set of maps in $\mathcal{F}(M_n(\mathbb{C}))$ which exactly factor through $M_n(\mathbb{C}) \otimes A$, where A is a finite dimensional C^*-algebra is convex, for all $n \geq 3$. We do not know if this set is closed.

Let A, B be unital C^*-alg, $\pi : A \otimes_{\max} B \to A \otimes_{\min} B$ the canonical surj. Let $S(A \otimes_{\max} B)$ be the state space of $A \otimes_{\max} B$, and set

$$S(A \otimes_{\min} B) = \Big\{ \varphi \circ \pi : \varphi \text{ state on } A \otimes_{\min} B \Big\},$$

$$S_*(A \otimes_{\max} B) = \Big\{ \varphi \in S(A \otimes_{\max} B) : \exists \pi_A : A \to B(H_A), \pi_B : B \to B(H_B)$$
$$*\text{–reps}, \exists \psi \in H_A \otimes H_B : \varphi(x) = \langle (\pi_a \otimes \pi_B)(x)\psi, \psi \rangle \, \forall x \Big\}$$

- We have $S_*(A \otimes_{\max} B) \subseteq S(A \otimes_{\min} B) \subseteq S(A \otimes_{\max} B)$.
- $S_*(A \otimes_{\max} B)$ is weak*-dense in $S(A \otimes_{\min} B)$.

Question: Give examples where $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$.
(Also relates to Slofstra's result.)

Let A, B be unital C^*-alg, $\pi : A \otimes_{\max} B \to A \otimes_{\min} B$ the canonical surj. Let $S(A \otimes_{\max} B)$ be the state space of $A \otimes_{\max} B$, and set

$$S(A \otimes_{\min} B) = \left\{ \varphi \circ \pi : \varphi \text{ state on } A \otimes_{\min} B \right\},$$

$$S_*(A \otimes_{\max} B) = \Big\{ \varphi \in S(A \otimes_{\max} B) : \exists \pi_A\colon A \to B(H_A), \pi_B\colon B \to B(H_B) \\ *\text{-reps}, \exists \psi \in H_A \otimes H_B : \varphi(x) = \langle (\pi_a \otimes \pi_B)(x)\psi, \psi \rangle \; \forall x \Big\}$$

- We have $S_*(A \otimes_{\max} B) \subseteq S(A \otimes_{\min} B) \subseteq S(A \otimes_{\max} B)$.
- $S_*(A \otimes_{\max} B)$ is weak*-dense in $S(A \otimes_{\min} B)$.

Question: Give examples where $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$. (Also relates to Slofstra's result.)

Let A, B be unital C^*-alg, $\pi : A \otimes_{\max} B \to A \otimes_{\min} B$ the canonical surj. Let $S(A \otimes_{\max} B)$ be the state space of $A \otimes_{\max} B$, and set

$$S(A \otimes_{\min} B) = \Big\{ \varphi \circ \pi : \varphi \text{ state on } A \otimes_{\min} B \Big\},$$

$$S_*(A \otimes_{\max} B) = \Big\{ \varphi \in S(A \otimes_{\max} B) : \exists \pi_A \colon A \to B(H_A), \pi_B \colon B \to B(H_B) \; *\text{–reps}, \exists \psi \in H_A \otimes H_B : \varphi(x) = \langle (\pi_a \otimes \pi_B)(x)\psi, \psi \rangle \; \forall x \Big\}$$

- We have $S_*(A \otimes_{\max} B) \subseteq S(A \otimes_{\min} B) \subseteq S(A \otimes_{\max} B)$.
- $S_*(A \otimes_{\max} B)$ is weak*-dense in $S(A \otimes_{\min} B)$.

Question: Give examples where $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$.
(Also relates to Slofstra's result.)

Let $A = B = C^*(\Gamma)$, where $\Gamma = \mathbb{Z}_m * \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ (k factors).
Set $M = \mathrm{Span}\left\{ e_a^x \otimes e_b^y : a, b, x, y \right\} \subseteq A \otimes_{\max} B$ finite dimens.

$$\begin{aligned}
\mathcal{Q}_s^o &= \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right]_{a,b,x,y} : \varphi \in S_*(A \otimes_{\max} B) \right\} \\
&= \{ \varphi_{|M} : \varphi \in S_*(A \otimes_{\max} B) \} = S_*(A \otimes_{\max} B)_{|M} \\
\mathcal{Q}_s &= \mathrm{cl}(\mathcal{Q}_s^o) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right]_{a,b,x,y} : \varphi \in S(A \otimes_{\min} B) \right\} \\
&= S(A \otimes_{\min} B)_{|M} \\
\mathcal{Q}_c &= S(A \otimes_{\max} B)_{|M} .
\end{aligned}$$

Slofstra showed $\mathcal{Q}_s^o \neq \mathcal{Q}_c$, i.e., $S_*(A \otimes_{\max} B)_{|M} \neq S(A \otimes_{\max} B)_{|M}$.
In particular, $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$.

Problem: Find explicit $\varphi \in S(A \otimes_{\max} B) : \varphi_{|M} \notin S_*(A \otimes_{\max} B)_{|M}$.

- **Brown-M-Rørdam**: Explicit example of $\omega \in S(A \otimes_{\max} B)$ such that $\omega \notin S_*(A \otimes_{\max} B)$. However, $\omega_{|M} \in S_*(A \otimes_{\max} B)_{|M}$.

Let $A = B = C^*(\Gamma)$, where $\Gamma = \mathbb{Z}_m * \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ (k factors).
Set $M = \mathrm{Span}\left\{ e_a^x \otimes e_b^y : a, b, x, y \right\} \subseteq A \otimes_{\max} B$ finite dimens.

$$\begin{aligned}
\mathcal{Q}_s^o &= \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right]_{a,b,x,y} : \varphi \in S_*(A \otimes_{\max} B) \right\} \\
&= \{ \varphi_{|M} : \varphi \in S_*(A \otimes_{\max} B) \} = S_*(A \otimes_{\max} B)_{|M} \\
\mathcal{Q}_s &= \mathrm{cl}(\mathcal{Q}_s^o) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right]_{a,b,x,y} : \varphi \in S(A \otimes_{\min} B) \right\} \\
&= S(A \otimes_{\min} B)_{|M} \\
\mathcal{Q}_c &= S(A \otimes_{\max} B)_{|M} .
\end{aligned}$$

Slofstra showed $\mathcal{Q}_s^o \neq \mathcal{Q}_c$, i.e.,$S_*(A \otimes_{\max} B)_{|M} \neq S(A \otimes_{\max} B)_{|M}$.
In particular, $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$.

Problem: Find explicit $\varphi \in S(A \otimes_{\max} B) : \varphi_{|M} \notin S_*(A \otimes_{\max} B)_{|M}$.

- **Brown-M-Rørdam**: Explicit example of $\omega \in S(A \otimes_{\max} B)$ such that $\omega \notin S_*(A \otimes_{\max} B)$. However, $\omega_{|M} \in S_*(A \otimes_{\max} B)_{|M}$.

Let $A = B = C^*(\Gamma)$, where $\Gamma = \mathbb{Z}_m * \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ (k factors).
Set $M = \mathrm{Span}\Big\{ e_a^x \otimes e_b^y : a, b, x, y \Big\} \subseteq A \otimes_{\max} B$ finite dimens.

$$\begin{aligned} \mathcal{Q}_s^o &= \Big\{ \Big[\varphi(e_a^x \otimes e_b^y)\Big]_{a,b,x,y} : \varphi \in S_*(A \otimes_{\max} B) \Big\} \\ &= \{\varphi_{|M} : \varphi \in S_*(A \otimes_{\max} B)\} = S_*(A \otimes_{\max} B)_{|M} \\ \mathcal{Q}_s &= \mathrm{cl}(\mathcal{Q}_s^o) = \Big\{ \Big[\varphi(e_a^x \otimes e_b^y)\Big]_{a,b,x,y} : \varphi \in S(A \otimes_{\min} B) \Big\} \\ &= S(A \otimes_{\min} B)_{|M} \\ \mathcal{Q}_c &= S(A \otimes_{\max} B)_{|M} . \end{aligned}$$

Slofstra showed $\mathcal{Q}_s^o \neq \mathcal{Q}_c$, i.e., $S_*(A \otimes_{\max} B)_{|M} \neq S(A \otimes_{\max} B)_{|M}$.
In particular, $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$.

Problem: Find explicit $\varphi \in S(A \otimes_{\max} B) : \varphi_{|M} \notin S_*(A \otimes_{\max} B)_{|M}$.

- **Brown-M-Rørdam**: Explicit example of $\omega \in S(A \otimes_{\max} B)$ such that $\omega \notin S_*(A \otimes_{\max} B)$. However, $\omega_{|M} \in S_*(A \otimes_{\max} B)_{|M}$.

Let $A = B = C^*(\Gamma)$, where $\Gamma = \mathbb{Z}_m * \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ (k factors).
Set $M = \mathrm{Span}\left\{ e_a^x \otimes e_b^y : a, b, x, y \right\} \subseteq A \otimes_{\max} B$ finite dimens.

$$\begin{aligned} \mathcal{Q}_s^o &= \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right]_{a,b,x,y} : \varphi \in S_*(A \otimes_{\max} B) \right\} \\ &= \{ \varphi_{|M} : \varphi \in S_*(A \otimes_{\max} B) \} = S_*(A \otimes_{\max} B)_{|M} \end{aligned}$$

$$\begin{aligned} \mathcal{Q}_s &= \mathrm{cl}(\mathcal{Q}_s^o) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right]_{a,b,x,y} : \varphi \in S(A \otimes_{\min} B) \right\} \\ &= S(A \otimes_{\min} B)_{|M} \end{aligned}$$

$$\mathcal{Q}_c = S(A \otimes_{\max} B)_{|M} \,.$$

Slofstra showed $\mathcal{Q}_s^o \neq \mathcal{Q}_c$, i.e.,$S_*(A \otimes_{\max} B)_{|M} \neq S(A \otimes_{\max} B)_{|M}$.
In particular, $S_*(A \otimes_{\max} B) \neq S(A \otimes_{\max} B)$.

Problem: Find explicit $\varphi \in S(A \otimes_{\max} B) : \varphi_{|M} \notin S_*(A \otimes_{\max} B)_{|M}$.

- **Brown-M-Rørdam**: Explicit example of $\omega \in S(A \otimes_{\max} B)$ such that $\omega \notin S_*(A \otimes_{\max} B)$. However, $\omega_{|M} \in S_*(A \otimes_{\max} B)_{|M}$.